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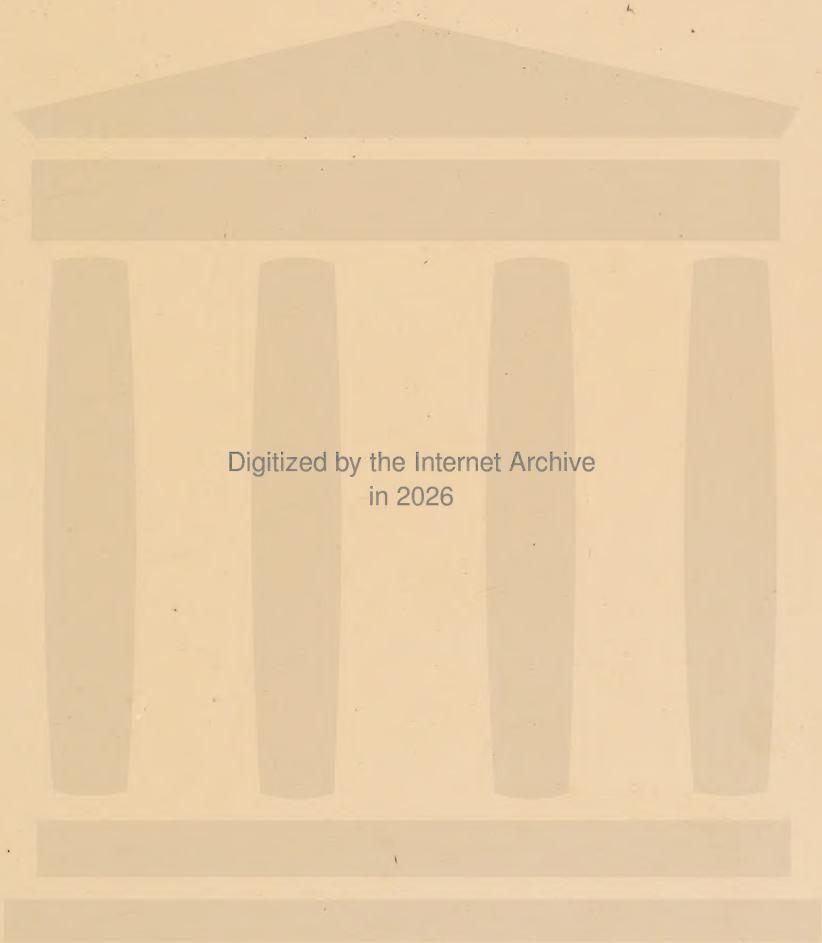

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PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY

SECOND SERIES

VOLUME 22

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RECORDS OF PROCEEDINGS AT MEETINGS.

Annual General Meeting.

Thursday, November 9th, 1922.

Mr. H. W. RICHMOND, President, in the Chair, and later
Prof. W. H. YOUNG, President, in the Chair.

Present thirty members and two visitors.

The President announced that the Manchester Mathematical Society had made a donation of £10 to the Society.

The President announced that a reciprocal agreement had been made with the American Mathematical Society for a reduction of subscription, as follows :—

1.—Members of the London Mathematical Society not residing in the United States or Canada will be admitted to full membership in the American Mathematical Society by payment of the entrance fee of $5 and one-half of the full annual dues (at present $6), such membership carrying with it the *Bulletin* without extra charge. If any such member desires the *Transactions* in place of the *Bulletin*, he shall pay annual dues of $5; if he desires both the *Bulletin* and *Transactions*, he shall pay half the full annual dues plus $5.

2.—Members of the London Mathematical Society not residing in the United States or Canada, who are life members of the American Mathematical Society, may obtain the *Transactions* by payment of $5 annually.

3.—Members of the American Mathematical Society not residing in Great Britain and Ireland will be admitted to full membership in the London Mathematical Society by payment of the entrance fee of £2. 2s. and one-half of the full annual subscription (at present £2. 2s.).

4.—Any person, who is a member of both Societies, shall not be entitled to the reductions above mentioned as regards both Societies, but shall pay full dues or subscriptions to that one of the two Societies of which he first became a member.

5.—The above arrangements take effect as regards the London Mathematical Society for the year beginning November 1st, 1921, and as regards the American Mathematical Society for the year beginning January 1st, 1922.

The Treasurer of the London Mathematical Society will, so far as practicable, give effect to these arrangements when he next applies for subscriptions due. But any member who is entitled to reduction in the London Mathematical Society's subscription, and who may be asked to pay the full subscription, is requested to write to the Treasurer, as cases will probably arise where the Treasurer will be unaware of rights to reduced subscription.

Members of the London Mathematical Society not residing in the United States or Canada, and who are at present subscribing members of the American Mathematical Society, are requested to communicate direct with the latter Society at 501 West 116th Street, New York, U.S.A., with regard to their subscriptions, and also as regards their options to receive the *Transactions* or the *Bulletin*, or both.

The Treasurer presented his Report for the Session 1921–22, which was duly received.

Lt.-Col. Cunningham was appointed Auditor.

Mr. L. C. Piggott was elected a member of the Society.

Messrs. W. E. Baker, P. N. Dé, A. E. Ingham, R. E. C. Kennedy, H. Thomas, E. C. Titchmarsh, A. G. Willox, Sir John Heaton, and Mrs. E. B. Veitch were nominated for election.

Prof. Dr. A. Korn, of Berlin, was nominated for election under By-law 20B.

Mr. W. L. Ferrar was admitted into the Society.

The meeting proceeded to the election of a Council and Officers. The list is as follows :—President, W. H. Young ; Vice-Presidents, A. L. Dixon, A. E. Jolliffe, H. W. Richmond ; Treasurer, A. E. Western ; Secretaries, G. H. Hardy, G. N. Watson ; other members of the Council, J. E. Campbell, L. N. G. Filon, H. Hilton, Miss H. P. Hudson, J. E. Littlewood, A. E. H. Love, E. A. Milne, L. J. Mordell, F. B. Pidduck.

The retiring President then vacated the Chair, which was taken by Prof. Young, and delivered an address on " The Mathematical Problems of Shell-Flight."

The following papers were communicated by title from the Chair :—

*Asymptotic Expansions and Generalized Hypergeometric Functions : W. R. Burwell.

*Determinants whose Elements are Determinants : W. L. Ferrar.

*Transformation of Variables in a Multiple Integral : B. von Kerékjártó.

(i) The Properties of certain Numbers, (ii) Contribution to the Evaluation of Persymmetric Determinants : C. Krishnamachari and M. Bhimasena Rao.

†On Trigonometric Series involving Algebraic Numbers : L. J. Mordell.

†Analogues of Waring's Problem for Rational Numbers : H. W. Richmond.

*Weber's Integral Theorem : E. C. Titchmarsh.

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* Printed in this volume.

† Printed in Volume 21.

NOTES AND ABSTRACTS.

Determinants whose Elements are Determinants

W. L. FERRAR.

Let Δ denote the determinant formed by taking the first n rows of an array of n columns and $n+m$ rows (columns a, b, ..., k),

A_r the minor of a_r in Δ,

$\Delta_{1,p}$ the determinant obtained by replacing in Δ the p-th row by the $(n+1)$-th row of the array,

$A_r^{1,p}$ the minor of a_r in this determinant, with similar notation for other determinants.

The paper uses the relation

$$A_p^{t,s}\Delta \equiv A_p\Delta_{t,s} - A_s\Delta_{t,p}, \tag{1}$$

and its analogues to express a determinant such as

$$|A_x, \ldots, D_w, E_p^{r,s}, \ldots, K_q^{v,t}|$$

as a product of the Δ's.

Theorem C of Prof. E. T. Whittaker's paper (*Proc. Edin. Math. Soc.*, Vol. 36, 1918, pp. 107–115) gives a particular case.

The relation (1) is also used to give a short proof of a theorem given by Cayley (without proof).*

In conclusion, an extension of Bazin's theorem is obtained by use of previous results, and Theorem B of the paper quoted above shown to follow easily from the mode of proof here used.

Commutative Ordinary Differential Operators †

J. L. BURCHNALL *and* T. W. CHAUNDY.

We consider the problem of the form of two operators $\phi(D)$, $\psi(D)$ polynomial in D, such that their alternant vanishes identically. We solve the problem for the case of $\phi(D)$ a quadratic operator, hyperelliptic functions being in general necessary; when, in addition, $\psi(D)$ is cubic, then $\phi(D)\,y = 0$ is effectively Lamé's equation.

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* Muir, *History of Determinants*, Vol. 2, p. 15.

† Read at the Meeting of June 8th, 1922; printed in Vol. 21.

It is also shown that any two such commutative operators are linked by an algebraic identity $f\{\phi(D), \psi(D)\} \equiv 0$, and that from $\phi(D)$, $\psi(D)$ can be developed a group of commutative pairs obeying the same identity. The close connection of the theory with the theory of adjoint operators is exhibited.

Weber's Integral Theorem

E. C. TITCHMARSH.

This paper contains a proof of the following theorem :—

If $x > a > 0$ and $\sqrt{t}\, f(t)$ is summable in the infinite interval (a, ∞), and $f(t)$ is of bounded variation in a neighbourhood of $t = x$, then

$$\int_0^\infty \frac{C_\nu(xu, au)}{J_\nu^2(au) + Y_\nu^2(au)}\, u\, du \int_a^\infty C_\nu(tu, au)\, t f(t)\, dt = \tfrac{1}{2}\{f(x+0) + f(x-0)\},$$

where J_ν and Y_ν are the Bessel functions of the first and second kinds of order ν, and

$$C_\nu(xu, au) = J_\nu(xu)\, Y_\nu(au) - Y_\nu(xu)\, J_\nu(au).$$

A Theorem concerning Summable Series

G. H. HARDY *and* J. E. LITTLEWOOD.

The results concerning power series and Fourier series, stated in a former communication to the Society,* depend on the following general arithmetic theorem, which has many interesting applications.

The necessary and sufficient condition that the series Σa_n *should be summable* (C, r) *is that the series* Σb_n, *where*

$$b_n = \frac{a_n}{n+1} + \frac{a_{n+1}}{n+2} + \ldots \quad (C, r-1)$$

should be summable $(C, r-1)$.

This holds for all positive integral values of r; and, if a series Σa_n is said to be summable $(C, -1)$ when it converges and $na_n = o(1)$, then it holds for $r = 0$.

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* See *Records*, June 8th, 1922.

Thursday, December 14th, 1922.

Prof. W. H. YOUNG, President, in the Chair.

Present nineteen members and a visitor.

Messrs. W. E. Baker, P. N. Dé, A. E. Ingham, R. E. C. Kennedy, H. Thomas, E. C. Titchmarsh, A. G. Willox, Sir John Heaton, and Mrs. E. B. Veitch were elected members of the Society.

Prof. Dr. A. Korn, of Berlin, was elected under By-law 20B.

Messrs. J. P. Gabbatt, S. F. Grace, and C. G. F. James were nominated for election.

Prof. Watson reported that the number of members on November 1st, 1921, was 349. 45 new members were elected in the Session 1921–1922 (including 7 under By-law 20B), and 6 members died. The number of members on November 1st, 1922, was therefore 388.

The Auditor's report was received, and a vote of thanks to the Auditor (Lt.-Col. Cunningham) was carried unanimously.

Prof. A. E. Jolliffe read two papers (i) "Collinear Apolar Triads on Cubic Curves," (ii) "The Inflexions and Inflexional Tangents of the Two-Cusped Quartic." Prof. Hilton raised various questions in connection with the papers.

Prof. G. H. Hardy read a paper* (written in collaboration with Mr. J. E. Littlewood) "Some Problems of Partitio Numerorum (V): A further contribution to the study of Goldbach's Problem." Dr. Western, Mr. Mordell, Prof. Watson, Mr. Brown, and Mrs. Young took part in a discussion of the paper.

Prof. G. N. Watson read a paper* "The Theorems of Clausen and Cayley on Products of Hypergeometric Functions." Prof. Hardy and the President raised questions concerning the paper.

The following papers were communicated by title from the Chair:—

*Poncelet's Poristic Polygons: T. W. Chaundy.

*Some Limiting Cases in the Theory of Integral Equations: A. C Dixon.

Some Formulæ of Ramanujan: G. H. Hardy.

The Rational Parametric Solutions of $x_1^4+x_2^4+x_3^4+x_4^4 = 2A^2w^2$: T. Stuart.

A Class of Expansions in Series of Bessel Functions: E. C. Titchmarsh.

*Ternary Perpetuant Types: A. Young.

* Printed in this volume.

NOTES AND ABSTRACTS.

Collinear Apolar Triads on Cubic Curves

A. E. JOLLIFFE.

The polar of a point A with respect to the polar conic of a point B with respect to a cubic is the same as the polar of B with respect to the polar conic of A with respect to the cubic and is called the dipolar of A and B with respect to the cubic. If A, B, C are three collinear points on a cubic U and V is any cubic of the system passing through the intersections of U and its Hessian, then the dipolar of A, B with respect to V passes through C. If it coincides with the tangent to U at C, then the dipolars of B, C and C, A with respect to V coincide with the tangents to U at A and B. In this case A, B, C are said to form a collinear triad on U apolar with respect to V, and the straight line ABC has an envelope which we may call the apolar envelope of U with respect to V. If V coincides with U the envelope is the Cayleyan of U. If V is the Hessian of U, then the envelope is the contravariant of U called by Cayley the Quippian. These results are given in a paper by W. P. Milne and D. G. Taylor, *Proc. London Math. Soc.*, Ser. 2, Vol. 20, pp. 101–105. The canonical form of the cubic

$$U \equiv x^3+y^3+z^3+6kxyz = 0$$

was not used in this paper. When this is used the equation of the apolar envelope takes a simple form, and some geometrical properties of the contravariant of U,

$$\Pi \equiv -k\,(\xi^3+\eta^3+\zeta^3)+3\xi\eta\zeta = 0,$$

may be obtained. The principal results are the following.

If the equation of the Cayleyan of U_1 and U_2 is written in the form

$$\lambda^3 C_{11}+\lambda^2\mu C_{12}+\lambda\mu^2 C_{21}+\mu^3 C_{22} = 0,$$

$C_{11} = 0$ and $C_{22} = 0$ being, of course, the Cayleyans of U_1 and U_2, then $C_{12} = 0$ is the apolar envelope of U_1 with respect to U_2, with a similar meaning for $C_{21} = 0$.

There are two cubics of the system which have the same Cayleyan as U. The apolar envelope of either with respect to the other is $\Pi = 0$.

There are two cubics of the system which have the same Hessian as U. The apolar envelope of either of these with respect to U is $\Pi = 0$.

If the apolar envelope of U with respect to V is the Π-contravariant of V', the apolar envelope of U with respect to V' is the Π-contravariant

of V. The relation between V and V' is involutory, the double elements being the two cubics which have the same Hessian as U.

If any tangent to $\Pi = 0$ meets U in the points A, B, C, then the dipolar of A, B with respect to U and the tangent to U at C are harmonic conjugates with respect to the dipolars of A, B with respect to the two cubics which have the same Hessian as U.

The Inflexions and Inflexional Tangents of a Plane Quartic with Two Cusps

A. E. JOLLIFFE.

Such a curve has eight inflexions and one double-tangent. It is of the sixth class.

Its coordinates can be expressed parametrically in the form

$$x = \wp u - \wp w, \qquad y = \wp v - \wp w,$$

where w is a constant and $u+v+w = 0$, the equation of the curve being

$$(xy - \tfrac{1}{2}a_2)^2 = a_1^2(x+y+3a),$$

where a, a_1, a_2 denote respectively $\wp w$, $\wp' w$, $\wp'' w$.

The eight inflexions do not lie on a conic except in the case $a_2 = 0$, and the conic is not a proper conic but the pair of straight lines

$$\sqrt{3}(x+y+4a) \pm (x-y) = 0.$$

It is evident from symmetry that in the general case a cubic can be drawn to pass through the inflexions and cusps.

Since the class of the curve is 6, the inflexional tangents can never touch a single conic, and it is evident from symmetry that a single class-cubic can be found to touch the inflexional tangents and the cuspidal tangents. The most interesting curve, however, which touches the inflexional tangents is the class-quartic

$$4(6alm - l - m)^2 + 3(l-m)^2(2a_2 lm - 1) = 0.$$

This curve touches the cuspidal tangents at the cusps and has the double tangent and the line joining the cusps as double-tangents. It also passes through the points of contact of the double-tangent.

When $a_2 = 0$ it breaks up into two conics. The line of four of the inflexions and the four corresponding inflexional tangents touch one of these conics, and the other line and the other four tangents touch the other.

Some Problems of "Partitio Numerorum" (V): A further contribution to the study of Goldbach's Problem

G. H. HARDY *and* J. E. LITTLEWOOD.

In this paper it is shown that, if a certain hypothesis is true, then *almost all even numbers are sums of two primes.* The meaning of this assertion is as follows: if $N(x)$ is the number of numbers less than x, and expressible in the form required, then

$$N(x) \sim \tfrac{1}{2}x,$$

when $x \to \infty$. More precisely

$$N(x) = \tfrac{1}{2}x + O(x^{\frac{1}{2}+\epsilon})$$

for every positive ϵ.

The hypothesis referred to is this, that no zero of any of Dirichlet's functions $L(s)$, associated with the primes in an arithmetical progression, has a real part exceeding $\frac{1}{2}$. This is the natural generalisation of Riemann's hypothesis concerning the zeros of $\zeta(s)$.

A similar method shows that *almost all numbers are sums of five cubes, fifteen biquadrates, two squares and a cube, or two squares and a prime.* Of these results, only the last depends on the hypothesis; the others are true without reservation.

Poncelet's Poristic Polygons

T. W. CHAUNDY.

Explicit forms of the condition are already known for polygons of 3, 4, 5, 6, 8 sides. New forms of the condition are established for these cases, and formulæ obtained also for polygons of 9, 10, 12, 15, 16, 18, 20, 24, 30 sides. Simplicity is obtained by presenting not the full condition, but a sufficiently representative irrational factor of this condition. In particular it is shown how the condition for a $(4n+2)$-agon may be at once obtained from the condition for a $(2n+1)$-agon.

Some Formulæ of Ramanujan

G. H. HARDY.

In my notice of Ramanujan (*Proceedings*, Ser. 2, Vol. 19, pp. xl–lviii) I quoted three curious summations of numerical series, viz.:

$$(1)\quad 1-5\left(\frac{1}{2}\right)^3+9\left(\frac{1.3}{2.4}\right)^3-13\left(\frac{1.3.5}{2.4.6}\right)^3+\ldots=\frac{2}{\pi},$$

$$(2)\quad 1+9\left(\frac{1}{4}\right)^4+17\left(\frac{1.5}{4.8}\right)^4+25\left(\frac{1.5.9}{4.8.12}\right)^4+\ldots=\frac{2\sqrt{2}}{\sqrt{\pi}\left\{\Gamma\left(\frac{3}{4}\right)\right\}^2},$$

$$(3)\quad 1-5\left(\frac{1}{2}\right)^5+9\left(\frac{1.3}{2.4}\right)^5-13\left(\frac{1.3.5}{2.4.6}\right)^5+\ldots=\frac{2}{\left\{\Gamma\left(\frac{3}{4}\right)\right\}^4}.$$

Of these (1) and (2) may be deduced from a formula given by Dr. J. Dougall in Vol. 25 of the *Proceedings of the Edinburgh Mathematical Society*, a formula which was found independently by Ramanujan, and which I have discussed in a note "A chapter from Ramanujan's note-book" presented to the Cambridge Philosophical Society.* The first is quite well known, and was given first by Bauer.† The third seems to stand on a different footing; at any rate, I cannot deduce it from the Dougall-Ramanujan formula; and I give an independent proof.

In the formula‡

$$\frac{1}{\sqrt{(1-x^2)}}=\tfrac{1}{2}\pi\left\{1+5\left(\frac{1}{2}\right)^2P_2(x)+9\left(\frac{1.3}{2.4}\right)^2P_4(x)+\ldots\right\},$$

write $z=xx'-\sqrt{\{(1-x^2)(1-x'^2)\}}\cos\omega$ for x, and integrate with respect to ω over the range $0, \pi$. Using the formula§

$$\frac{1}{\pi}\int_0^{\pi}P_n(z)d\omega=P_n(x)\,P_n(x'),$$

and then putting $x'=0$, we obtain

$$1-5\left(\frac{1}{2}\right)^3P_2(x)+\ldots=\frac{4}{\pi^2}\int_0^{\frac{1}{2}\pi}\frac{d\omega}{\sqrt{\{1-(1-x^2)\sin^2\omega\}}}=\frac{4}{\pi^2}K'(x).$$

---

* *Proc. Camb. Phil. Soc.*, Vol. 21 (1923), pp. 492–504.

† G. Bauer, "Von den Coefficienten der Reihen von Kugelfunctionen einer Variabeln", *Journal für Math.*, Vol. 56 (1859), pp. 101–121. See also J. W. L. Glaisher, "On series for $1/\pi$ and $1/\pi^2$", *Quarterly Journal*, Vol. 37 (1906), pp. 173–198, and E. Heine, *Kugelfunctionen* (1878), Vol. 1, p. 85.

‡ Heine, *l.c.*, p. 85.

§ Heine, *l.c.*, p. 313; or Whittaker and Watson, *Modern Analysis*, p. 322.

Write $\cos\theta$ for x, and integrate over the range $(0, \pi)$. Observing that

$$\int_0^\pi P_{2n}(\cos\theta)\,d\theta = \pi\left(\frac{1.3\ldots 2n-1}{2.4\ldots 2n}\right)^2,$$

we obtain

$$1-5\left(\frac{1}{2}\right)^5+9\left(\frac{1.3}{2.4}\right)^5-\ldots = \frac{8}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{d\theta\,d\omega}{\sqrt{(1-\sin^2\theta\sin^2\omega)}}$$

$$= \frac{2}{\pi}\left\{1+\left(\frac{1}{2}\right)^3+\left(\frac{1.3}{2.4}\right)^3+\ldots\right\},$$

which gives the result, the sum of the last series being known.*

Another such formula given by Ramanujan is

$$(4)\qquad 1-\left(\frac{1}{2}\right)^3+\left(\frac{1.3}{2.4}\right)^3-\left(\frac{1.3.5}{2.4.6}\right)^3+\ldots = \left\{\frac{\Gamma\left(\frac{9}{8}\right)}{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{7}{8}\right)}\right\}^2.$$

This may be proved by combining a formula of Clausen† with other known formulæ in the theory of hypergeometric series.

A Class of Expansions in Series of Bessel Functions

E. C. TITCHMARSH.

An arbitrary function $f(x)$, defined for $0 < a < x < b$, can be expanded in a series of the form

$$(1)\qquad f(x) = \sum_{m=1}^{\infty} a_m C_\nu(\gamma_m x, \gamma_m b),$$

where $$C_\nu(\alpha, \beta) \equiv J_\nu(\alpha)\,Y_\nu(\beta) - Y_\nu(\alpha)\,J_\nu(\beta),$$

and γ_m is the m-th positive root of the equation

$$C_\nu(az, bz) = 0.$$

---

* See F. Morley, "On a series of cubes", *Proc. London Math. Soc.*, Ser. 1, Vol. 34 (1902), pp. 397–402.

† Th. Clausen, "Ueber die Fälle u. s. w.", *Journal für Math.*, Vol. 3 (1828), pp. 89–95: the result is stated as an example by Whittaker and Watson (p. 292, Ex. 13).

We have, in fact,

$$(2) \qquad \int_a^b x C_\nu(\gamma_m x,\ \gamma_m b)\, C_\nu(\gamma_n x,\ \gamma_n b)\, dx = \frac{2}{\pi^2 \gamma_m^2} \left\{ 1 - \frac{J_\nu^2(\gamma_m b)}{J_\nu^2(\gamma_m a)} \right\},$$

if $m = n$, while the integral vanishes if $m \neq n$; so that we can calculate the coefficients by the ordinary Fourier method, which gives

$$(3) \qquad a_m = \frac{\pi^2 \gamma_m^2}{2} \frac{J_\nu^2(\gamma_m a)}{J_\nu^2(\gamma_m a) - J_\nu^2(\gamma_m b)} \int_a^b C_\nu(\gamma_m t,\ \gamma_m b)\, t f(t)\, dt.$$

The expansion has been used in the solution of physical problems,* but no mathematical theory of it appears to exist. I have constructed a proof of the theorem, that *if $f(t)$ is summable in (a, b) and of bounded variation in the neighbourhood of the point $t = x$, then the series* (1), *formed with the coefficients* (3) (*the order ν being any real number*), *converges to the sum* $\frac{1}{2}\{f(x+0)+f(x-0)\}$.

The proof is on the lines of Prof. Watson's proof of the analogous theorem for Fourier-Bessel series.† It is based on the methods of the calculus of residues. We have first to investigate the behaviour of the sum

$$(4) \qquad R_n(t, x) = \sum_{m=1}^{n} \frac{\pi^2 \gamma_m^2}{2} \frac{J_\nu^2(\gamma_m a)}{J_\nu^2(\gamma_m a) - J_\nu^2(\gamma_m b)} C_\nu(\gamma_m x,\ \gamma_m b)\, C_\nu(\gamma_m t,\ \gamma_m b)$$

for large values of n. If $\gamma_n < A_n < \gamma_{n+1}$, we find that

$$(5) \qquad R_n(t, x) = \frac{1}{2i} \int_{A_n - i\infty}^{A_n + i\infty} \frac{z C_\nu(xz,\ az)\, C_\nu(tz,\ bz)}{C_\nu(az,\ bz)}\, dz,$$

if $a \leqslant x < t \leqslant b$; and we must interchange x and t on the right if $a \leqslant t < x < b$.‡ From this formula we deduce the inequalities

$$|\, R_n(t, x)\,| < \frac{K}{|\, t - x\,|},$$

$$\left| \int_t^b t^{\nu+1} R_n(t, x)\, dt \right| < \frac{K}{A_n(b-x)} + \frac{K}{A_n(t-x)} \qquad (x < t),$$

$$\left| \int_a^t t^{\nu+1} R_n(t, x)\, dt \right| < \frac{K}{A_n(x-t)} + \frac{K}{A_n(x-a)} \qquad (x > t).$$

---

* *E.g.* Carslaw, *Mathematical Theory of the Conduction of Heat in Solids*, 2nd ed., § 62.

† *Theory of Bessel Functions*, Ch. xviii.

‡ Cf. Watson, *l.c.*, § 18.21, (7), (8).

We deduce from these that if $a \leqslant A < B \leqslant b$, and x lies outside the interval (A, B), then

$$\lim_{n\to\infty} \int_A^B t f(t)\, R_n(t, x)\, dt = 0. \tag{6}$$

A new application of the calculus of residues gives the formula

$$\int_a^b t^{\nu+1} R_n(t, x)\, dt = x^\nu + \frac{1}{\pi i} \int_{A_n - i\infty}^{A_n + i\infty} \frac{b^\nu C_\nu(xz, az) - a^\nu C_\nu(xz, bz)}{z C_\nu(az, bz)}\, dz,$$

and it follows that

$$\lim_{n\to\infty} \int_a^b t^{\nu+1} R_n(t, x)\, dt = x^\nu. \tag{7}$$

We observe that this gives at once the particular expansion

$$x^\nu = \sum_{m=1}^{\infty} \pi J_\nu(\gamma_m a)\, \frac{b^\nu J_\nu(\gamma_m a) - a^\nu J_\nu(\gamma_m b)}{J_\nu^2(\gamma_m a) - J_\nu^2(\gamma_m b)}\, C_\nu(\gamma_m x, \gamma_m b).$$

We also obtain the result

$$\lim_{n\to\infty} \int_a^x t^{\nu+1} R_n(t, x)\, dt = \tfrac{1}{2} x^\nu. \tag{8}$$

We can now prove the main theorem. The sum of the first n terms of the series (1) is

$$\int_a^b R_n(t, x)\, t f(t)\, dt,$$

and it follows from (4), (5) and (6) that, when $n \to \infty$, this tends to the limit stated.

The result may also be obtained in a different way. We form the Fourier-Bessel series

$$\sum_{m=1}^{\infty} \frac{2}{b J_{\nu+1}^2(j_m)}\, J_\nu\left(\frac{x j_m}{b}\right) \int_a^b J_\nu\left(\frac{t j_m}{b}\right) t f(t)\, dt \tag{9}$$

of the function equal to $f(x)$ in (a, b) and zero in $(0, a)$. Here j_m denotes the m-th positive zero of $J_\nu(z)$. We prove that *if* $a < x < b$, *the series* (1) *converges whenever* (9) *converges, and to the same sum.*

For, if $Q_n(t, x)$ denotes the difference

$$\sum_{m=1}^{n} \frac{\pi^2 \gamma_m^2}{2}\, \frac{J_\nu^2(\gamma_m a)}{J_\nu^2(\gamma_m a) - J_\nu^2(\gamma_m b)}\, C_\nu(\gamma_m x, \gamma_m b)\, C_\nu(\gamma_m t, \gamma_m b) - \sum_{m=1}^{N} \frac{2}{b J_{\nu+1}^2(j_m)}\, J_\nu\left(\frac{x j_m}{b}\right) J_\nu\left(\frac{t j_m}{b}\right),$$

where $$\gamma_n < D_n < \gamma_{n+1}, \quad j_N < b D_n < j_{N+1},$$

we find

$$Q_n(t,x) = \frac{1}{2i}\int_{D_n-i\infty}^{D_n+i\infty} \frac{J_\nu(az)}{J_\nu(bz)}\,\frac{C_\nu(xz,bz)\,C_\nu(tz,bz)}{C_\nu(az,bz)}\,z\,dz. \tag{10}$$

We deduce from this the inequalities

$$|Q_n(t,x)| < \frac{K}{x+t-2a},$$

$$\left|\int_a^t t^{\nu+1}Q_n(t,x)\,dt\right| < \frac{K}{D_n(x+t-2a)} + \frac{K}{D_n(x-a)};$$

and from these it follows that

$$\lim_{n\to\infty}\int_a^b Q_n(t,x)\,t\,f(t)\,dt = 0, \tag{11}$$

for any value of x in (a, b). The theorem stated is simply an expression of this last result. Hence we can deduce the theory of the series under consideration from the known theory of Fourier-Bessel series. In the latter we must take $\nu > -1$, but, since C_ν is an even function of ν, the results hold in the former case for all values of ν.

Ternary Perpetuant Types

ALFRED YOUNG.

A generating function

$$\left[\frac{1}{r-3}\,\Sigma\binom{r-3}{a+1}\binom{r-3}{a}x^a\right](1-x)^{-2r+5}$$

is obtained for pure ternary perpetuant types of degree r, the word "pure" indicating the absence of line coordinate symbols.

From this a reduced generating function is obtained giving the number of perpetuant types of particular weight and degree which are linearly independent of perpetuants having one of the quantics as a factor.

Symbolical expressions are then obtained for types of the first few degrees, and of the lowest weights, the generating function confirming the results obtained, and indicating what to expect. Thus a syzygy of weight four is indicated both for degree 7 and degree 8, and these are obtained.

Thursday, January 18th, 1923.

Prof. W. H. YOUNG, President, in the Chair.

Present thirty members.

The minutes of the last meeting were read and confirmed.

Messrs. J. P. Gabbatt, S. F. Grace, and C. G. F. James were elected members of the Society.

Prof. G. A. Bliss and Prof. O. E. Glenn were nominated for election.

Messrs. W. E. Baker, P. N. Dé, Sir John Heaton, A. E. Ingham, E. C. Titchmarsh, F. P. White, and R. F. Whitehead were admitted into the Society.

Mr. L. J. Mordell delivered a lecture on "An Introduction to the Arithmetic Theory of Algebraic Numbers and its Modern Developments."

The following papers were communicated by title from the Chair :—

*The Rational Linear Algebras of Maximum and Minimum Ranks: L. E. Dickson.

The Zeros of Mathieu Functions: Einar Hille.

*Plane Homaloidal Families of General Degree: Miss H. P. Hudson.

*Abel's Theorem and its Converse (II): G. H. Hardy and J. E. Littlewood.

A Linear Differential Equation with Periodic Coefficients: E. L. Ince.

A Property of the 24 Inflexional Tangents of the Non-Singular Plane Quartic: A. E. Jolliffe.

The Algebraic Two-Two Correspondence: R. Vaidyanathaswami.

*An Application of Pfeiffer's Method to a Problem of Hardy and Littlewood: B. M. Wilson.

On Successions with Sub-Sequences Converging to an Integral: W. H. Young.

NOTES AND ABSTRACTS.

The Zeros of Mathieu Functions

EINAR HILLE.

A Mathieu function is defined as a solution of the differential equation

$$w''+(a+b\cos 2z)\,w = 0.$$

* Printed in this volume.

The main purpose of the paper is to describe the distribution in the complex plane of the zeros of these functions when a and b are real. Assume $b < 0$, and divide up the plane in simply infinite parallel strips by means of the lines $y = 0$ and $x = (2k+1)\frac{1}{2}\pi$, where k takes on all integral values. Then the general solution has infinitely many zeros in each strip, the zeros forming a point-set of simple structure. For each strip there are two linearly independent solutions which are free from zeros in the strip. These solutions have interesting asymptotic properties. The periodic solutions which include the *functions of the elliptic cylinder* are treated at some length.

The investigation is based partly upon a transformed equation, asymptotic to a sine-equation in the sense of Boutroux, and partly upon integral equalities of Sturmian nature.

Note on a Theorem concerning Fourier Series

G. H. HARDY *and* J. E. LITTLEWOOD.

Dr. Marcel Riesz has indicated to us a material simplification in the proof of Theorem X of our paper "Abel's Theorem and its Converse" in Vol. 18 (1918) of the *Proceedings* (see in particular pp. 229–231).

It is a question of proving that, *if*

$$A_n = O\left(\frac{1}{n}\right),$$

and ΣA_n *converges to* A, *then*

$$\Phi(\alpha) = \Sigma A_n \frac{\sin n\alpha}{n\alpha} \to A,$$

when $\alpha \to 0$.

We may suppose $A = 0$ and $|nA_n| < 1$ for all values of n. Given δ, then, we can choose μ so that

$$\left|\sum_1^{\mu} A_n\right| < \delta \tag{1}$$

and

$$\left|\sum_{\nu}^{\nu'} A_n\right| < \delta, \tag{2}$$

if $\mu \leqslant \nu < \nu'$. We write

$$\Phi = \sum_1^{n} + \sum_{\mu+1}^{m} + \sum_{m+1}^{\infty} = \Phi_1 + \Phi_2 + \Phi_3,$$

where $$m = \left[\frac{K}{a}\right].$$

We have first

$$|\Phi_3| < \frac{1}{a} \sum_{m+1}^{\infty} \frac{1}{n^2} < \frac{1}{ma} < \frac{2}{K} < \epsilon, \tag{3}$$

by choice of K. When K has been fixed, the function $(\sin na)/na$ has, in the range of summation covered by Φ_2, a finite number $k(K)$ of maxima and minima, all numerically less than unity. Hence $|\Phi_2|$ is less than the sum of $k(K)$ terms of the type (2); *i.e.*

$$|\Phi_2| < \delta k(K) < \epsilon, \tag{4}$$

by choice of δ (*i.e.* of μ). We may obviously suppose $\delta < \epsilon$. Finally, when μ is fixed, we can choose a_0 so that

$$|\Phi_1| \leqslant \left| \sum_{1}^{\mu} A_n \right| + \epsilon < \delta + \epsilon < 2\epsilon, \tag{5}$$

for $0 < a \leqslant a_0$, by (1). From (3), (4), and (5) our conclusion follows.

Thursday, February 8th, 1923.

Prof. W. H. YOUNG, President, in the Chair.

Present sixteen members.

The minutes of the last meeting were read and confirmed.

Prof. G. A. Bliss and Prof. O. E. Glenn were elected members of the Society.

The President announced that Sir A. B. Kempe had left a legacy of £200 to the Society.

Mr. T. A. Brown read a paper on "A Class of Factorial Series."

Miss H. P. Hudson read a paper* on "Plane Homaloidal Families of General Degree."

Prof. A. E. Jolliffe read a paper on "A Property of the 24 Inflexional Tangents of the Non-Singular Plane Quartic."

Prof. G. H. Hardy read a paper* (written in collaboration with Mr. J. E. Littlewood) on "Abel's Theorem and its Converse (II)."

Prof. G. N. Watson gave an account of a paper* by Mr. T. A. Lumsden on "A certain Fourier-Bessel Series."

* Printed in this volume.

The following papers were communicated by title from the Chair :—

The Hypergeometric Function with k Denominators: D. M. Wrinch.

On Successions with Sub-Sequences converging to an Integral: W. H. Young.

A Note on the Riesz-Fischer Theorem in the Theory of Trigonometrical Series: E. C. Titchmarsh.

NOTES AND ABSTRACTS.

On a Class of Factorial Series

T. ARNOLD BROWN.

Necessary and sufficient conditions are obtained that an analytic function $f(z)$ may be expressible in the form

$$e_0+e_1 z+e_2\frac{z(z-1)}{2!}+e_3\frac{(z+1)z(z-1)}{3!}+\ldots.$$

A certain interpolation formula of de la Vallée Poussin is generalised and its relation to the above series is exhibited.

Runge's solution of a certain interpolation problem is also extended.

The methods employed depend upon Carlson's researches on Taylor's series. Carlson has discussed the factorial series

$$\Sigma e_\nu\frac{z(z-1)\ldots(z-\nu+1)}{\nu!}$$

in a similar way. It is shown that, in certain respects, the two series differ widely in their behaviour.

A Note on the Riesz-Fischer Theorem in the Theory of Trigonometrical Series

E. C. TITCHMARSH.

The Riesz-Fischer theorem states that if a sequence (a_n) is such that $\Sigma|a_n|^2$ is convergent, then there is a function of summable square of which the given numbers are the Fourier coefficients. It would be natural to suppose that, if $\Sigma|a_n|^\alpha$, where $\alpha>2$, is convergent, the series

is necessarily the Fourier series of a function of which some lower power is summable. But Hardy and Littlewood* have proved that this is untrue, by showing that the series

$$\Sigma n^{-\alpha} \cos(n^2\pi x), \quad \Sigma n^{-\alpha} \sin(n^2\pi x) \quad (\alpha \leqslant \tfrac{1}{2}),$$

are not Fourier series, or even "generalised" Fourier series, since they are not convergent or summable by any of Cesàro's means for any irrational value of x. They are merely formal series with which no sum has been associated.

I have constructed a trigonometrical series which proves the same point, but in a quite different way. My series is everywhere convergent, and is in fact the "generalised" Fourier series of its sum. Further, the convergence exponent of the sequence of coefficients is 2. But the series is not a Fourier series, since its sum is not summable.

The analysis necessary to prove these points is direct and elementary, whereas that of Hardy and Littlewood rests on the theory of the linear transformation of the Theta-functions.

The function $f(t)$ is defined in $(-\pi, \pi)$ as follows:—

$$f(t) = \log p \cos(tp^2 \log p)$$

for $$\pi/\log(p+1) < t \leqslant \pi/\log p \quad (p = 3, 4, 5, \ldots),$$

$f(t)$ being zero elsewhere, and $f(-t) = -f(t)$. The series

$$\sum_{p=3}^{\infty} \log p \int_{\pi/\log(p+1)}^{\pi/\log p} |\cos(tp^2 \log p)| \, dt$$

is divergent, so that $f(t)$ is not absolutely integrable; but

$$\sum_{p=3}^{\infty} \int_{\pi/\log(p+1)}^{\pi/\log p} f(t)\,dt = \Sigma \log p \int_{\pi/\log(p+1)}^{\pi/\log p} \cos(tp^2 \log p)\,dt = \Sigma\, O\left(\frac{1}{p^2}\right)$$

is convergent, so that

$$\lim_{\eta \to 0} \int_{\eta}^{\pi} f(t)\,dt$$

exists, and so do the generalised Fourier coefficients

$$b_n = \frac{2}{\pi} \lim_{\eta \to 0} \int_{\eta}^{\pi} f(t) \sin nt\,dt.$$

* G. H. Hardy and J. E. Littlewood, "Some Problems of Diophantine Approximation", *Acta Mathematica*, Vol. 37, pp. 232–238.

Finally, it may be proved that

$$b_n = O\{n^{-\frac{1}{2}}(\log n)^2\},$$

so that $\Sigma |b_n|^{2+\epsilon}$ is convergent if $\epsilon > 0$.

The Hypergeometric Function with k Denominators

D. M. WRINCH.

In this paper, the hypergeometric function with one denominator $f(z, p)$ is defined by the contour integral

$$f(z, p) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^t t^{-p-1} e^{z/t} dt,$$

and it is deduced that

$$f(z, p) = \sum_{s=0}^{\infty} \frac{z^s}{\Gamma(s+1)\,\Gamma(p+s+1)}.$$

Evidently

$$f\left(-\frac{z^2}{4}, p\right) = \left(\frac{z}{2}\right)^{-p} J_p(z).$$

Various properties of the function are obtained, some of which yield new results for Bessel functions.

The hypergeometric function with k denominators $f(z, p_1, p_2, \ldots, p_k)$ is defined by a contour integral having the generalised hypergeometric function with $k-1$ denominators in the integrand. Various properties of the function $f(z, p_1, p_2, \ldots, p_k)$ follow by induction from the properties of the function $f(z, p)$, including the recurrence formulæ and the fact that

$$f(z, p_1, p_2, \ldots, p_k) = \sum_{s=0}^{\infty} \frac{z^s}{\Gamma(s+1)\,\Gamma(p_1+s+1) \ldots \Gamma(p_k+s+1)}.$$

Some of the properties of the function when one or more of the parameters $p_1, p_2, \ldots, p_k$ is zero or a positive or negative integer are discussed. It is established that when $p_1, p_2, \ldots, p_k$ are severally equal either to zero or a positive or negative integer, the function $f(z, p_1, p_2, \ldots, p_k)$ is the coefficient of $t_1^{p_1} t_2^{p_2} \ldots t_k^{p_k}$ in the expansion in a k-multiple series in powers of $t_1, t_2, \ldots, t_k$ of the function

$$e^{t_1+t_2+\ldots+t_k+z/(t_1 t_2 \ldots t_k)}.$$

Solution of a certain Differential Equation

T. C. HUDSON.

The solution of the equation

$$d/d\theta\,[d^2\lambda/d\theta^2+\lambda(1+4E\cos\theta)/(1+E\cos\theta)] = d\xi/d\theta+2\eta,$$

in which E is constant, may be written in the form

$$\lambda = S(1+EC)\,[\lambda_1+\int C(1+EC)^{-1}\,\xi\,d\theta]$$
$$+C(1+EC)\lambda_2/2E+C^2(2+EC)(1+EC)^{-1}\int \eta\,d\theta$$
$$-C(1+EC)^{-1}[\lambda_3+\int \{C(2+EC)\eta+S(1+EC)\xi\}\,d\theta]$$
$$+2S(1+EC)\int C(1+EC)^{-3}[E\lambda_3+\int \{(1+EC)^2\,\eta+ES(1+EC)\,\xi\}\,d\theta]d\theta,$$

in which $C = \cos\theta$ and $S = \sin\theta$, and λ_1, λ_2, λ_3 are constants.

Thursday, March 8th, 1923.

Prof. W. H. YOUNG, President, and later Mr. H. W. RICHMOND, Ex-President, in the Chair.

Present twenty members.

Dr. Einar Hille, Mr. H. O. Newboult, and Miss Enid M. Turner were nominated for election.

Prof. A. E. H. Love read a paper* on "Some Electrostatic Distributions in Two Dimensions."

Prof. G. H. Hardy read a paper, written in collaboration with Mr. E. C. Titchmarsh, on "Solutions of some Integral Equations considered by Bateman, Kapteyn, Littlewood, and Milne."

Prof. W. H. Young read a paper on "Stokes's Theorem."

The following papers were communicated by title from the Chair:—

(i) *Functions of Intervals, (ii) *The Problem of Area: J. C. Burkill.

On some Integral Equations connected with the Elliptic Cylinder Functions: S. C. Dhar.

A Hyperspatial Analogue of Feuerbach's Theorem: J. P. Gabbatt.

Génération et étude d'une surface du troisième ordre particulière: M. Merker.

The Order of Magnitude of the Coefficients in a Generalised Fourier Series: E. C. Titchmarsh.

*The General Symbolic Notation for the Principle of Duality, and its Application to Determinants: H. W. Turnbull.

On Lattice-Points in Regions of Two or Three Dimensions: I. Vinogradow.

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* Printed in this volume.

NOTES AND ABSTRACTS.

The General Symbolic Notation for the Principle of Duality, and its Application to Determinants

H. W. TURNBULL

This paper generalises results well known in the symbolic treatment of ternary forms, and includes a proof that, for n variables, all the fundamental symbolic identities may be reciprocated. The notation of a previous investigation* is employed thus:—

If symbols a, b, c, a_1, &c., have currency one†, the simplest identity between $n+1$ such symbols is

$$(a_1 a_2 \dots a_n)a_x = (aa_2 \dots a_n)a_{1x} + (a_1 aa_3 \dots a_n)a_{2x} + \dots \text{ to } n \text{ terms.}$$

The dual identity is

$$(a_1 a_2 \dots a_n)u_a = (aa_2 \dots a_n)u_{a_1} + \dots,$$

where, for example, u_a^2 is the reciprocal of the quadratic a_x^2. By introducing suitable types of bracket factors, analogous to the ternary factor $(ab\,.\,\rho\sigma)$ defined by

$$(ab\,.\,\rho\sigma) = a_\rho b_\sigma - a_\sigma b_\rho,$$

it is shown that *any* identity or syzygy may be reciprocated letter for letter. A symbol of currency r reciprocates into one of currency $n-r$.

The notation readily adapts itself to the theory of determinants; for the bracket factors are determinants of orders n, $2n$, $3n$, ... In the general identities the rows or columns of these determinants are arbitrary. By giving special values to the symbols involved many well known theorems emerge as special cases of something more general.

Some Electrostatic Distributions in Two Dimensions

A. E. H. LOVE

The condenser problem is solved for all cases of two thin plates, with straight parallel edges and of infinite length, provided that their planes are parallel, whether the rectilinear segments in which the plates cut a transverse plane are equal or unequal, and whether the line which bisects one segment at right angles bisects the other or not. Formulae are given for the correction to the capacity on account of displacement. The problem of influence for one such plate, in presence of a charged line

* "The irreducible concomitants of two quadratics in n variables", *Camb. Phil. Trans.*, Vol. 21, pp. 197–209.

† Cf. *Proc. London Math. Soc.*, Ser. 2, Vol. 21, p. 381.

parallel to its edges, is solved, and the solution for a free charge on a circular cylindrical sheet with edges along two generators is given. A general theorem is obtained to the effect that the problem of given free charges on two cylindrical conductors, external to each other, can always be solved when the condenser problem can be solved for the same two conductors, and the example of two circular cylinders is worked out.

The Order of Magnitude of the Coefficients in a Generalised Fourier Series

E. C. TITCHMARSH

If a function $f(t)$ is integrable (in the sense of Lebesgue) over the range $(\epsilon, 2\pi)$ for all positive values of ϵ, and the limit

$$\lim_{\epsilon \to 0} \int_{\epsilon}^{2\pi} f(t)\,dt$$

exists, then we may define the generalised Fourier coefficients of the function by the formulæ

$$\pi a_n = \lim_{\epsilon \to 0} \int_{\epsilon}^{2\pi} f(t) \cos nt\,dt, \quad \pi b_n = \lim_{\epsilon \to 0} \int_{\epsilon}^{2\pi} f(t) \sin nt\,dt.$$

It is easily seen on integrating by parts that $a_n = o(n)$, $b_n = o(n)$: it is indeed well known that this is true under very much more general conditions, for example whenever the integrals defining the coefficients are of the general Denjoy type.

It is the object of the present note to show that no more than this is generally true. More precisely:—

Let the function $\lambda(n)$ be positive and decrease steadily to zero as $n \to \infty$. However slowly $\lambda(n) \to \infty$, there is a function $f(t)$ of the type described such that

$$a_n \neq o\{n\lambda(n)\}, \quad b_n \neq o\{n\lambda(n)\}.$$

The method will be sufficiently illustrated by giving an example in which

$$J_n = \lim_{\epsilon \to 0} \int_{\epsilon}^{\pi} \sin nt\, f(t)\,dt \sim \tfrac{1}{2}\pi n\lambda(n),$$

when $n \to \infty$ through a particular sequence of values. Let

$$f(t) = p^2 n_p \lambda(n_p) \sin n_p t, \quad \pi/(p+1) < t \leqslant \pi/p,$$

where $p = 1, 2, 3, \ldots$ and n_p is an integer depending on p, which tends to infinity so rapidly that $n_p > 2n_{p-1}$ and $\Sigma p^2 \lambda(n_p)$ is convergent. We have

$$J_{n_q} = \sum_{p=1}^{\infty} p^2 n_p \lambda(n_p) \int_{\pi/(p+1)}^{\pi/p} \sin n_p t \sin n_q t\,dt.$$

It is easily seen that the qth term of this sum is

$$\tfrac{1}{2}\pi n_q \lambda(n_q)+o(1),$$

and the modulus of the remainder is less than

$$\sum_{p=1}^{\infty} \frac{p^2 n_p \lambda(n_p)}{n_p+n_q}+\sum_{p=1}^{q-1} \frac{p^2 n_p \lambda(n_p)}{n_q-n_p}+\sum_{p=q+1}^{\infty} \frac{p^2 n_p \lambda(n_p)}{n_p-n_q}.$$

Each of these sums tends to zero, so that the result follows.

Solutions of some Integral Equations considered by Bateman, Kapteyn, Littlewood, and Milne

G. H. HARDY *and* E. C. TITCHMARSH

The typical equations are

$$\text{(K)} \quad f'(x) = \tfrac{1}{2}\int_0^{\infty} \frac{J_1(t)}{t}\{f(x+t)-f(x-t)\}\,dt,$$

$$\text{(M)} \quad f'(x) = \tfrac{1}{2}\int_0^{\infty} \frac{e^{-t}}{t}\{f(x+t)-f(x-t)\}\,dt.$$

It is shown that the latter equation has no solution, other than the obvious solution

$$f(x) = ax^2+2bx+c,$$

of less rapid increase than an exponential of the type $e^{A|x|}$ ($A < 1$). The equation (K), on the other hand, has a multitude of solutions. These may be characterised roughly (*a*) as functions defined by definite integrals of the type

$$\int_0^1 \{\lambda(u)\cos xu+\mu(u)\sin xu\}\,du,$$

or (*b*) as integral functions of the complex variable x subject to certain inequalities.

Functions of Intervals

J. C. BURKILL

A *function of intervals* is defined by a set of rules correlating a number $g(I)$ with each interval I of an aggregate of intervals. The only functions of intervals which have hitherto been studied have been supposed to be *additive;* this restriction is not made in the present paper.

We define the *integral* $\int_R g(I)$ by subdividing R into intervals $I_1, \ldots I_n$, and taking the limit of $\sum_{i=1}^{i=n} g(I_i)$ as the dimensions of I_i tend to zero. If $g(I)$ is *absolutely continuous*, we prove that its integral may be taken over a measurable set; this forms a generalisation of the classical theory of measure.

The *derivative* of $g(I)$ at a point x is the limit of $g(I)$ divided by the measure of I, for intervals of a *regular* family approaching x. The main result is that the integral over a measurable set of an absolutely continuous function of intervals is equal to the Lebesgue integral of its derivative. This includes as a special case the fundamental theorem of Lebesgue integration.

The Problem of Area

J. C. BURKILL

Given a surface defined by

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v),$$

we seek an expression for its area as a double integral analogous to the integral formula for the length of a curve.

The formula in terms of the ordinary Jacobian J holds only when the derivatives $\partial x/\partial u$, etc., satisfy certain conditions. We introduce here a *modified Jacobian.*

We first define the *increment of* (x, y) *over a rectangle* R in the (u, v) plane as

$$\Delta(x, y) = \tfrac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_4 - x_4y_3 + x_4y_1 - x_1y_4),$$

where 1, 2, 3, 4 denote the corners of R in anti-clockwise order, and x_1 is the value of $x(u, v)$ at the point 1, etc.

The modified Jacobian of (x, y) with respect to (u, v) is then defined as the limit of the ratio of the increment of (x, y) over a rectangle in the (u, v) plane to the area of the rectangle as the sides of the rectangle tend to zero.

We obtain the integral formula for the area of a surface in terms of modified Jacobians under the conditions that $\Delta(y, z)$, $\Delta(z, x)$, $\Delta(x, y)$ are absolutely continuous functions of rectangles. These conditions are analogous to those for the validity of the formula for the length of a curve.

Thursday, April 26th, 1923.

Mr. H. W. RICHMOND, Ex-President, and later Prof. A. E. H. LOVE, Ex-President, in the Chair.

Present twenty-four members and one visitor.

Dr. E. Hille, Mr. H. O. Newboult, and Miss E. M. Turner were elected members of the Society.

Sir George Greenhill and Mr. J. Henderson were nominated for election.

Mr. Balak Ram was admitted into the Society.

The Chairman referred briefly to the death of Dr. J. G. Leathem, and announced that Dr. Nicholson's lecture would take place on May 17th.

Dr. H. Jeffreys gave an account of four papers:—

(i) On Certain Approximate Solutions of Linear Differential Equations of the Second Order.

(ii) On Certain Solutions of Mathieu's Equation.

(iii) On the Modified Mathieu's Equation.

(iv) On the Oscillations of an Elliptic Lake.

Col. Hippisley read a paper on "Two Theorems on Centroids."

Mr. H. W. Richmond read two papers*:—

(i) On the Electrostatic Field of a Plane Grating with Thick Rounded Bars.

(ii) Notes on the Use of the Schwarz-Christoffel Transformation in Electrostatics and Hydrodynamics.

The following papers were communicated by title from the chair:—

On the Solution of a Pair of Diophantine Equations connected with the Nuptial Number of Plato: Grace Chisholm Young.

*On the Discontinuous Motion produced in an Infinite Stream by Two Plane Obstacles: E. G. C. Poole.

Two Mean-value Theorems concerning Riemann's Zeta-Function: A. E. Ingham.

The Inflexions of the Non-Singular Quartic: A. E. Jolliffe.

Transversal Problems in Hyperspace: R. Vaidyanathaswamy.

Diophantine Equations: T. Stuart.

*Sur l'équivalence de certaines méthodes de sommation: Marcel Riesz.

On the Perturbations of the Orbit of the Valency-electron in the Generalized Hydrogen-unlike Atom: K. Basu.

* Printed in this volume.

Cyclotomic Heptasection: Pandit Oudh Upadhyaya.
Diffusion of Heat along a Moving Cylinder: S. P. Owen.
Dougall's Theorem on Generalized Hypergeometric Functions: C. T. Preece.
On some Complexes of Conics in Space of Three Dimensions: C. G. F. James.

NOTES AND ABSTRACTS.

Two Mean-Value Theorems concerning Riemann's Zeta-Function

A. E. INGHAM.

This paper contains proofs of the following theorems.

THEOREM 1.—*Let us write* (α, β *real*, $T \geqslant 1$)

$$\int_1^T \xi(\alpha+it)\,\xi(\beta-it)\,dt = F(\alpha+\beta, T)+R(\alpha, \beta, T),$$

where
$$F(\sigma, t) = \zeta(\sigma)(t-1)+(2\pi)^{\sigma-1}\zeta(2-\sigma)\frac{t^{2-\sigma}-1}{2-\sigma}$$

(F *being defined by continuity when* $\sigma = 1$ *and* $\sigma = 2$). *Then, if* $h > 0$ *is fixed, we have, as* $T \to \infty$,

$$R(\alpha, \beta, T) = O(T^{\nu(\alpha)+\nu(\beta)}\log^2 T)$$

uniformly in $-h \leqslant \alpha \leqslant h, \quad -h \leqslant \beta \leqslant h,$

where $\nu(\sigma) = \mathrm{Max}\left(0, \frac{1}{2}(1-\sigma), \frac{1}{2}-\sigma\right).$

THEOREM 2.—*As* $T \to \infty$

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4\,dt \sim \frac{1}{2\pi^2}T\log^4 T.$$

The first theorem is a refinement of one due to Littlewood. With the above notation his result* may be stated in the following form:

As $T \to \infty$

$$R(\sigma, \sigma, T+H)-R(\sigma, \sigma, T) = O(T^{1-\sigma+\epsilon})+O(T^{\epsilon})+O(HT^{-\frac{1}{2}\sigma+\epsilon})$$

uniformly in $0 \leqslant H \leqslant T, \qquad \frac{1}{2} \leqslant \sigma \leqslant 2.$

* See *Records of Proceedings at Meetings*, Feb. 10, 1921 (Vol. 20, p. xxii).

Theorem 2 is a companion to the following results recently proved by Hardy and Littlewood in these *Proceedings*.*

(i) *If* $\frac{1}{2} < \sigma < 1$ (σ *fixed*) *and* $T \to \infty$, *then*

$$\frac{1}{2T}\int_{-T}^{T} |\zeta(\sigma+it)|^4\, dt \sim \sum_{1}^{\infty} \frac{[d(n)]^2}{n^{2\sigma}} = \frac{[\zeta(2\sigma)]^4}{\zeta(4\sigma)}.$$

(ii) *We have, uniformly in* $\frac{1}{2} \leqslant \sigma \leqslant 2$,

$$\int_{2}^{T} |\zeta(\sigma+it)|^4\, dt = O(T \log^4 T).$$

The authors deduce these results from their approximate functional equation for $\zeta(s)$. It does not appear easy, however, to derive Theorem 2 from the same source, and the proof given here makes use of an analogous "approximate functional equation" for the *square* of $\zeta(s)$: a result also due to Hardy and Littlewood, but as yet unpublished.

The Inflexions of the Non-Singular Quartic

A. E. JOLLIFFE.

The twenty-four inflexions of a non-singular quartic are its intersections with its Hessian, and that appears to be all that is known about them at present. It might be expected, from the analogy of the cubic, whose nine inflexions lie in several ways on three straight lines, that the inflexions of the quartic might lie on two cubics or three conics, or at any rate that some theorem of this character might be true; but this seems hardly probable, for the eight inflexions of a 2-cusped quartic do not lie on a conic, and the twelve inflexions of a 2-nodal quartic do not lie on a cubic. I have found a relation between the inflexions and any Steiner complex of the quartic, though not so simple and striking as that which I found between the inflexional tangents and the twelve bitangents of a Steiner complex. It is that an infinite number of sextics can be drawn through the twenty-four inflexions and the six points of intersection of the six pairs of bitangents of any Steiner's complex.

On some Complexes of Conics in Space of Three Dimensions

C. G. F. JAMES.

In this paper we consider certain complexes (systems α^3) of conics in space, namely such as can be birationally represented on a space, and we examine the nature of this representation, and its applications. More

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* Ser. 2, Vol. 21, p. 59.

especially we consider those systems in which the conics through a point are represented by points on a line.

The Uniformity of $\operatorname{sn} u$

T. W. CHAUNDY.

The Jacobian elliptic function $\operatorname{sn} u$ is supposed defined for a complex argument by inversion of the elliptic integral. The uniformity is established by showing that the integral between any two points cannot vanish. This is effected by introducing two systems of bicircular quartics along which the integral is respectively always purely real or purely imaginary. We can integrate between any two points along arcs of a curve of each system. The integral can vanish only if both arcs also vanish.

Two Theorems on Centroids

COL. R. L. HIPPISLEY, C.B.

The two theorems are—

I. *If a poristic polygon be inscribed and circumscribed to any two conics, the centroids of the points where it meets these curves lie respectively on two other fixed conics, similar and similarly situated to those on which the points themselves lie.*

II. *If a series of points be taken on a three-bar sextic such that their elliptic arguments increase by common differences of* $2K/n$ *or* $4K/n$, *where* n *is an integer, their centroid lies on another fixed* 3*-bar sextic with a similar and similarly situated focal triangle.*

The first theorem is proved for the case of a polygon circumscribed to a circle and inscribed in a conic. The coordinates of the points are expressible as the sum or difference of two linear conjugate functions of $\operatorname{tn} u$, in virtue of Arnold Emch's research into the connection between poristic polygons and the deformable quadrilateral, and of Darboux' representation of the variables of the latter as elliptic functions of u. The vector summations are effected by means of corresponding linear substitutions in Jacobi's equations for the transformation of elliptic functions of u, k into those of $u/M, \lambda$. It is then shown that any conical projection of the polygon, circle, and conic has the same property.

Mr. Richmond has found a simpler solution not involving elliptic functions; but one of the objects of the author was to call attention to the method of linear substitutions in the transformation equations.

The second theorem is a consequence of certain facts brought to light as the results of the linear substitutions used in the first. The sextic locus reduces to a point locus in certain cases.

The Nodes of the 3-bar Sextic.

COL. R. L. HIPPISLEY, C.B.

I desire to correct an error in the opening sentence of my article "Note on the Nodes of the 3-bar Sextic," on p. 410, vol. 21, of the *Proceedings*.

The fact that the two principal chords through a node terminate on the opposite side of the nodal triangle was proved by Mr. Bennett on p. 73, vol. 20.

Thursday, May 17th, 1923.

Prof. W. H. YOUNG, President, in the Chair.

Present twenty members and two visitors.

Mr. J. Vint was nominated for election.

Sir George Greenhill and Mr. J. Henderson were elected members of the Society.

Miss E. M. Turner was admitted into the Society.

The President referred to the death of Prof. Charles Niven in his 78th year.

Dr. J. W. Nicholson delivered a lecture on "Future Problems of the Quantum Theory."

The following papers were communicated by title from the Chair :—

A New Proof of a Formula of A. C. Dixon : G. N. Watson.

Relations between Hypergeometric Series : F. J. W. Whipple.

NOTES AND ABSTRACTS.

Dixon's Theorem on Generalized Hypergeometric Functions

G. N. WATSON.

One of the comparatively few cases in which a hypergeometric function of the third order, with last element equal to unity, ${}_3F_2(\alpha, \beta, \gamma ; \delta, \epsilon ; 1)$, is expressible in terms of Gamma-functions was discovered by A. C. Dixon.* His formula is

$$ {}_3F_2(\alpha, \alpha-\delta+1, \alpha-\epsilon+1 ; \delta, \epsilon ; 1) = \frac{\Gamma(\delta)\Gamma(\epsilon)\Gamma(\delta+\epsilon-\frac{3}{2}\alpha-1)\Gamma(\frac{1}{2})}{2^{\alpha}\Gamma(\frac{1}{2}\alpha+\frac{1}{2})\Gamma(\delta-\frac{1}{2}\alpha)\Gamma(\epsilon-\frac{1}{2}\alpha)\Gamma(\delta+\epsilon-\alpha-1)}, $$

* *Proc. London Math. Soc.*, Vol. 35 (1903), pp. 284–289.

it being supposed that $R(3a-2\delta-2\epsilon+2) < 0$, in order that the series may be convergent.

Dixon's proof made use of rather elaborate transformations of contour integrals; but I have recently noticed that a simple proof can be obtained by rearrangement of repeated series combined with the use of the ordinary formula

$$F(a,\ \beta;\ \gamma;\ 1) = \frac{\Gamma(\gamma)\,\Gamma(\gamma-a-\beta)}{\Gamma(\gamma-a)\,\Gamma(\gamma-\beta)}$$

together with a formula due to Kummer*

$$F(a,\ \beta;\ a-\beta+1;\ -1) = \frac{\Gamma(a-\beta+1)\,\Gamma(\tfrac{1}{2})}{2^a\,\Gamma(\tfrac{1}{2}a+\tfrac{1}{2})\,\Gamma(\tfrac{1}{2}a-\beta+1)},$$

which may easily be proved in a variety of ways. My proof of Dixon's theorem is as follows:

$$\frac{\Gamma(a)\,\Gamma(a-\delta+1)\,\Gamma(a-\epsilon+1)}{\Gamma(\delta)\,\Gamma(\epsilon)} \cdot {}_3F_2(a,\ a-\delta+1,\ a-\epsilon+1;\ \delta,\ \epsilon;\ 1)$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\,\Gamma(a-\delta+n+1)\,\Gamma(a-\epsilon+n+1)}{n!\,\Gamma(\delta+n)\,\Gamma(\epsilon+n)}$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\,\Gamma(a-\delta+n+1)\,\Gamma(a-\epsilon+n+1)}{n!\,\Gamma(a+2n+1)\,\Gamma(\delta+\epsilon-a-1)} \times F(a-\delta+n+1,\ a-\epsilon+n+1;\ a+2n+1;\ 1)$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(a+n)\,\Gamma(a-\delta+m+n+1)\,\Gamma(a-\epsilon+m+n+1)}{n!\,m!\,\Gamma(a+2n+m+1)\,\Gamma(\delta+\epsilon-a-1)}$$

$$= \sum_{p=0}^{\infty}\sum_{n=0}^{p} \frac{\Gamma(a+n)\,\Gamma(a-\delta+p+1)\,\Gamma(a-\epsilon+p+1)}{n!\,(p-n)!\,\Gamma(a+n+p+1)\,\Gamma(\delta+\epsilon-a-1)}$$

$$= \sum_{p=0}^{\infty} \frac{\Gamma(a)\,\Gamma(a-\delta+p+1)\,\Gamma(a-\epsilon+p+1)}{p!\,\Gamma(a+p+1)\,\Gamma(\delta+\epsilon-a-1)}\ F(a,\ -p,\ a+p+1;\ -1)$$

$$= \sum_{p=0}^{\infty} \frac{\Gamma(a)\,\Gamma(\ -\delta+p+1)\,\Gamma(a-\epsilon+p+1)\,\Gamma(\tfrac{1}{2})}{p!\,2^a\,\Gamma(\delta+\epsilon-a-1)\,\Gamma(\tfrac{1}{2}a+\tfrac{1}{2})\,\Gamma(\tfrac{1}{2}a+p+1)}$$

$$= \frac{\Gamma(a)\,\Gamma(a-\delta+1)\,\Gamma(a-\epsilon+1)\,\Gamma(\tfrac{1}{2})}{2^a\,\Gamma(\delta+\epsilon-a-1)\,\Gamma(\tfrac{1}{2}a+\tfrac{1}{2})\,\Gamma(\tfrac{1}{2}a+1)} \times F(a-\delta+1,\ a-\epsilon+1;\ \tfrac{1}{2}a+1;\ 1),$$

and Dixon's formula is at once obtained from this.

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* *Journal für Math.*, Vol. 15 (1836), p. 78, formula (53).

On the Partitions of Multipartite Numbers

P. A. MACMAHON.

In the identity $$\prod_{n=2}^{\infty} \frac{1}{1-n^{-s}} = \sum_{n=1}^{\infty}{}' a_n n^{-s},$$

if on the right-hand side n, expressed in prime factors, be

$$p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots,$$

then the coefficient a_n is equal to the number of partitions of the multipartite numbers

$$(\pi_1\ \pi_2\ \pi_3\ \ldots).$$

The ordinary partitions of unipartite numbers π_1 are enumerated by the coefficient of n^{-s}, where $n = p_1^{\pi_1}$. Thus

$$\prod_{n=2}^{\infty} \frac{1}{1-n^{-s}}$$

is seen to be a generating function for all multipartite numbers, and to provide (apparently) a new starting point.

A Pair of Inversion Formulæ

E. C. TITCHMARSH.

The theory of Schlömilch series gives a method of expressing an arbitrary function as a series of the functions $J_\nu(x)$, $\mathbf{H}_\nu(x)$, the former being the Bessel function of the first kind, and the latter Struve's function.* Similar methods can be applied to the corresponding expressions of a function as an integral. In the former case we are led indirectly, as we should expect, to Hankel's integral formula.† In the latter case we obtain the reciprocal formulæ

$$f(x) = \int_0^\infty \mathbf{H}_\nu(xt)\, t\, F(t)\, dt, \tag{1}$$

$$F(x) = \int_0^\infty Y_\nu(xt)\, t\, f(t)\, dt. \tag{2}$$

Here $Y_\nu(x)$ is the Bessel function of the second kind.‡

* See Watson, *Theory of Bessel Functions*, p. 328.
† *Loc. cit.*, p. 456.
‡ *Loc. cit.*, p. 64.

The formulæ are obtained as follows. We have without difficulty from (1)

$$\frac{d}{dx}\int_0^{\frac{1}{2}\pi} x^{1-\nu} f(x\sin\theta)\sin^{\nu+1}\theta\cos^{-2\nu}\theta\,d\theta = 2^{-\nu}\pi^{-\frac{1}{2}}\Gamma(\tfrac{1}{2}-\nu)\int_0^\infty t^{\nu+1}F(t)\sin xt\,dt.$$

Hence, by Fourier's sine integral formula,

$$(3)\qquad (\tfrac{1}{2}x)^{\nu+1}\Gamma(\tfrac{1}{2}-\nu)\Gamma(\tfrac{1}{2})F(x)$$
$$= \int_0^\infty \sin xt\,dt\int_0^{\frac{1}{2}\pi}\frac{d}{dt}\{t^{1-\nu}f(t\sin\theta)\}\sin^{\nu+1}\theta\cos^{-2\nu}\theta\,d\theta.$$

Writing $\phi(z) = z^{1-\nu}f(z)$, we have

$$\int_0^{\frac{1}{2}\pi}\frac{d}{dt}\{t^{1-\nu}f(t\sin\theta)\}\sin^{\nu+1}\theta\cos^{-2\nu}\theta\,d\theta = t^{-1}\int_0^t \phi'(u)(t^2-u^2)^{-\nu-\frac{1}{2}}u^{2\nu+1}du.$$

Substitute this in the right-hand side of (3), invert the order of integration, and put $t = zu$; we find

$$(\tfrac{1}{2}x)^{\nu+1}\Gamma(\tfrac{1}{2}-\nu)\Gamma(\tfrac{1}{2})F(x) = \int_0^\infty \phi'(u)\,du\int_1^\infty \frac{\sin xuz}{z(z^2-1)^{\nu+\frac{1}{2}}}dz.$$

Integrating by parts and observing that*

$$\frac{d}{du}\int_1^\infty \frac{\sin xuz}{z(z^2-1)^{\nu+\frac{1}{2}}}dz = -(\tfrac{1}{2}x)^{\nu+1}\Gamma(\tfrac{1}{2}-\nu)\Gamma(\tfrac{1}{2})u^\nu Y_\nu(xu),$$

we obtain (2).

The result is thus proved if the various steps can be justified. This is certainly the case if $-\frac{1}{2} \leqslant \nu < \frac{1}{2}$, and the integrals

$$\int_0^A t^{\frac{1}{2}}F(t)\,dt, \qquad \int_A^\infty t^{\nu+1}F(t)\,dt, \qquad \int_0^\infty \phi'(u)\,du$$

are absolutely convergent. These are very artificial conditions, bearing partly on F and partly on f; but they are only "conditions of inequality," and the result can probably be obtained under very general hypotheses. I have not yet considered the question of the generality of the result.

It may be observed that in the case $\nu = -\frac{1}{2}$ the formulæ reduce to Fourier's sine integral formula. The result fails in the case $\nu = \frac{1}{2}$.

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* *Loc. cit.*, p. 170, § 6.13 (4).

Thursday, June 14th, 1923.

Prof. W. H. YOUNG, President, in the Chair.

Present eighteen members.

Dr. J. Marshall and Mr. J. Vint were elected members of the Society.

Mr. W. M. H. Greaves was nominated for election.

Mr. J. Henderson was admitted into the Society.

The President announced that the De Morgan medal had been awarded to Major P. A. MacMahon.

Major P. A. MacMahon read two papers:

(i) *The Partitions of Multipartite Numbers.

(ii) Note on Euler's Theorem in regard to certain Polyhedra: $S-E+F=2$.

Prof. E. W. Hobson read a paper*, "On the Summability of Generalized Fourier Series."

Prof. G. H. Hardy read a paper, written in collaboration with Mr. J. E. Littlewood, on "The Allied Series of a Fourier Series."

The following papers were taken as read:—

On Dirichlet Multiplication of Infinite Series: T. S. Broderick.

The Classification of Cubic Irrationalities: W. E. H. Berwick.

A Contribution to the Theory of Fourier Transforms: E. C. Titchmarsh.

The Equivalence of certain Integral Means: G. H. Hardy and J. E. Littlewood.

The Conformal Representation of certain Associated Figures: J. Hodgkinson.

The Break-up of the Orthoptic Locus: A. Narasinja Rao.

On the Quadrics and Complexes Determined by a Rational Norm-curve: R. Vaidyanathaswamy.

Über Taubersche Bedingungen: L. Neder (communicated by G. H. Hardy).

A Note on Knots: F. V. Morley.

Orthogonal Developments whose Sum is everywhere different from the Developed Function: H. Looman (communicated by G. H. Hardy).

A Hyperspatial Analogue of Feuerbach's Theorem: J. P. Gabbatt.

* Printed in this volume.

NOTES AND ABSTRACTS.

Note on Euler's Theorem in Regard to Certain Polyhedra: $S-E+F=2$

Major P. A. MACMAHON.

1. The dihedral angle between two planes is usually written in circular measure in the discussion of polyhedra.

The angle between two planes is, in fact, a solid angle and should be expressed in spherical measure.

We see at once that if Ω be the solid angle between two planes (expressed in spherical measure) and ω the dihedral angle between the same two planes (expressed in circular measure) then

$$\Omega = 2\omega.$$

2. An edge of a polyhedron defines a solid angle, viz. that between the two faces which determine it.

2.1. DEFINITION.—The "hemispherical deficiency" of a solid angle Ω is

$$2\pi - \Omega.$$

2.2. THEOREM.—*In any polyhedron, for which the relation of Euler is satisfied, the sum of the hemispherical deficiencies of the solid angles at the summits is equal to the sum of the hemispherical deficiencies of the solid angles at the edges.*

2.3. This theorem is equivalent to the theorem of Euler from which it is readily deducible.

It proves to be useful in the discussion of repeating polyhedra in space of three dimensions in analogy with that of repeating polygons in space of two dimensions.

A Note on Knots

F. V. MORLEY.

The construction of a regular pentagon by tying a simple knot in a strip of paper leads to a generalization for the construction of any regular polygon with five or more sides.

Take the odd polygons first. The construction for the pentagon, familiar by oral tradition, is as follows:

1. Hold vertical a strip of paper, sides cut parallel and even, say half an inch wide and ten inches long; the lower end a between the thumb and forefinger of the left hand; the upper end b in the right hand.

2. Carry the end b forward and pass it, from the right, behind a, catching the double thickness between the left thumb and forefinger. We have thus a simple loop, with the long end b projecting to the left.

3. Carry b forward again, and pass it from the left axially through the loop.

4. Pull the ends a and b. Folding the paper neatly flat when the knot is tight, we have the pentagon.

If, with a somewhat thinner strip of the same length, we start from stage 3 in the above process, and, instead of pulling the ends immediately, pass b again back and behind the double thickness so as now to catch three thicknesses between the left thumb and forefinger, we have a double loop, with the long end b projecting to the left. Now carry b forward and pass it from the left axially through the double loop. Pulling both ends and flattening when the knot is tight, we have the regular heptagon.

We may so construct any odd regular polygon of a greater number of sides merely by increasing the loopage of the knot.

For the even polygons, starting with the regular hexagon, take two strips, and make a simple loop in each. Turn the second loop over. Pass the ends of each looped strip through the loop of the other. Pull all four ends and the knot will flatten into a regular hexagon. If, instead of using two simple loops, we combine in this way a double loop and a single loop, the result is the regular octagon; if a triple loop and a single loop, the regular decagon; and so on.

If we tie a simple pentagonal knot in a strip with one complete twist, so that b presents the same face as a, we are led to conjecture that a simple knot tied in a helical strip may give a pentagonal pyramid. I have seen it happen, but I have no strict proof.

The Derivates of Functions of Intervals

J. C. BURKILL.

We investigate the possible values of the derivates of a function $g(I)$, assuming only that $\int_R g(I)$ exists. The work may be compared with that

of Denjoy and Grace Chisholm Young on the derivates of a function $f(x)$. We prove that:—

If $\int_R g(I)$ *exists, then, except at a set of measure zero, either*

(1) $$u(x) = +\infty, \quad l(x) = -\infty,$$

or

(2) *a finite* $g'(x)$ *exists.*

Orthogonal Developments whose Sum is everywhere different from the Developed Function

H. LOOMAN (communicated by G. H. HARDY).

Mr. S. Banach* has recently given an example of an orthogonal development

$$\Sigma a_n \phi_n(x),$$

where the a_n's are the "Fourier coefficients" of a function $f(x)$, so that

$$a_n = \int f(x)\phi_n(x)dx \Big/ \int \{\phi_n(x)\}^2 dx,$$

which converges everywhere to a sum which differs from $f(x)$. An example to illustrate this possibility may also be constructed in the following manner.

The functions

$$\phi_n(x) = e^{-\frac{1}{2}x} L_n(x) = \frac{1}{n!} e^{\frac{1}{2}x} \left(\frac{d}{dx}\right)^n \left(e^{-x}x^n\right)$$

form an orthogonal system for the interval $(0, \infty)$. We take

$$f(x) = e^{\frac{1}{2}x} g(x) = e^{\frac{1}{2}x - x^{\frac{1}{2}}} \sin x^{\frac{1}{2}},$$

except that, at the zeros of $\sin x^{\frac{1}{2}}$, we attribute to $f(x)$ an arbitrary value other than 0. Then

$$\int_0^\infty f(x)\phi_n(x)dx = \int_0^\infty e^{-x^{\frac{1}{2}}} \sin x^{\frac{1}{2}} L_n(x)dx = 0,$$

since, after Stieltjes,

$$\int_0^\infty e^{-x^{\frac{1}{2}}} \sin x^{\frac{1}{2}} x^n dx = 0 \quad (n = 0, 1, 2, \ldots).$$

Thus $a_n = 0$ for every n, and $\Sigma a_n \phi_n(x)$ converges uniformly to zero, although $f(x)$ is never zero.

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* S. Banach, *Proc. Lond. Math. Soc.*, Vol. 21 (1922), p. 95.

A Hyperspatial Analogue of Feuerbach's Theorem

J. P. GABBATT.

In a recent paper* the writer has shown that a considerable part (including Feuerbach's theorem) of the theory of the circles associated with the Euclidean plane triangle consists of special cases of theorems on the general plane cubic and the associated loci, particularly poloconics. M. Malgouzou† has given a different generalization of Feuerbach's theorem, also involving poloconics. In space of n dimensions, the poloconic is represented by a hypersurface of order n. The writer has thus been led to inquire how far circles may be considered as two-dimensional representatives of n-tic hypersurfaces in n-dimensional space.

In the present paper, which forms an instalment of the results of that inquiry, the hyperspatial analogue of Malgouzou's theorem is given, and a theorem (concerning Euclidean space of n dimensions) of which that of Feuerbach is the two-dimensional case is deduced. The three-dimensional case of this theorem may be stated as follows:—Let the two tetrahedra inscribed to any given parallelopiped be denoted by $[A]$, $[B]$; and let the isogonal transformation, with respect to any tetrahedron, of the plane at infinity be termed the pedal surface of that tetrahedron. Then there are eight cubic surfaces, each of which touches all the edges of $[A]$, is circumscribed to $[B]$, and contains the cubic at infinity, which is anallagmatic for the isogonal transformation with respect to either tetrahedron; and each of the eight surfaces touches the pedal surface of $[B]$ along one of eight twisted cubics.

The theorem in n dimensions concerns an n-tic hypersurface which is touched by each of 2^n other n-tic hypersurfaces at ∞^{n-2} points, the loci of the points of contact being 2^n varieties of order $\frac{1}{2}n(n-1)$.

The Equivalence of certain Integral Means

G. H. HARDY *and* J. E. LITTLEWOOD.

The equivalence of Hölder's and Cesàro's methods for the summation of divergent series was established by Knopp and Schnee, and later in a

* J. P. Gabbatt, *Proc. Camb. Math. Soc.*, Vol. 21 (1922), p. 297.
† Malgouzou, *Nouv. Ann.* (4), Vol. 19 (1919), p. 210.

more elegant manner by Schur;* and the corresponding theorem for integrals has been proved by Landau.† If $a > 0$, and $F(x)$ is integrable (in the sense of Lebesgue) in any finite interval (a, X), and

$$(1.1) \qquad F_r(x) = \frac{1}{x}\int_a^x \frac{dx_1}{x_1}\int_a^{x_1}\frac{dx_2}{x_2}\cdots\frac{dx_{r-1}}{x_{r-1}}\int_a^{x_{r-1}} F(x_r)dx_r \to A$$

when $x \to \infty$, then we say that

$$(1.2) \qquad F(x) \to A \quad (H, r);$$

while if

$$(2.1) \qquad \mathrm{F}_r(x) = \frac{r}{x^r}\int_a^x F(y)(x-y)^{r-1}\,dy \to A,$$

we say that

$$(2.2) \qquad F(x) \to A \quad (C, r);$$

and Landau's theorem is substantially as follows: *if* $F(x) \to A$ (H, r), *then* $F(x) \to A$ (C, r), *and conversely.*‡

In the theory of trigonometrical series we are often concerned with integral mean values of a somewhat different kind, viz.,

$$(3.1) \qquad G_r(t) = \frac{1}{t}\int_0^t \frac{dt_1}{t_1}\int_0^{t_1}\frac{dt_2}{t_2}\cdots\frac{dt_{r-1}}{t_{r-1}}\int_0^{t_{r-1}} G(t_r)\,dt_r$$

and

$$(4.1) \qquad \mathrm{G}_r(t) = \frac{r}{t^r}\int_0^t G(u)(t-u)^{r-1}du.$$

Here $t \to 0$, $G(t)$ is integrable (in the sense of Lebesgue) in any interval $0 < \tau \leqslant t \leqslant T$, and the integrals down to 0 exist either as Lebesgue integrals or as elementary generalized integrals of the type

$$\lim_{\epsilon \to 0}\int_\epsilon .$$

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* K. Knopp, *Dissertation* (Berlin, 1907), pp. 19–22; W. Schnee, *Math. Annalen*, Vol. 67 (1909), pp. 110–125; I. Schur, *Math. Annalen*, Vol. 74 (1913), pp. 447–458 (see also Knopp, *ibid.*, pp. 459–461). Another proof was given by W. B. Ford, *American Journal of Math.*, Vol. 32 (1910), pp. 315–326.

† E. Landau, *Leipziger Berichte*, Vol. 65 (1913), pp. 131–138.

‡ We have modified Landau's enunciation in two respects. Landau takes $a = 0$, and he assumes more about the behaviour of $F(x)$ for finite values of x, viz. that it is the Riemann integral of a bounded function. What is essential is, of course, the behaviour of $F(x)$ when $x \to \infty$.

It would be natural to say that $G(t)\to A$ (H,r) if $G_r(t)\to 0$ and $G(t)\to A$ (C,r) if $G_r(t)\to 0$; and analogy would lead us to expect these two generalizations of the notion of a limit also to be equivalent to one another.

The theorem thus suggested is in fact true, but a little consideration shows that it is not quite so obvious an analogue of Landau's theorem as we might have anticipated. In fact, if in (3.1) and (4.1) we write

$$t=\frac{1}{x},\ t_1=\frac{1}{x_1},\ \ldots,\ G(t)=G\left(\frac{1}{x}\right)=F(x),$$

we obtain

$$F_r(x)=x\int_x^\infty \frac{dx_1}{x_1}\int_{x_1}^\infty \frac{dx_2}{x_2}\ldots\frac{dx_{r-1}}{x_{r-1}}\int_{x_{r-1}}^\infty \frac{F(x_r)}{x_r^2}\,dx_r\to A \tag{3.2}$$

and

$$\mathrm{F}_r(x)=rx\int_x^\infty F(y)(y-x)^{r-1}\frac{dy}{y^{r+1}}\to A\ ; \tag{4.2}$$

and these means are obviously of a different type from (1.1) and (2.1). If (3.2) is true, we say that

$$F(x)\to A\quad (M,r), \tag{3.3}$$

and if (4.2) is true, we say that

$$F(x)\to A\quad (N,r)\ ; \tag{4.3}$$

and we have to consider whether these definitions are equivalent to (1.2), (2.2), and to one another.

The results are as follows:—

(i) *the H means are equivalent to the C means*;

(ii) *the M means are equivalent to the N means*;

(iii) *the H (or C) means are never less general than the M (or N) means; if* $F(x)\to A$ (M,r) *then* $F(x)\to A$ (H,r);

(iv) *if* $r=1$, *all four sets of means are equivalent; but*

(v) $F(x)\to A$ (H,r) *does not necessarily imply* $F(x)\to A$ (M,r) *for any value of r greater than* 1.

We prove first that, if $F(x)$ is any integrable function, and $a>0$, there are integrable functions $g(x)$ such that

$$F(x)-F(a)=G(x)-G(a)-xg(x)+ag(a)=\int_a^x g(t)\,dt-xg(x)+ag(a) \tag{5}$$

for all values of x for which $F(x)$ is defined. We suppose, as we may do without loss of generality, that $G(a)=F(a)=0$, and we then show that

the necessary and sufficient condition that $F(x) \to A$ (M, r) *is that there should be a* $g(x)$ *for which*

$$G(x) = o(x), \quad G(x) \to A \quad (M, r-1),$$

and that this remains true when M is replaced by N. From this the equivalence of the M and N means follows by induction. A similar proposition holds for H or C means; we have only to omit the provision that $G(x) = o(x)$, and our remaining theorems follow without difficulty.

Combining our theorem of equivalence with the necessary and sufficient criterion for the summability of a Fourier series which we enunciated recently in the *Proceedings*,* we obtain the theorem: *the necessary and sufficient condition that the Fourier series of* $f(x)$ *should be summable by Cesàro's means to sum* S *is that*

$$\frac{r}{t^r}\int_0^t \phi(u)\,(t-u)^{r-1}\,du,$$

where

$$\phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)-2S\},$$

should tend to zero with t *for some value of* r.

The Allied Series of a Fourier Series

G. H. HARDY *and* J. E. LITTLEWOOD.

1. The methods which we have applied to the solution of the "summability problem" for a Fourier's series $\frac{1}{2}a_0+\Sigma(a_n \cos nx+b_n \sin nx)$ may be applied also to the corresponding problem for the allied or conjugate series $\Sigma(b_n \cos nx-a_n \sin nx)$.

Suppose that $f(t)$ is integrable (in the sense of Lebesgue) in (ϵ, a) for every positive ϵ, that

$$\chi(\epsilon) = \int_\epsilon^a f(t)\,dt, \quad \chi_1(\epsilon) = \int_0^\epsilon \chi(\epsilon_1)\,d\epsilon_1, \quad \chi_2(\epsilon) = \int_0^\epsilon \chi_1(\epsilon_1)\,d\epsilon_1, \ldots,$$

and that

$$\chi_k(\epsilon) \sim A\epsilon^k$$

for some value of k. Then we say that

$$\int_0^a f(t)\,dt = A \ (C),$$

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* See *Records*, &c. for 8 June, 1922 (*Proc. Lond. Math. Soc.* (2), Vol. 21, pp. xxxi–xxxii). The detailed proofs of the theorems there stated will appear shortly in the *Math. Zeitschrift*.

or that the integral exists in the Cesàro sense. This being so, *the necessary and sufficient condition that the allied series of $f(x)$ should be summable (C) is that*

$$(1) \qquad \int_0^a \frac{f(x+t)-f(x-t)}{t}\,dt$$

should exist in the Cesàro sense.

2. It is known that any Fourier series is summable (C, δ) for every positive δ and almost all values of x. The same is true of the allied series; but the theorem lies a little deeper and has not, so far as we know, been stated generally, though the corresponding theorem for summability $(C, 1)$ is proved in a recent dissertation of Plessner.*

It was first proved by Young† that the allied series is summable $(C, 1)$ whenever (i) the integral (1) exists (as an "elementary generalized integral") and (ii) the integral

$$\int_0^u |f(x+t)-f(x-t)|\,dt$$

has, for $u = 0$, a differential coefficient which is zero; and there is no particular difficulty in replacing 1 by δ in Young's enunciation. The second condition is satisfied almost everywhere, and, in particular, in the set, whose complementary has measure zero, in which $|f(x)-q|$ is the derivative of its integral whatever be q.

We may call this set the Lebesgue set. In the Lebesgue set, the Fourier series of $f(x)$ is summable (C, δ); but this is not generally true of the allied series, since (i) is not necessarily true in the Lebesgue set or even at a point of continuity of the function. There is, however, an interesting property of the allied series which is true in the Lebesgue set, whether (i) be true or not, and which is the strict analogue of the property of the Fourier series. This property is that *if the series is summable by* any *Cesàro mean, or, more generally, if it is summable by Abel's limit, then it is summable (C, δ) for every positive δ.*

The question remains whether (i) is true almost everywhere, and this question is answered by the work of Fatou‡ and Plessner. It is equivalent to the question whether Abel's limit for the allied series exists for almost all values of x; the existence of this limit being, for points of the Lebesgue

* A. Plessner, "Zur Theorie der konjugierten trigonometrischen Reihen", Giessen, 1922.

† *Proc. London Math. Soc.* (2), Vol. 10 (1911), pp. 271–272.

‡ P. Fatou, *Acta Math.*, Vol. 30 (1906), pp. 335–400.

set, a necessary and sufficient condition for the existence of the integral (1). This connection was first remarked by Fatou as holding at all points of continuity of $f(x)$,* and is established in full generality in Plessner's dissertation. An equivalent condifion is the summability $(C, 1)$ of the allied series. This may be inferred from the work of Young, and is proved directly and explicitly by Plessner; and the theorem quoted in the last paragraph shows that we may replace $(C, 1)$ by (C, δ). To settle the main question, however, Plessner has recourse again to Abel's limit, and proves, by a direct adaptation of one of Fatou's theorems, that this limit does in fact exist (though not necessarily in the Lebesgue set) for almost all values of x. Combining these results we reach our final conclusion, that *the allied series is almost everywhere summable* (C, δ) *for every positive* δ.

We should refer in conclusion to two notes of Priwaloff.† The subject matter of these notes is extremely interesting, but the indications of demonstrations are insufficient.

* *Loc. cit.*, p. 360.

† I. Priwaloff, *Comptes Rendus*, Vol. 162 (1916), pp. 123–126, and Vol. 165 (1917), pp. 96–99.

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PAPERS

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SURFACE WAVES ON LIMITED SHEETS OF WATER

By J. Vint.

(*Communicated by* H. R. Hassé.)

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Introduction.

The problem of determining surface waves on limited sheets of water has been solved only in a comparatively small number of cases. These solutions,* as far as the writer is aware, are:—

(*a*) Waves in a cylindrical basin of circular section and uniform depth, of which the complete solution was given by Rayleigh.†

(*b*) Waves in a canal bounded by two planes at right angles and equally inclined to the vertical, of which the two-dimensional solution was given by Kirchhoff.‡ The propagation of waves with a plane wave-front along this type of canal was given by Kelland§ and A. G. Greenhill.‖

* Lamb, *Hydrodynamics*, 3rd edition, §§ 254–258.

† Rayleigh, *Collected Papers*, Vol. 1, pp. 251 *et seq.*

‡ Kirchhoff, *Ges. Abhand.*, p. 428.

§ Kelland, *Trans. Royal Soc. Edin.*, Vol. 14 (1839).

‖ A. G. Greenhill, *American Journal of Math.*, Vol. 9 (1887).

(*c*) Waves in a canal bounded by two planes each inclined at an angle of 60° to the vertical plane through their intersection. The two-dimensional solution was given by Greenhill, while that for waves propagated along the canal with a plane front was given by H. M. Macdonald.*

In the present paper, solutions are obtained for the following cases:—

(A) Surface waves in the pyramidal basin bounded by four symmetrically situated planes each inclined to the vertical at an angle of 45°, and whose section by any horizontal plane is a square.

(B) Symmetrical surface waves along a canal which are of a more general type than those already enumerated under (*c*).

(A) is really a three-dimensional analogue of the two-dimensional case under (*b*). Four different solutions are obtained which are analogous in pairs to the symmetrical and asymmetrical solutions given by Kirchhoff for the two-dimensional case. In addition, there is a corresponding limiting form. If rectangular axes (x, y, z) are taken with z measured vertically upwards, and the basin is bounded by the planes $y = \pm z$, $x = \pm z$, one of the solutions is given by the velocity potential function

$$\phi = A \cos(\sigma t + \epsilon) \sum_{x, y, z} [(\cosh k_1 z \cos k_2 y + \cosh k_1 y \cos k_2 z) \cos k_3 x],$$

where $$k_1^2 = k_2^2 + k_3^2.$$

The corresponding symmetrical solution obtained by Kirchhoff is

$$\phi = A \cos(\sigma t + \epsilon)(\cosh kz \cos ky + \cosh ky \cos kz).$$

It may also be noted that the solutions given under (*b*) are readily obtained by the methods of this paper.

(B) gives an infinite number of forms for the free surface corresponding to any particular wave-length along the canal, while the solutions given by Greenhill and Macdonald are particular cases.

Statement of the Problem and Method of Solution.

The solution of the problem consists in determining a velocity potential ϕ which satisfies the following conditions

$$\nabla^2 \phi = 0,$$

$$\frac{\partial \phi}{\partial n} = 0$$

* H. M. Macdonald, *Proc. London Math. Soc.*, Ser. 1, Vol. 25 (1894).

over the fixed boundaries, and

$$\sigma^2\phi = g\frac{\partial\phi}{\partial z}$$

at the free surface, where ϕ contains the time-factor $\cos(\sigma t+\epsilon)$, and the axes (x, y, z) are rectangular with z measured vertically upwards.

The method employed in this paper may be described as an "image" method, and consists in determining a function ϕ which satisfies the fixed boundary condition, and then adjusting ϕ to fit the remaining conditions of the problem by means of the arbitrary constants involved in the function.

The function $\psi_2(x, y, z)$ is defined as the image of the function $\psi_1(x, y, z)$ with respect to the plane $z = my$, when the equation

$$\frac{\partial}{\partial z}(\psi_1+\psi_2) = m\frac{\partial}{\partial y}(\psi_1+\psi_2)$$

is satisfied at all points on this plane. If ψ_1 and ψ_2 are velocity potentials, then $(\psi_1+\psi_2)$ is a velocity potential which gives zero normal velocity everywhere over the plane $z = my$, so that the plane fulfils the condition for a fixed boundary.

(A) *Pyramidal Basin.*

The fixed boundaries are given by the planes

$$x = \pm z, \quad y = \pm z,$$

and the undisturbed free surface by $z = h$.

The image of the function

$$\psi_1 = e^{k_1 z}\cos k_2 y,$$

with respect to $y = z$ is $\psi_2 = e^{k_1 y}\cos k_2 z$,

that of ψ_2 with respect to $y = -z$ is

$$\psi_3 = e^{-k_1 z}\cos k_2 y,$$

that of ψ_3 with respect to $y = z$ is

$$\psi_4 = e^{-k_1 y}\cos k_2 z,$$

and that of ψ_4 with respect to $y = -z$ is ψ_1 again. It easily follows that the function

$$\tfrac{1}{2}(\psi_1+\psi_2+\psi_3+\psi_4) = (\cosh k_1 z\cos k_2 y+\cosh k_1 y\cos k_2 z), \qquad (1)$$

is one which satisfies the conditions

$$\frac{\partial}{\partial y} = \pm \frac{\partial}{\partial z},$$

when $y = \pm z$ respectively.

Consider the function

$$\phi_1 = A\,(\cosh k_1 z \cos k_2 y + \cosh k_1 y \cos k_2 z) \cos k_3 x, \tag{2}$$

where $$k_1^2 = k_2^2 + k_3^2.$$

It is a velocity potential which gives zero normal velocity across the planes $y = \pm z$. The function

$$\left.\begin{aligned} \phi = A \sum_{x,\,y,\,z} [(\cosh k_1 z \cos k_2 y + \cosh k_1 y \cos k_2 z) \cos k_3 x] \\ \text{where} \qquad k_1^2 = k_2^2 + k_3^2 \end{aligned}\right\}, \tag{3}$$

is also a velocity potential which gives zero normal velocity across the planes

$$x = \pm y, \quad y = \pm z, \quad z = \pm x.$$

This result may be verified independently.

The function ϕ gives a solution for standing waves in the basin

$$y = \pm z, \quad x = \pm z.$$

It may be noted that this is also a solution for the more elementary basin

$$x = \pm y, \quad y = z.$$

In order that the condition

$$\sigma^2 \phi = g \frac{\partial \phi}{\partial z} \quad \text{when} \quad z = h,$$

may be satisfied, the following conditions must hold

$$\sigma^2 \cosh k_1 h = \quad g k_1 \sinh k_1 h,$$

$$\sigma^2 \cos k_2 h = -g k_2 \sin k_2 h,$$

$$\sigma^2 \cos k_3 h = -g k_3 \sin k_3 h.$$

As yet, the quantities σ^2, k_1, k_2, k_3 are arbitrary, except that

$$k_1^2 = k_2^2 + k_3^2.$$

So there are four equations to determine these four quantities.

Writing $$k_1 h = \alpha, \quad k_2 h = \beta, \quad k_3 h = \gamma,$$

we find that these "period" equations are

$$\left.\begin{aligned} \frac{\sigma^2 h}{g} = \alpha \tanh \alpha = -\beta \tan \beta = -\gamma \tan \gamma \\ \alpha^2 = \beta^2 + \gamma^2 \end{aligned}\right\}. \qquad (4)$$

The determination of k_1, k_2, k_3 is equivalent to finding the intersections of the curve

$$\alpha \tanh \alpha = -\beta \tan \beta = -\gamma \tan \gamma,$$

with the cone

$$\alpha^2 = \beta^2 + \gamma^2,$$

or, what comes to the same thing, the intersection of the curves

$$\sqrt{(\beta^2+\gamma^2)} \tanh \sqrt{(\beta^2+\gamma^2)} + \beta \tan \beta = 0,$$

and

$$\beta \tan \beta - \gamma \tan \gamma = 0.$$

There is an infinite number of real positive solutions to (4).

One set of solutions is given by

$$\beta = \gamma = \alpha/\sqrt{2},$$

where α is determined by

$$\sqrt{2} \tanh \alpha + \tan (\alpha/\sqrt{2}) = 0.$$

The lowest root of this equation other than zero is 3·095 ..., while all higher roots are given by

$$\tan (\alpha/\sqrt{2}) = -\sqrt{2} \text{ approximately.}$$

The corresponding value of σ is given by

$$\sigma^2 h/g = \alpha \tanh \alpha.$$

For moderately large values* of α,

$$\tanh \alpha = 1 \text{ approximately,}$$

so that, for practical purposes, equations (4) become

$$\left.\begin{aligned} \alpha = -\beta \tan \beta = -\gamma \tan \gamma \\ \alpha^2 = \beta^2 + \gamma^2 \end{aligned}\right\}. \qquad (5)$$

On eliminating α, we see that the intersections of the curves

$$\gamma^2 = \beta^2(\tan^2 \beta - 1),$$

$$\beta^2 = \gamma^2(\tan^2 \gamma - 1),$$

* $\tanh 5 = 0{\cdot}99991 \ldots$.

include the solutions of (5), and these particular solutions are readily picked out from these curves. For example,

$$\gamma = \tfrac{1}{2}\pi + \epsilon, \qquad \beta = n\pi - \tfrac{1}{4}\pi,$$

where n is a large integer, is a suitable set of solutions which give

$$\sigma^2 h/g = (n\pi - \tfrac{1}{4}\pi) \text{ approximately.}$$

By replacing $\cos k_3 x$ by $\cosh k_3 x$ in equation (3), we obtain another form which satisfies the necessary conditions

$$\left.\begin{aligned} \phi &= A \sum_{x,y,z} [(\cosh k_1 z \cos k_2 y + \cosh k_1 y \cos k_2 z) \cosh k_3 x] \\ \text{where} \qquad & k_1^2 = k_2^2 - k_3^2 \end{aligned}\right\}. \tag{6}$$

Retaining the same notation, the corresponding period equations are

$$\sigma^2 h/g = \alpha \tanh \alpha = -\beta \tan \beta = \gamma \tanh \gamma,$$

and

$$\alpha^2 = \beta^2 - \gamma^2.$$

In the particular case, where

$$\alpha = \beta/\sqrt{2} = \gamma,$$

α is given by

$$\tanh \alpha + \sqrt{2} \tan (\sqrt{2}\,\alpha) = 0.$$

All the positive values of α, except one, are given by

$$\sqrt{2} \tan \sqrt{2}\,\alpha = -1 \text{ approximately.}$$

When $e^{k_1 z} \sin k_2 y$ is taken for ψ_1, and the above method applied, it is easy to verify that a solution analogous to the asymmetrical two-dimensional case is

$$\left.\begin{aligned} \phi' &= A \sum_{x,y,z} [(\sinh k_1 z \sin k_2 y + \sinh k_1 y \sin k_2 z) \sin k_3 x] \\ \text{where} \qquad & k_1^2 = k_2^2 + k_3^2 \end{aligned}\right\}. \tag{7}$$

The corresponding period equations are

$$\left.\begin{aligned} \sigma^2 h/g = \alpha \coth \alpha &= \beta \cot \beta = \gamma \cot \gamma \\ \alpha^2 &= \beta^2 + \gamma^2 \end{aligned}\right\}, \tag{8}$$

and may be examined in exactly the same manner as the previous sets.

When $\sin k_3 x$ is replaced by $\sinh k_3 x$ in ϕ', then

$$\phi' = A \sum_{x,y,z} [(\sinh k_1 z \sin k_2 y + \sinh k_1 y \sin k_2 z) \sinh k_3 x], \tag{9}$$

satisfies the conditions, where

$$\sigma^2 h/g = \alpha \coth \alpha = \beta \cot \beta = \gamma \coth \gamma,$$

$$\alpha^2 = \beta^2 - \gamma^2.$$

Slowest Mode.—A solution of (8) is given by

$$\alpha = \beta = \gamma = 0, \qquad \sigma^2 h/g = 1.$$

In (7) replace A by $B/(3k_1 k_2 k_3)$ and approximate to ϕ' to the first order. When k_1, k_2, k_3 tend to zero, the limiting value obtained for the velocity potential is

$$\phi = Bxyz.$$

This is an algebraic solution which gives $\sigma^2 = g/h$, and it may be verified independently. It is analogous to, and has the same period as, the slowest mode in two dimensions where the free surface is always plane.*

Form of Free Surface.—If ζ denotes the elevation of the free surface above the level $z = h$, at the time t,

$$\zeta = \frac{1}{g}\left[\frac{\partial \phi}{\partial t}\right]_{z=h}.$$

By introducing the factor $\cos(\sigma t + \epsilon)$ into the forms for the velocity potential, the general value for the elevation of the free surface at the time t may be written down.

In this connection, it may be noted that when the hyperbolic functions have comparatively large values, the constant A should be considered relatively small in order that the hypothesis of small motions may not be violated.

The case of most interest is the slowest mode where

$$\phi = Bxyz \cos(\sigma t + \epsilon), \qquad \sigma^2 = g/h,$$

for which

$$\zeta = -\frac{\sigma h}{g} Bxy \sin(\sigma t + \epsilon)$$

$$= -\frac{B}{\sigma} xy \sin(\sigma t + \epsilon)$$

at the time t. The contours at any time are rectangular hyperbolæ, and the straight lines given by the intersections of $x = 0$ and $y = 0$, with $z = h$, always lie on the free surface which is a hyperbolic paraboloid.

* Lamb, *Hydrodynamics*, 3rd edition, § 255, 1°.

Limiting Form for Rectangular Bay. — When we change the direction of the axis of z to vertically downwards, (3) and (4) are unaltered, while the free surface condition naturally becomes

$$\sigma^2\phi = -g\frac{\partial\phi}{\partial z} \quad \text{at} \quad z = -h.$$

Also, transferring the origin to the corner of the basin at $(-h, -h, -h)$, the solution becomes

$$\phi = A \sum_{x,y,z} [\{\cosh k_1(z-h)\cos k_2(y-h) + \cosh k_1(y-h)\cos k_2(z-h)\} \times \cos k_3(x-h)], \quad (10)$$

together with

$$\sigma^2/g = k_1 \tanh k_1 h = -k_2 \tan k_2 h = -k_3 \tan k_3 h,$$

$$k_1^2 = k_2^2 + k_3^2.$$

On expanding and taking a new constant

$$C = A \cosh k_1 h \cos k_2 h \cos k_3 h,$$

(10) becomes

$$\phi = C \sum_{x,y,z} [\{(\cosh k_1 z - \tanh k_1 h \sinh k_1 z)(\cos k_2 y + \tan k_2 h \sin k_2 y) + (\cosh k_1 y - \tanh k_1 h \sinh k_1 y)(\cos k_2 z + \tan k_2 h \sin k_2 z)\} \times \{\cos k_3 x + \tan k_3 h \sin k_3 x\}].$$

When h is large the period equations give

$$\tan k_2 h = -k_1/k_2, \qquad \tan k_3 h = -k_1/k_3,$$

$$k_1^2 = k_2^2 + k_3^2.$$

Hence it follows that, when h tends to infinity, the limiting form obtained for ϕ is

$$\phi = \frac{C}{k_2 k_3} \sum_{x,y,z} [\{(\cosh k_1 z - \sinh k_1 z)(k_2 \cos k_2 y - k_1 \sin k_2 y) + (\cosh k_1 y - \sinh k_1 y)(k_2 \cos k_2 z - k_1 \sin k_2 z)\} \times \{k_3 \cos k_3 x - k_1 \sin k_3 x\}],$$

which may be written

$$\left.\begin{aligned} \phi = D \sum_{x,y,z} [e^{-k_1 z}\{(k_2 \cos k_2 y - k_1 \sin k_2 y)(k_3 \cos k_3 x - k_1 \sin k_3 x) \\ + (k_2 \cos k_2 x - k_1 \sin k_2 x)(k_3 \cos k_3 y - k_1 \sin k_3 y)\}] \\ \text{where} \qquad k_1^2 = k_2^2 + k_3^2 \end{aligned}\right\}. \quad (11)$$

It is easy to verify that this expression for ϕ satisfies the conditions

$$\nabla^2\phi = 0,$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{when} \quad x = z, \quad y = z, \quad \text{and} \quad z = x,$$

as well as $$\sigma^2\phi = -g\frac{\partial\phi}{\partial z} \quad \text{when} \quad z = 0.$$

The last condition gives $\sigma^2 = gk_1$, a result which is evident from the period equations.

(11) gives the solution for waves in a rectangular bay of infinite extent bounded by the planes

$$x = z, \qquad y = z,$$

and also in the infinite triangular bay bounded by the planes

$$x = y, \qquad x = z.$$

The motions everywhere are small, and the velocity at a great depth tends to zero in accordance with the theory of waves on deep water.

The same limiting form for ϕ may be obtained from (7), but (6) and (9) are unsuitable.

Deduction of known Solutions.—If the functions given by (1) and the corresponding function deduced from $e^{k_1 z}\sin k_2 y$ are made to satisfy

$$\frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0,$$

it follows that $k_1 = k_2 = k$, and the cases due to Kirchhoff follow. Again (2) and the other corresponding expression give the cases for waves propagated along the canal whose sides are

$$y = \pm z.$$

(B) *Canal of Triangular Section.*

It is now proposed to apply this image process to the canal bounded by the planes

$$y = \pm\sqrt{3}\,.\,z,$$

where the axis of z is again taken vertically upwards. Commencing with

$$\psi_1 = e^{k_1 z}\cos k_2 y,$$

and working on the same lines, the function

$$[e^{k_1 z}\cos k_2 y + e^{\frac{1}{2}k_1(\sqrt{3}.y-z)}\cos \tfrac{1}{2}k_2(y+\sqrt{3}.z)] + e^{-\frac{1}{2}k_1(\sqrt{3}.y+z)}\cos \tfrac{1}{2}k_2(y-\sqrt{3}.z)], \quad (12)$$

is built up to satisfy the conditions

$$\frac{\partial}{\partial y} = \pm\sqrt{3}\,\frac{\partial}{\partial z},$$

when $y = \pm\sqrt{3}.z$ respectively.

If $e^{k_1 z}\sin k_2 y$ is taken for ψ_1, it is found that twice the number of images are required before ψ_1 is repeated, while the function given by the sum of these images is identically zero.

Multiplying (12) by $\cos k_3 x$ gives

$$\phi_1 = [e^{k_1 z}\cos k_2 y + e^{\frac{1}{2}k_1(\sqrt{3}.y-z)}\cos \tfrac{1}{2}k_2(y+\sqrt{3}.z) + e^{-\frac{1}{2}k_1(\sqrt{3}.y+z)}\cos \tfrac{1}{2}k_2(y-\sqrt{3}.z)]\cos k_3 x \ldots,$$

which satisfies the fixed boundary conditions, and will also satisfy

$$\nabla^2\phi = 0,$$

provided $k_1^2 = k_2^2 + k_3^2$.

On attempting to fit ϕ_1 to the condition

$$\sigma^2\phi_1 = g\,\frac{\partial\phi_1}{\partial z} \quad \text{when} \quad z = h,$$

it is found that there are more conditions to be satisfied than variables for disposal. However, an additional function ϕ_2 is given by taking $-k_1$ instead of k_1, and the velocity potential ϕ gives a solution of the problem, where

$$\left.\begin{aligned}\phi &= A\phi_1 + B\phi_2 \\ &= \cos k_3 x\,\{(Ae^{k_1 z} + Be^{-k_1 z})\cos k_2 y \\ &\qquad + [Ae^{\frac{1}{2}k_1(\sqrt{3}.y-z)} + Be^{-\frac{1}{2}k_1(\sqrt{3}.y-z)}]\cos \tfrac{1}{2}k_2(y+\sqrt{3}.z) \\ &\qquad + [Ae^{-\frac{1}{2}k_1(\sqrt{3}.y+z)} + Be^{\frac{1}{2}k_1(\sqrt{3}.y+z)}]\cos \tfrac{1}{2}k_2(y-\sqrt{3}.z)\}\end{aligned}\right\}. \quad (13)$$

and $k_1^2 = k_2^2 + k_3^2$

Further ϕ must satisfy the surface condition, and this demands that the

following four equations should be satisfied

$$\left.\begin{aligned}
&\sigma^2(Ae^{k_1h}+Be^{-k_1h}) = gk_1(Ae^{k_1h}-Be^{-k_1h})\\
&2\sigma^2(Ae^{-\frac{1}{2}k_1h}+Be^{\frac{1}{2}k_1h})\cos\tfrac{1}{2}\sqrt{3}.k_2h\\
&\qquad = g\left[-k_1(Ae^{-\frac{1}{2}k_1h}-Be^{\frac{1}{2}k_1h})\cos\tfrac{1}{2}\sqrt{3}.k_2h\right.\\
&\qquad\qquad \left.-\sqrt{3}.k_2(Ae^{-\frac{1}{2}k_1h}+Be^{\frac{1}{2}k_1h})\sin\tfrac{1}{2}\sqrt{3}.k_2h\right]\\
&2\sigma^2(-Ae^{-\frac{1}{2}k_1h}+Be^{\frac{1}{2}k_1h})\sin\tfrac{1}{2}\sqrt{3}.k_2h\\
&\qquad = g\left[k_1(Ae^{-\frac{1}{2}k_1h}+Be^{\frac{1}{2}k_1h})\sin\tfrac{1}{2}\sqrt{3}.k_2h\right.\\
&\qquad\qquad \left.-\sqrt{3}.k_2(Ae^{-\frac{1}{2}k_1h}-Be^{\frac{1}{2}k_1h})\cos\tfrac{1}{2}\sqrt{3}.k_2h\right]\\
&k_1^2 = k_2^2+k_3^2
\end{aligned}\right\}. \qquad (14)$$

These four equations are sufficient to determine the quantities B/A, k_1, k_2, σ, suitable to any particular value of k_3. Hence (13) gives the solution for standing waves of any given wave-length along the canal bounded by the planes $y = \pm\sqrt{3}.z$.

Of course, $\cos k_3x$ may be replaced by $\sin k_3x$ in (13), and the superposition of two properly chosen standing waves will give a progressive wave along the canal.

Writing $$A = e^{\beta}, \quad B = e^{-\beta},$$

$$\frac{\sigma^2h}{g} = \mu, \quad k_1h = v, \quad \sqrt{3}.k_2h = w, \quad k_3h = \frac{a}{\sqrt{3}},$$

where a is constant, (14) become on simplification

$$\left.\begin{aligned}
\mu\cosh(v+\beta) &= v\sinh(v+\beta)\\
2\mu\cosh(\tfrac{1}{2}v-\beta)\cos\tfrac{1}{2}w &= v\sinh(\tfrac{1}{2}v-\beta)\cos\tfrac{1}{2}w - w\cosh(\tfrac{1}{2}v-\beta)\sin\tfrac{1}{2}w\\
2\mu\sinh(\tfrac{1}{2}v-\beta)\sin\tfrac{1}{2}w &= v\cosh(\tfrac{1}{2}v-\beta)\sin\tfrac{1}{2}w + w\sinh(\tfrac{1}{2}v-\beta)\cos\tfrac{1}{2}w\\
3v^2-w^2 &= a^2
\end{aligned}\right\}. \qquad (15)$$

These may be written

$$\left.\begin{aligned}
\mu &= v\tanh[\tfrac{3}{2}v-(\tfrac{1}{2}v-\beta)]\\
2\mu+w\tan\tfrac{1}{2}w &= v\tanh(\tfrac{1}{2}v-\beta)\\
2\mu-w\cot\tfrac{1}{2}w &= v\coth(\tfrac{1}{2}v-\beta)\\
3v^2-w^2 &= a^2
\end{aligned}\right\}. \qquad (16)$$

The second and third equations of (16) give

$$(2\mu+w\tan\tfrac{1}{2}w)(2\mu-w\cot\tfrac{1}{2}w) = v^2,$$

while the first two give

$$v[\mu - v\tanh\tfrac{3}{2}v] = [2\mu + w\tan\tfrac{1}{2}w][\mu\tanh\tfrac{3}{2}v - v].$$

Writing $\tanh\frac{3}{2}v = T, \quad \tan\frac{1}{2}w = t,$

these last two equations take the forms

$$\left.\begin{aligned} 4\mu^2 t + 2\mu(wt^2 - w) - t(w^2 + v^2) &= 0 \\ 2\mu^2 T + \mu(wtT - 3v) + v(vT - wt) &= 0 \end{aligned}\right\}. \qquad (17)$$

The elimination of μ from (17) then gives

$$w^3 tT(t^2+1) + vw^2(3t^2T^2 + 2T^2 - 3t^2 - 2t^4)$$
$$- 3v^2 w(t^2+1)tT + 9v^3(T^2-1)t^2 = 0. \qquad (18)$$

Regarded as a curve, the intersections of (18) with the hyperbola

$$3v^2 - w^2 = a^2$$

will give the values of $k_1 h$ and $k_2 h$ suitable to any wave length.

Special Cases.—If in (13) $k_1 = k_2 = k$, and $k_3 = 0$, the two-dimensional case due to Greenhill is given. Putting $a = 0$, and

$$w = \sqrt{3}.v = \sqrt{3}.kh,$$

the equation to determine k given by (18) leads to

$$\cosh 3kh = -\cos\sqrt{3}.kh + 2\sec\sqrt{3}.kh.$$

The second period equation given by Greenhill*

$$3\cosh 3kh = -\cos\sqrt{3}.kh - 2\sec\sqrt{3}.kh$$

does not arise, and has been introduced by his method of elimination.

When $w = \sqrt{3}.k_2 h = 0$ and $k_1 = k_3 = k,$

(13) gives the solution due to Macdonald, while the corresponding period equations† given by (15) are then

$$\sigma^2\cosh(kh+\beta) = gk\sinh(kh+\beta),$$
$$2\sigma^2\cosh(\tfrac{1}{2}kh-\beta) = gk\sinh(\tfrac{1}{2}kh-\beta),$$

* Greenhill, *American Journal of Math.*, Vol. 9, p. 98.
† Macdonald, *Proc. London Math. Soc.*, Ser. 1, Vol. 25, p. 108.

or $$\frac{kg}{\sigma^2} = 2\coth(\tfrac{1}{2}kh-\beta) = \coth(kh+\beta).$$

The Period Equation.—If we replace t and T by their values $\tan\frac{1}{2}w$ and $\tanh\frac{3}{2}v$ in (17), that equation becomes

$$w(w^2-3v^2)\sin w\sinh 3v - 9v^3\sin^2 w + w^2v(4\cos w\cosh 3v - 4 - \sin^2 w) = 0,$$

where $$3v^2-w^2 = a^2.$$

To find the possible values of v and w write

$$\sqrt{3}.v+w = a\theta,$$

$$\sqrt{3}.v-w = a/\theta,$$

or $$\sqrt{3}.v = \frac{a}{2}\left(\theta+\frac{1}{\theta}\right),$$

$$w = \frac{a}{2}\left(\theta-\frac{1}{\theta}\right).$$

When we substitute these values for v and w the equation to determine θ is

$$\frac{\sqrt{3}}{3}\left(\theta-\frac{1}{\theta}\right)^2\left(\theta+\frac{1}{\theta}\right)\left[4\cos\frac{a}{2}\left(\theta-\frac{1}{\theta}\right)\cosh\frac{a\sqrt{3}}{2}\left(\theta+\frac{1}{\theta}\right)\right.$$

$$\left.-4-\sin^2\frac{a}{2}\left(\theta-\frac{1}{\theta}\right)\right]$$

$$-4\left(\theta-\frac{1}{\theta}\right)\sin\frac{a}{2}\left(\theta-\frac{1}{\theta}\right)\sinh\frac{a\sqrt{3}}{2}\left(\theta+\frac{1}{\theta}\right)$$

$$-\sqrt{3}\left(\theta+\frac{1}{\theta}\right)^3\sin^2\frac{a}{2}\left(\theta-\frac{1}{\theta}\right) = 0.$$

The roots of this equation will give the corresponding values for v and w. An obvious root of this equation is $\theta = 1$, which gives the Macdonald solution. If $f(\theta)$ denotes the left-hand side of this equation, $f(\theta)$ is obviously a continuous function. Also, when θ is greater than unity and $\frac{a}{2}\left(\theta-\frac{1}{\theta}\right)$ greater than 2π, the sign of $f(\theta)$ is governed by the term in $\cosh\frac{a\sqrt{3}}{2}\left(\theta+\frac{1}{\theta}\right)$. In fact, $f(\theta)$ is positive when

$$\frac{a}{2}\left(\theta-\frac{1}{\theta}\right) = 2n\pi,$$

and negative when $$\frac{a}{2}\left(\theta-\frac{1}{\theta}\right)=(2n+1)\pi,$$

where n is a positive integer. Consequently there is an infinite number of positive values of v and w suitable to any particular wave-length $2\pi/k_3$. Hence the first equation in (17) has always one positive root for μ, and therefore the problem has always a solution.

When $a=2$, and therefore $k_3h=2/\sqrt{3}$, the smallest value of θ greater than unity is 4·8457. The corresponding values for the other quantities are

$$\mu = \sigma^2h/g = 2{\cdot}915,$$

$$v = k_1h = 2{\cdot}917,$$

$$w = \sqrt{3}.k_2h = 4{\cdot}639,$$

$$B/A = 0{\cdot}098.$$

WEBER'S INTEGRAL THEOREM

By E. C. TITCHMARSH.

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1. Dr. J. W. Nicholson has recently* applied to a problem of heat conduction a formula due to H. Weber,† which gives an expression of an arbitrary function $f(x)$ as a repeated integral,

$$\text{(A)} \qquad f(x) = \int_0^\infty \frac{C_\nu(xu, u)}{J_\nu^2(u) + Y_\nu^2(u)}\, u\, du \int_1^\infty C_\nu(tu, u)\, t f(t)\, dt.$$

[I write $$C_\nu(\alpha, \beta) \equiv J_\nu(\alpha)\, Y_\nu(\beta) - Y_\nu(\alpha)\, J_\nu(\beta),$$

my general notation being that of Prof. G. N. Watson's *Theory of Bessel Functions*.]

Weber, in fact, gave two formulæ of this kind, the other one being

$$\text{(B)} \qquad f(x) = \int_1^\infty C_\nu(xu, x) u\, du \int_0^\infty \frac{C_\nu(tu, t)}{J_\nu^2(t) + Y_\nu^2(t)}\, t f(t)\, dt.$$

The two formulæ are closely connected; each of them provides the solution of an integral equation; but if we know either result, the other is at once suggested. This is Weber's procedure. He proves (B), and then proposes the integral equation

$$f(x) = \int_0^\infty C_\nu(xu, u)\, g(u)\, du.$$

Multiplying by $xC_\nu(\xi x, \xi)$ and integrating from 1 to ∞, and using (B), we get

$$g(\xi) = \frac{\xi}{J_\nu^2(\xi) + Y_\nu^2(\xi)} \int_1^\infty C_\nu(\xi x, \xi)\, x f(x)\, dx.$$

These inversion formulæ contain both (A) and (B). But to justify the procedure, we should have to know that $g(u)$ was in fact a function to

* *Proc. Royal Soc.*, A, Vol. 100 (1921), pp. 233–240.

† *Math. Annalen*, Vol. 6 (1873), p. 154.

which (B) could be applied, whereas the given function is here not $g(u)$ but $f(x)$. So, as Weber points out, all we have got is a formal solution, the truth of which still remains to be proved. Weber made no attempt to prove it. The result embodied in (A) was rediscovered by Orr,* by a method of contour integration. This was completely independent of (B); but it is not at all clear from Orr's discussion what convergence or other conditions $f(x)$ must satisfy, and his general method would require considerable justification. A rigorous proof of (B) is given by Prof. Watson.†

The essential difference between the two formulæ is that in (A) the kernel, $C_\nu(xu, u)$, is linear in Bessel functions of x, whereas the kernel of (B), $C_\nu(xu, x)$, contains products of these functions. Hence it is (A) that can be used for problems such as those of heat conduction, and Weber's sole object in obtaining (B) appears to have been to deduce (A) from it.

As regards (A) I propose to prove the following theorem :—

If $x > a > 0$, and $\sqrt{t}.f(t)$ is summable in the infinite interval (a, ∞), and $f(t)$ is of bounded variation in a neighbourhood of $t = x$, then

$$\int_0^\infty \frac{C_\nu(xu, au)}{J_\nu^2(au)+Y_\nu^2(au)}\, u\,du \int_a^\infty C_\nu(tu, au)\, t\, f(t)\,dt = \tfrac{1}{2}\{f(x+0)+f(x-0)\},$$

the order ν being any real number.

The proof is on the lines of Prof. Watson's proof of Hankel's integral theorem.‡ It differs from the latter in the following respects :—

(i) I use the Lebesgue theory of integration.

(ii) There appears to be no indefinite integral connected with (A) corresponding to that of $tJ_\nu(\alpha t)\,J_\nu(\beta t)$ used in the proof of Hankel's theorem; so it has been necessary to prove a number of lemmas leading up to the use of the general theorem of § 6.

(iii) In § 8, in inverting the order of integration, I have simply appealed to uniformity of convergence, instead of justifying the step directly.§

(iv) In § 9, apart from the formal differences, the details of the argument have been simplified.‖

* *Proc. Royal Irish Acad.*, Vol. 27 (1909), Sect. A, No. 11, p. 222.

† *Theory of Bessel Functions*, § 14.52.

‡ *Loc. cit.*, § 14.4, etc.

§ Cf. *loc. cit.*, § 14.42.

‖ Cf. *loc. cit.*, § 14.44.

In the proof of the theorem it is generally assumed that $\nu > 0$. The case $\nu = 0$ requires only a number of trivial changes, certain expressions being replaced by their limits when $\nu \to 0$. To deal with negative values of ν, we need merely observe that the functions involved are even functions of ν.

2. We have first to evaluate some definite integrals. The results amount to a verification of the theorem in the case in which $f(t) = t^\nu$ in an arbitrary interval, and zero elsewhere.

The contour of integration consists of the real axis with an indentation above the origin, and a large semi-circle above the real axis with centre at the origin.

Consider $$\int \frac{H_\nu^{(1)}(xu)}{H_\nu^{(1)}(au)} \frac{du}{u} \quad (x > a),$$

taken round the contour. The integrand is regular inside the contour, since $H_\nu^{(1)}(au)$ has no zeros on or above the real axis. When $|u| \to 0$,

$$\frac{H_\nu^{(1)}(xu)}{H_\nu^{(1)}(au)} = \frac{x^{-\nu}}{a^{-\nu}} + o(1),$$

so that when the radius of the indentation tends to zero, the integral round it tends to $-\pi i a^\nu x^{-\nu}$. When $|u| \to \infty$,

$$\frac{H_\nu^{(1)}(xu)}{H_\nu^{(1)}(au)} = e^{i(x-a)u} \frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}} (1 + O|u|^{-1}),$$

so that this integral tends to zero when the radius tends to infinity.

Hence $$P\int_{-\infty}^{\infty} \frac{H_\nu^{(1)}(xu)}{H_\nu^{(1)}(au)} \frac{du}{u} = \pi i \frac{a^\nu}{x^\nu},$$

$$\int_0^\infty \frac{C_\nu(xu, au)}{J_\nu^2(au) + Y_\nu^2(au)} \frac{du}{u} = -\frac{\pi}{2} \frac{a^\nu}{x^\nu}. \tag{2.1}$$

Consider

$$\int \frac{H_\nu^{(1)}(xu)}{H_\nu^{(1)}(au)} \{J_{\nu+1}(tu)\, Y_\nu(au) - Y_{\nu+1}(tu)\, J_\nu(au)\}\, du \quad (x > t > a),$$

round the same contour. When $|u| \to 0$,

$$J_{\nu+1}(tu)\, Y_\nu(au) - Y_{\nu+1}(tu)\, J_\nu(au) = u^{-1}[2a^\nu/(\pi t^{\nu+1}) + o(1)].$$

Hence the integrand is equal to

$$u^{-1}[2a^\nu/(\pi x^\nu t^{\nu+1}) + o(1)],$$

and the integral round the indentation tends to $-2ai^{2\nu}x^{-\nu}t^{-\nu-1}$.

When $|u| \to \infty$,

$$J_{\nu+1}(tu)\,Y_\nu(au) - Y_{\nu+1}(tu)\,J_\nu(au)$$
$$= \pi^{-1}u^{-1}a^{-\frac{1}{2}}t^{-\frac{1}{2}}\left\{e^{i(t-a)u}(1+O|u|^{-1}) + e^{-i(t-a)u}(1+O|u|^{-1})\right\},$$

so that the integrand is

$$\pi^{-1}u^{-1}x^{-\frac{1}{2}}t^{-\frac{1}{2}}\left\{e^{i(x+t-2a)u}(1+O|u|^{-1}) + e^{i(x-t)u}(1+O|u|^{-1})\right\},$$

and the integral round the large semi-circle tends to zero. Hence

$$P\int_{-\infty}^{\infty} \frac{H_\nu^{(1)}(xu)}{H_\nu^{(1)}(au)}\left\{J_{\nu+1}(tu)\,Y_\nu(au) - Y_{\nu+1}(tu)\,J_\nu(au)\right\}du = \frac{2ia^{2\nu}}{x^\nu t^{\nu+1}},$$

$$(2.2)\quad \int_0^\infty \frac{C_\nu(xu, au)}{J_\nu^2(au) + Y_\nu^2(au)}\left\{J_{\nu+1}(tu)\,Y_\nu(au) - Y_{\nu+1}(tu)\,J_\nu(au)\right\}du = -\frac{a^{2\nu}}{x^\nu t^{\nu+1}}.$$

Now consider the same integrand as before, but with $t = x$. Similar considerations apply at the origin, and the integral round the indentation tends to $-2ia^{2\nu}x^{-2\nu-1}$. When $|u| \to \infty$, the integrand is

$$\pi^{-1}u^{-1}x^{-1}\left\{e^{2i(x-a)u}(1+O|u|^{-1}) + 1 + O|u|^{-1}\right\}.$$

The integral of $(\pi ux)^{-1}$ is

$$\int_0^\pi (\pi x r e^{i\theta})^{-1}\, ri\, e^{i\theta}\, d\theta = ix^{-1},$$

and the integral of the rest tends to zero. Hence

$$P\int_{-\infty}^{\infty} \frac{H_\nu^{(1)}(xu)}{H_\nu^{(1)}(au)}\left\{J_{\nu+1}(xu)\,Y_\nu(au) - Y_{\nu+1}(xu)\,J_\nu(au)\right\}du = \frac{2ia^{2\nu}}{x^{2\nu+1}} - \frac{i}{x},$$

and consequently

$$(2.3)\quad \int_0^\infty \frac{C_\nu(xu, au)}{J_\nu^2(au) + Y_\nu^2(au)}\left\{J_{\nu+1}(xu)\,Y_\nu(au) - Y_{\nu+1}(xu)\,J_\nu(au)\right\}du$$
$$= -\frac{a^{2\nu}}{x^{2\nu+1}} + \frac{1}{2x}.$$

Finally, take $$\int \frac{H_{\nu+1}^{(1)}(tu)}{H_\nu^{(1)}(au)}\,C_\nu(xu, au)\,du,$$

where $t > x > a$, round the same contour.

When $|u| \to 0$,

$$\frac{H_{\nu+1}^{(1)}(tu)}{H_\nu^{(1)}(au)} = \frac{1}{u}\left\{\frac{2\nu a^\nu}{t^{\nu+1}} + o(1)\right\},$$

$$C_\nu(xu, au) = -\frac{1}{\nu\pi}\left\{\left(\frac{x}{a}\right)^\nu - \left(\frac{a}{x}\right)^\nu\right\} + O|u|.$$

The integral round the indentation tends to

$$\frac{2ia^\nu}{t^{\nu+1}}\left\{\left(\frac{x}{a}\right)^\nu-\left(\frac{a}{x}\right)^\nu\right\},$$

and hence

$$P\int_{-\infty}^{\infty}\frac{H_{\nu+1}^{(1)}(tu)}{H_\nu^{(1)}(au)}C_\nu(xu,au)\,du=-\frac{2ia^\nu}{t^{\nu+1}}\left\{\left(\frac{x}{a}\right)^\nu-\left(\frac{a}{x}\right)^\nu\right\},$$

so that

$$(2.4)\qquad \int_0^\infty \frac{J_{\nu+1}(tu)\,Y_\nu(au)-Y_{\nu+1}(tu)J_\nu(au)}{J_\nu^2(au)+Y_\nu^2(au)}C_\nu(xu,au)\,du = \frac{a^\nu}{t^{\nu+1}}\left\{\left(\frac{x}{a}\right)^\nu-\left(\frac{a}{x}\right)^\nu\right\}.$$

Preliminary Lemmas.

3. *If* $x>a>0$, $\nu>0$, *the function of* t *and* u,

$$\frac{C_\nu(xu,au)\,C_\nu(tu,au)}{J_\nu^2(au)+Y_\nu^2(au)}\,u\sqrt{t},$$

is bounded for $u>0$, $t>a$.

The function considered is the sum of

$$\sqrt{(tu)}\,J_\nu(tu)\,\frac{C_\nu(xu,au)\,Y_\nu(au)\sqrt{u}}{J_\nu^2(au)+Y_\nu^2(au)}$$

and

$$-\sqrt{(tu)}\,Y_\nu(tu)\,\frac{C_\nu(xu,au)\,J_\nu(au)\sqrt{u}}{J_\nu^2(au)+Y_\nu^2(au)}.$$

In the first part, $\sqrt{(tu)}\,J_\nu(tu)$ is bounded for all values of tu. When $u\to\infty$, the other factor is bounded (using the asymptotic expansions). When $u\to 0$, the numerator is $O(u^{-\nu+\frac{1}{2}})$, and the reciprocal of the denominator is $O(u^{2\nu})$. Hence the first part is always bounded.

It is equally obvious that the second part is bounded, except when $tu\to 0$. Then $u\to 0$, since $t>a$, and the expression is $O\{(ut)^{\frac{1}{2}-\nu}.u^{\frac{1}{2}+3\nu}\}$. But $u=O(ut)$, since $t>a$. Hence the expression is $O(ut)^{1+2\nu}$, and so it is bounded.

4. *If* $x>a>0$, $\nu>0$, *the function of* t *and* λ,

$$\sqrt{t}\int_0^\lambda \frac{C_\nu(xu,au)\,C_\nu(tu,au)}{J_\nu^2(au)+Y_\nu^2(au)}\,u\,du,$$

is bounded for $t>a$, $\lambda>0$, $|x-t|>$ *const.*

For any fixed value of λ, this follows at once from the previous lemma.

Also, when $u \to \infty$,

$$C_\nu(tu, au)$$

$$= -\frac{2}{\pi u\sqrt{(at)}}\left\{\sin u(t-a)+\sin u(t-a).O\left(\frac{1}{u}\right)+\cos u\,(t-a).O\left(\frac{1}{u}\right)\right\},$$

$$\frac{C_\nu(xu, au)}{J_\nu^2(au)+Y_\nu^2(au)}$$

$$= -\sqrt{\left(\frac{a}{x}\right)}\left\{\sin u(x-a)+\sin u(x-a).O\left(\frac{1}{u}\right)+\cos u(x-a).O\left(\frac{1}{u}\right)\right\}.$$

On substituting these expressions, we get the integral

$$\int^\lambda \sin(x-a)u\,\sin(t-a)u\,du,$$

which is bounded, and integrals of the form

$$\int^\lambda \begin{matrix}\cos\\ \sin\end{matrix}(x-a)u\ \begin{matrix}\cos\\ \sin\end{matrix}(t-a)u.O\left(\frac{1}{u}\right)du,$$

which converge to bounded functions of t, when $\lambda \to \infty$.

5. *If* $a \leqslant A \leqslant \alpha < \beta \leqslant B$, *and* x *is not a point of the closed interval* (A, B), *then when* $\lambda \to \infty$,

$$\int_\alpha^\beta t^{\nu+1}dt\int_0^\lambda \frac{C_\nu(xu, au)\,C_\nu(tu, au)}{J_\nu^2(au)+Y_\nu^2(au)}\,u\,du \to 0,$$

uniformly in α *and* β.

Changing the order of integration, we get

$$\int_0^\lambda \frac{C_\nu(xu, au)u\,du}{J_\nu^2(au)+Y_\nu^2(au)}\int_\alpha^\beta t^{\nu+1}C_\nu(tu, au)\,du$$

$$= \int_0^\lambda \frac{C_\nu(xu, au)}{J_\nu^2(au)+Y_\nu^2(au)}\left\{\beta^{\nu+1}[J_{\nu+1}(\beta u)Y_\nu(au)-Y_{\nu+1}(\beta u)J_\nu(au)]\right.$$

$$\left.-\alpha^{\nu+1}[J_{\nu+1}(au)\,Y_\nu(au)-Y_{\nu+1}(au)J_\nu(au)]\right\}du.$$

The integral converges uniformly when $a \leqslant A \leqslant \alpha < \beta \leqslant B$, as is seen by substituting the asymptotic expansions; in fact, it behaves like

$$m_1\int_1^\infty \frac{\sin(x+\beta-2a)u}{u}\,du+m_2\int^\infty \frac{\sin(x-\beta)u}{u}\,du+m_3\int^\infty \frac{\sin(x+\alpha-2a)u}{u}\,du$$

$$+m_4\int^\infty \frac{\sin(x-\alpha)u}{u}\,du,$$

where the m's are bounded.

We have thus only to prove that it converges to zero; there are two cases:

(i) $x > \beta > a \geqslant a$. Then, by 2.1 or 2.2, it converges to

$$\beta^{\nu+1}\left(-\frac{a^{2\nu}}{x^{\nu}\beta^{\nu+1}}\right)-a^{\nu+1}\left(-\frac{a^{2\nu}}{x^{\nu}a^{\nu+1}}\right)=0.$$

(ii) $\beta > a > x$. Then, by 2.4, it converges to

$$\beta^{\nu+1}\left[\frac{a^{\nu}}{\beta^{\nu+1}}\left\{\left(\frac{x}{a}\right)^{\nu}-\left(\frac{a}{x}\right)^{\nu}\right\}\right]-a^{\nu+1}\left[\frac{a^{\nu}}{a^{\nu+1}}\left\{\left(\frac{x}{a}\right)^{\nu}-\left(\frac{a}{x}\right)^{\nu}\right\}\right]=0.$$

6. *If $\psi(t)$ is summable in the interval (A, B); $\phi(\lambda, t)$ is summable in (A, B) for all values of λ, and bounded uniformly with respect to λ in (A, B); and*

$$\int_{\alpha}^{\beta}\phi(\lambda, t)\,dt\to 0,$$

*when $\lambda\to\infty$, uniformly in α and β, for $A\leqslant\alpha<\beta\leqslant B$; then**

$$\lim_{\lambda\to\infty}\int_{A}^{B}\phi(\lambda, t)\,\psi(t)\,dt=0.$$

The theorem is proved in three stages:

(i) Suppose that $\psi(t)$ is absolutely continuous in (A, B). Let

$$\int_{A}^{t}\phi(\lambda, \tau)\,d\tau=\phi_1(\lambda, t).$$

Then, having given ϵ, we can find λ_0, so that

$$|\phi_1(\lambda, t)|<\epsilon \text{ for } \lambda>\lambda_0, \quad A\leqslant t\leqslant B.$$

Now ϕ_1 and ψ being absolutely continuous, we have

$$\int_{A}^{B}\phi(\lambda, t)\,\psi(t)\,dt=\phi_1(\lambda, B)\,\psi(B)-\int_{A}^{B}\phi_1(\lambda, t)\,D\psi(t)\,dt.$$

Hence, for $\lambda>\lambda_0$,

$$\left|\int_{A}^{B}\phi(\lambda, t)\,\psi(t)\,dt\right|<\epsilon\left\{|\psi(B)|+\int_{A}^{B}|D\psi(t)|\,dt\right\},$$

and the expression in brackets being independent of λ, the theorem follows in this case.

(ii) Suppose $\psi(t)$ is measurable and bounded in (A, B). Then having given two positive numbers ϵ and ω, we can define an absolutely continuous function $\chi(t)$ such that

$$|\psi(t)-\chi(t)|<\epsilon,$$

* This is a case of a theorem of Hobson, *Proc. London Math. Soc.*, Ser. 2, Vol. 12 (1912), p. 166. I add the proof for completeness.

except for the points of a set whose measure is less than ω.* Call this set ω, and its complement with respect to (A, B), $C\omega$. Then

$$\int_A^B \phi(\lambda, t)\,\psi(t)\,dt = \int_A^B \phi(\lambda, t)\,\chi(t)\,dt + \int_\omega \phi(\lambda, t)\,\{\psi(t)-\chi(t)\}\,dt + \int_{C\omega} \phi(\lambda, t)\,\{\psi(t)-\chi(t)\}\,dt.$$

The first integral on the right tends to zero when $\lambda \to \infty$, by the first part. The modulus of the second is less than

$$M \int_\omega |\,\psi(t)-\chi(t)\,|\,dt,$$

where M is the upper bound of $\phi(\lambda, t)$; and this is independent of λ and as small as we please with ω.

The modulus of the third integral is less than

$$(B-A)\,M\epsilon,$$

which is independent of λ and as small as we please with ϵ.

Hence the theorem is true in this case.

(iii) Suppose $\psi(t)$ summable. We may take it to be positive, since a summable function is the difference between two positive summable functions.

Let $$\psi_n(t) = \psi(t) \quad \text{where} \quad \psi(t) \leqslant n,$$
$$= \quad n \qquad\qquad \psi(t) > n;$$

$$\int_A^B \phi(\lambda, t)\,\psi(t)\,dt = \int_A^B \phi(\lambda, t)\,\psi_n(t)\,dt + \int_A^B \phi(\lambda, t)\,\{\psi(t)-\psi_n(t)\}\,dt.$$

The modulus of the last integral is less than

$$M \int_A^B \{\psi(t)-\psi_n(t)\}\,dt,$$

which is less than ϵ if n is sufficiently large. Having fixed n, we can, by the second part, find λ_0 so that

$$\left| \int_A^B \phi(\lambda, t)\,\psi_n(t)\,dt \right| < \epsilon \quad \text{for} \quad \lambda > \lambda_0.$$

Hence the general theorem is true.

* De la Vallée Poussin, *Cours d'Analyse*, t. 2, § 93.

7. *If $a \leqslant A < B$, and x is not a point of the closed interval (A, B), and $\sqrt{t}\,.f(t)$ is summable in (a, ∞), then*

$$\lim_{\lambda\to\infty} \int_A^B t f(t)\,dt \int_0^\lambda \frac{C_\nu(xu, au)\,C_\nu(tu, au)}{J_\nu^2(au)+Y_\nu^2(au)}\,u\,du = 0.$$

If B is finite this follows at once from the previous lemma, on putting

$$\phi(\lambda, t) = t^{\nu+1} \int_0^\lambda \frac{C_\nu(xu, au)\,C_\nu(tu, au)}{J_\nu^2(au)+Y_\nu^2(au)}\,u\,du,$$

$$\psi(t) = t^{-\nu} f(t).$$

If B is infinite we can choose C so that

$$\int_C^\infty \sqrt{t}\,|f(t)|\,dt < \epsilon,$$

$$\left| \int_C^\infty t f(t)\,dt \int_0^\lambda \frac{C_\nu(xu, au)\,C_\nu(tu, au)}{J_\nu^2(au)+Y_\nu^2(au)}\,u\,du \right| < K\epsilon,$$

where K is independent of λ, by lemma 4; and applying the first part to $\int_A^C$, the result follows.

8. We have so far considered repeated integrals in which only the second integration involves the arbitrary function $f(t)$. To obtain the desired result it is necessary to invert the order of integration. This can be done by the following lemma.

If $\sqrt{t}.f(t)$ is summable in (a, ∞), then

$$\int_0^\infty \frac{C_\nu(xu, au)}{J_\nu^2(au)+Y_\nu^2(au)}\,u\,du \int_a^\infty C_\nu(tu, au)\,t f(t)\,dt$$

$$= \lim_{\lambda\to\infty} \int_0^\infty t f(t)\,dt \int_0^\lambda \frac{C_\nu(xu, au)\,C_\nu(tu, au)}{J_\nu^2(au)+Y_\nu^2(au)}\,u\,du,$$

provided the limit on the right exists.

$C_\nu(tu, au)\sqrt{t}$ is bounded independently of u for $0 < u < \lambda$. Hence

$$\int_0^\infty C_\nu(tu, au)\,t f(t)\,dt$$

is uniformly convergent, and we can multiply by $\frac{C_\nu(xu, au)}{J_\nu^2(au)+Y_\nu^2(au)}\,u$, and

integrate with respect to u from 0 to λ under the integral sign. Hence

$$\int_0^\lambda \frac{C_\nu(xu, au)}{J_\nu^2(au)+Y_\nu^2(au)}\,u\,du \int_a^\infty C_\nu(tu, au)\,t f(t)\,dt$$
$$= \int_0^\infty t f(t)\,dt \int_0^\lambda \frac{C_\nu(xu, au)\,C_\nu(tu, au)}{J_\nu^2(au)+Y_\nu^2(au)}\,u\,du,$$

and making $\lambda \to \infty$ we have the result stated.

9. *The repeated integral*

$$\int_A^B \sqrt{t}.dt \int_0^\lambda \frac{C_\nu(xu, au)\,C_\nu(tu, au)}{J_\nu^2(au)+Y_\nu^2(au)}\,u\,du \quad (a \leqslant A < B)$$

tends to a limit when $\lambda \to \infty$.

We can omit $\int_A^B \int_0^h$, which is a mere constant. In the remainder, we have

$C_\nu(xu, au)$

$$= -\frac{2}{\pi u\sqrt{(ax)}} \left\{ \sin(x-a)u + \tfrac{1}{2}(\nu^2-\tfrac{1}{4}) \left(\frac{1}{x}-\frac{1}{a}\right) \cos(x-a)u + O\left(\frac{1}{u^2}\right) \right\},$$

with a similar result for $C_\nu(tu, au)$, in which the constant implied in O can be taken to be independent of t, since t is bounded below; and

$$J_\nu^2(au)+Y_\nu^2(au) = 2/(\pi au)+O(u^{-2}).$$

On substituting these expressions, all the integrals with respect to u are convergent when $\lambda \to \infty$ for all values of t, except

$$\int_h^\lambda \sin(x-a)u \sin(t-a)u\,du.$$

This contributes to the repeated integral a constant multiple of

$$\int_A^B dt \int_h^\lambda \sin(x-a)u \sin(t-a)u\,du,$$

and this differs by a constant only from the integral obtained by putting $h = 0$; and if we change the order of integration this is

$$\tfrac{1}{2}\int_A^B \left\{ \frac{\sin(x-t)\lambda}{x-t} - \frac{\sin(x+t-2a)\lambda}{x+t-2a} \right\} dt$$
$$= \tfrac{1}{2}\int_{\lambda(A-x)}^{\lambda(B-x)} \frac{\sin\theta}{\theta}\,d\theta - \tfrac{1}{2}\int_{\lambda(x+A-2a)}^{\lambda(x+B-2a)} \frac{\sin\theta}{\theta}\,d\theta,$$

which tends to a limit when $\lambda \to \infty$.

It is also evident from the above work that *the repeated integral is bounded, if A and B are bounded functions of* λ.

Proof of Weber's Integral Theorem.

10. It follows from §§ 7, 8 that, in evaluating the repeated integral, we need only consider the value of $f(t)$ in an arbitrarily small neighbourhood of $t = x$. Since $f(t)$ is of bounded variation in an interval of which x is an internal point, so also is $\sqrt{t}.f(t)$; so we may write, for the interval,

$$\sqrt{t}.f(t) = \chi_1(t) - \chi_2(t),$$

where $\chi_1(t)$, $\chi_2(t)$ are positive increasing functions of t.

Having chosen a positive number ϵ, take δ so small that $f(t)$ is of bounded variation in $(x-\delta, x+\delta)$, and also

$$\chi_1(x+\delta) - \chi_1(x+0) < \epsilon, \qquad \chi_1(x-0) - \chi_1(x-\delta) < \epsilon,$$

$$\chi_2(x+\delta) - \chi_2(x+0) < \epsilon, \qquad \chi_2(x-0) - \chi_2(x-\delta) < \epsilon.$$

By the second mean value theorem, there is a number ξ between 0 and δ such that

$$\int_x^{x+\delta} \chi_1(t)\sqrt{t}.dt \int_0^\lambda \frac{C_\nu(xu, au)\, C_\nu(tu, au)}{J_\nu^2(au) + Y_\nu^2(au)}\, u\, du$$

$$= \chi_1(x+0) \int_x^{x+\delta} \sqrt{t}.dt \int_0^\lambda \frac{C_\nu(xu, au)\, C_\nu(tu, au)}{J_\nu^2(au) + Y_\nu^2(au)}\, u\, du$$

$$+ \{\chi_1(x+\delta) - \chi_1(x+0)\} \int_{x+\xi}^{x+\delta} \sqrt{t}.dt \int_0^\lambda \frac{C_\nu(xu, au)\, C_\nu(tu, au)}{J_\nu^2(au) + Y_\nu^2(au)}\, u\, du.$$

It follows from § 9 that the first term on the right tends to a limit when $\lambda \to \infty$ (δ remaining fixed); and at the same time the modulus of the second term remains less than a constant multiple of ϵ. Hence

$$\lim_{\lambda\to\infty} \int_x^{x+\delta} \chi_1(t)\sqrt{t}.dt \int_0^\lambda \frac{C_\nu(xu, au)\, C_\nu(tu, au)}{J_\nu^2(au) + Y_\nu^2(au)}\, u\, du$$

$$= \chi_1(x+0) \lim_{\lambda\to\infty} \int_x^{x+\delta} \sqrt{t}.dt \int_0^\lambda \frac{C_\nu(xu, au)\, C_\nu(tu, au)}{J_\nu^2(au) + Y_\nu^2(au)}\, u\, du$$

$$= \chi_1(x+0)\, P.x^{-\frac{1}{2}},$$

say, where P is a function of x and a to be determined.

We can treat $\chi_2(t)$ in a similar way, and also apply similar reasoning to the interval $(x-\delta, x)$. Adding up the results, we have

$$\lim_{\lambda\to\infty} \int_{x-\delta}^{x+\delta} t f(t)\, dt \int_0^\lambda \frac{C_\nu(xu, au)\, C_\nu(tu, au)}{J_\nu^2(au) + Y_\nu^2(au)}\, u\, du = P f(x+0) + Q f(x-0),$$

where $$\lim_{\lambda\to\infty}\int_{x-\delta}^{x}\sqrt{t}.dt\int_0^\lambda \frac{C_\nu(xu,au)\,C_\nu(tu,au)}{J_\nu^2(au)+Y_\nu^2(au)}\,u\,du = Qx^{-\frac{1}{2}}.$$

To calculate P, put $f(t)=t^\nu$ in $(x, x+\delta)$ and zero elsewhere, in the above result. We get

$$\lim_{\lambda\to\infty}\int_x^{x+\delta} t^{\nu+1}dt\int_0^\lambda \frac{C_\nu(xu,au)\,C_\nu(tu,au)}{J_\nu^2(au)+Y_\nu^2(au)}\,u\,du = Px^\nu,$$

or, writing y for $x+\delta$ and inverting the order of integration,

$$\begin{aligned}Px^\nu &= \int_0^\infty \frac{C_\nu(xu,au)\,u\,du}{J_\nu^2(au)+Y_\nu^2(au)}\int_x^y C_\nu(tu,au)\,t^{\nu+1}dt\\
&= \int_0^\infty \frac{C_\nu(xu,au)u}{J_\nu^2(au)+Y_\nu^2(au)}\,\{y^{\nu+1}[J_{\nu+1}(yu)Y_\nu(au)-Y_{\nu+1}(yu)\,J_\nu(au)]\\
&\qquad -x^{\nu+1}[J_{\nu+1}(xu)\,Y_\nu(au)-Y_{\nu+1}(xu)\,J_\nu(au)]\}\,du\\
&= a^\nu\left\{\left(\frac{x}{a}\right)^\nu-\left(\frac{a}{x}\right)^\nu\right\}-x^{\nu+1}\left\{\frac{1}{2x}-\frac{a^{2\nu}}{x^{2\nu+1}}\right\}\quad[\text{by (2.3), (2.4)}]\\
&= \tfrac{1}{2}x^\nu,\end{aligned}$$

so that $P=\frac{1}{2}$.

Similarly, writing z for $x-\delta$, we have

$$\begin{aligned}Qx^\nu &= \int_0^\infty \frac{C_\nu(xu,au)u}{J_\nu^2(au)+Y_\nu^2(au)}\,\{x^{\nu+1}[J_{\nu+1}(xu)\,Y_\nu(au)-Y_{\nu+1}(xu)\,J_\nu(au)]\\
&\qquad -z^{\nu+1}[J_{\nu+1}(zu)\,Y_\nu(au)-Y_{\nu+1}(zu)\,J_\nu(au)]\}\,du\\
&= x^{\nu+1}\left\{\frac{1}{2x}-\frac{a^{2\nu}}{x^{2\nu+1}}\right\}-z^{\nu+1}\left\{-\frac{a^{2\nu}}{x^\nu z^{\nu+1}}\right\}\quad[\text{by (2.2), (2.3)}]\\
&= \tfrac{1}{2}x^\nu,\end{aligned}$$

so that $Q=\frac{1}{2}$ also. Hence

$$\lim_{\lambda\to\infty}\int_{x-\delta}^{x+\delta} t\,f(t)\,dt\int_0^\lambda \frac{C_\nu(xu,au)C_\nu(tu,au)}{J_\nu^2(au)+Y_\nu^2(au)}\,u\,du = \tfrac{1}{2}\{f(x+0)+f(x-0)\},$$

and, inverting the order of integration, we have Weber's theorem.

Note.—We have assumed throughout, as a convenient *sufficient* condition, that $\int_a^\infty \sqrt{t}\,|f(t)|\,dt$ exists; but there is no reason to suppose that this is *necessary*. On the other hand the formula, like others of a similar character, appears to break down if the inner integral (that with respect

to t) only exists as a generalized integral. A referee has pointed out to me that some of the results on p. 238 of Dr. Nicholson's paper are incorrect. This appears to be a consequence of his use of the Weber-Orr formula in the form (in our notation)

$$J_0(x\rho) = \int_0^\infty z\,dz\,\frac{C_0(z\rho,\ za)}{J_0^2(za)+Y_0^2(za)}\int_a^\infty r\,dr\,C_0(rz,\ az)\,J_0(rx).$$

The function $J_0(rx)$ violates the conditions which we have postulated, since

$$\int^\infty \sqrt{r}\,.\,J_0(rx)\,dr$$

is not convergent; and also the inner integral in the above formula is not convergent.

The correct form of the interesting definite integral which Dr. Nicholson deduces from the repeated integral is

$$\frac{J_0(x\rho)J_0(xa)+Y_0(x\rho)\,Y_0(xa)}{J_0^2(xa)+Y_0^2(xa)} = \frac{2}{\pi}\,P\int_0^\infty \frac{z\,dz}{x^2-z^2}\,\frac{J_0(z\rho)\,Y_0(za)-Y_0(z\rho)\,J_0(za)}{J_0^2(za)+Y_0^2(za)} \qquad (\rho > a),$$

and it is easily proved by considering

$$\int \frac{dz}{x-z}\,\frac{H_0^{(1)}(z\rho)}{H_0^{(1)}(za)}$$

round a contour consisting of the real axis (indented at the origin and at $z = x$), and a large semi-circle above it.

Added March 1923.—Weber's integral formula may also be deduced as a particular case of a formula due to Hilb* and Plancherel.† The work of these writers is based on the general theory of integral equations. The formula in question is‡

$$(1) \qquad f(s) = \frac{1}{\pi}\int_{l(0)}^\infty \frac{d\lambda}{\{\lambda-l(0)\}^{\frac{1}{2}}}\,u(\lambda,\ s)\int_0^k \frac{f(t)\,u(\lambda,\ t)}{\{U_0(\lambda,\ k)\}^2+\{U_1(\lambda,\ k)\}^2}\,\frac{dt}{t}.$$

Here $U_0(\lambda,\ s)$, $U_1(\lambda,\ s)$ are fundamental solutions of the equation

$$(2) \qquad \frac{d}{ds}\left(s\,\frac{du}{ds}\right)+\frac{\lambda-l(s)}{s}\,u = 0$$

and

$$u(\lambda,\ s) = U_1(\lambda,\ k)\ U_0(\lambda,\ s)-U_0(\lambda,\ k)\ U_1(\lambda,\ s).$$

* E. Hilb, *Math. Annalen*, Vol. 66 (1909), pp. 1–66.

† M. Plancherel, *Math. Annalen*, Vol. 67 (1909), pp. 519–534. The general theorem is that stated on p. 534, the relevant particular case is formula (5a), p. 522.

‡ Taking an interval (0, k) instead of Plancherel's interval (0, 1).

It is supposed that $l(s) > 1$; and the formula holds for $0 < s < k$.

We have to reduce Bessel's equation for $J_\nu(ax)$, viz.

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \frac{a^2x^2-\nu^2}{x}y = 0 \tag{3}$$

to the form (2). This is effected by putting

$$x = 1+\log s, \quad y = u(1+\log s)^{-\frac{1}{2}}, \quad a = -(\lambda-2)^{\frac{1}{2}}.$$

Then (3) takes the form (2), with

$$l(s) = 2+\frac{\nu^2-\frac{1}{4}}{(1+\log s)^2}.$$

To avoid the singularity of $l(s)$ at $s = e^{-1}$, it must be supposed that $k < e^{-1}$. The appropriate fundamental solutions are

$$U_0(\lambda, s) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}}(\lambda-2)^{\frac{1}{4}}(-1-\log s)^{\frac{1}{2}} J_\nu\{-(\lambda-2)^{\frac{1}{2}}(1+\log s)\},$$

$$U_1(\lambda, s) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}}(\lambda-2)^{\frac{1}{4}}(-1-\log s)^{\frac{1}{2}} Y_\nu\{-(\lambda-2)^{\frac{1}{2}}(1+\log s)\}.$$

Hence, writing $a = -1-\log k$, so that $a > 0$,

$$u(\lambda, s) = \tfrac{1}{2}\pi\{a(\lambda-2)(-1-\log s)\}^{\frac{1}{2}} C_\nu\{-(\lambda-2)^{\frac{1}{2}}(1+\log s),\ a(\lambda-2)^{\frac{1}{2}}\}.$$

Thus (1) is in this case

$$2f(s)(-1-\log s)^{-\frac{1}{2}}$$

$$= \int_2^\infty \frac{C_\nu\{-(\lambda-2)^{\frac{1}{2}}(1+\log s),\ a(\lambda-2)^{\frac{1}{2}}\}}{J_\nu^2\{a(\lambda-2)^{\frac{1}{2}}\} + Y_\nu^2\{a(\lambda-2)^{\frac{1}{2}}\}}\, d\lambda$$

$$\int_0^k C_\nu\{-(\lambda-2)^{\frac{1}{2}}(1+\log t),\ a(\lambda-2)^{\frac{1}{2}}\}(-1-\log t)^{\frac{1}{2}} f(t)\,\frac{dt}{t}.$$

Putting

$$\lambda = 2+u^2, \quad 1+\log s = -x, \quad 1+\log t = -z,$$

we obtain Weber's formula for the function $x^{-\frac{1}{2}}f(e^{-1-x})$.

It is assumed in the proof of (1) that $f(t)$ is of bounded variation in the neighbourhood of the point $t = x$, and that $t^{-1}f(t)$ is integrable over $(0, k)$. These are clearly equivalent to the hypotheses of the present paper.

DETERMINANTS WHOSE ELEMENTS ARE DETERMINANTS*

By W. L. FERRAR.

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1. From an array of n columns and $n+m$ rows we may obtain ${}^{n+m}C_n$ determinants of order n. This paper deals primarily with determinants whose elements are first minors of those determinants. For the case $m = 1$ the result is remarkably analogous to Jacobi's theorem on the minors of an adjugate determinant.

The determinant usually written as $|a_1\ b_2\ \dots\ k_n|$ or as $\begin{vmatrix} 1 & 2 & \dots & n \\ a & b & \dots & k \end{vmatrix}$ will, in this paper, be denoted by $|a_1\ a_2\ \dots\ a_n|$; *i.e.* the first row is used to indicate the determinant, and the remaining rows are implied but not written.

2. Consider an array of n columns and $n+1$ rows

$$\begin{matrix} a_1 & b_1 & \dots & k_1, \\ a_2 & b_2 & \dots & k_2, \\ \dots & \dots & & \dots \\ a_{n+1} & b_{n+1} & \dots & k_{n+1}. \end{matrix}$$

Let Δ_s denote the determinant

$$|a_{s+1}\ \dots\ a_{n+1}\ a_1\ \dots\ a_{s-1}|,$$

and A_r^s the co-factor of a_r in Δ_s.

Then it is easy to verify that

$$(n \text{ even})\ A_r^s \equiv -A_s^r, \quad (n \text{ odd})\ A_r^s \equiv (-)^{r+s+1} A_s^r. \qquad (2, a)$$

It may also be shown that

$$\left.\begin{aligned} &(n \text{ even})\ A_r^s \Delta_t \equiv A_r^t \Delta_s - A_s^t \Delta_r \\ &(n \text{ odd})\ A_r \Delta_t \equiv A_r \Delta_s + (-)^{r+s+1} A_s^t \Delta_r \end{aligned}\right\}. \qquad (2, b)$$

* The writer wishes to acknowledge the helpful criticism of the referees.

The identities (2, *b*) form the basis of proof of most of the results of this paper. The existence of some such identities is indicated by the expansion of certain zero determinants: *e.g.* ($n = 3$)

$$0 \equiv \begin{vmatrix} a_1 & b_1 & c_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 & b_4 & c_4 \\ . & . & . & b_4 & c_4 \end{vmatrix} \equiv A_2^3\Delta_1 + A_1^3\Delta_2 - A_1^2\Delta_3.$$

The signs in the identity (2, *b*) were determined by elementary methods other than direct Laplace expansion. The method used is shown in § 7.

2.1. LEMMA I.—*A determinant of order* $p\ (\leqslant n)$ *of the type*

$$|\, A_\alpha^\beta A_\beta^\gamma \ldots A_\kappa^\lambda A_\lambda^\alpha \,|$$

vanishes identically. [*The* $p-1$ *letters B, C, ..., F form the remaining rows.*]

This lemma is proved by induction and the use of (2, *b*).

DEFINITION.—A set of columns, such as the set here considered, of which the affixes and suffixes each form the same cycle of numbers, will be called a *cycle set.*

LEMMA II.—*In a determinant of order* n, $D \equiv |\, A_l^r A_m^s \ldots |$, *unless* ALL *the numbers* 1 *to* $n+1$ *actually occur,* D *contains a cycle set of columns and therefore vanishes.*

LEMMA III.—*If it be impossible, by the use of* (2, *a*), *to write* D, *above, with columns whose suffixes are all different, it vanishes identically.*

In dealing with determinants of order n, $|\, A_l^r A_m^s \ldots |$, which are not dealt with by the above lemmas, we may therefore confine our attention to the determinant $D \equiv |\, A_1^r A_2^s \ldots A_n^z |$, in which $(n+1)$ occurs at least once among the affixes and in which no cycle sets of columns occur.

By using (2, *b*) we obtain the following result for this determinant:—

THEOREM I.—*If* a, *one of the numbers* $1, \ldots, n+1$, *occurs in any one row* m *times, both affix and suffix positions being counted,*

$$D = \prod_{\alpha=1}^{\alpha=n+1} (\Delta_\alpha)^{m-1}.$$

The product is, of course, of degree $(n-1)$ in the Δ's.

2.2. If D be a determinant of order $n-k$, $|A_l^r A_m^s \ldots|$, the value of D is easily calculated in terms of the Δ's, but if $k > 2$ very few types give neat forms.

The general result for $k = 1$, with no row A in D, is of the form

$$D = (a_1\Delta_1 + \ldots + a_r\Delta_r) \times (\text{product of } \Delta\text{'s of degree } \overline{n-3}).$$

3. Consider an array of n columns and $n+m$ rows.

Δ denotes the determinant formed by the first n rows of the array.

A_r denotes the co-factor of a_r in Δ.

Any other determinant of order n which can be formed from the array, may be obtained by replacing rows in Δ, by rows of the array after the n-th. The principle of the notation here used is exemplified by

$\Delta_{12,\,pq} \equiv$ the determinant derived from Δ by replacing its p-th row by the $(n+1)$-th row of the array, and its q-th row by the $(n+2)$-th row of the array.

$A_r^{12,\,pq} \equiv$ the co-factor of a_r in $\Delta_{12,\,pq}$.

In this notation the identities of § 2 become, whether n be odd or even :—

$$A_p^{r,\,s} \equiv -A_s^{r,\,p}, \tag{3, a}$$

$$\Delta A_p^{t,\,s} \equiv A_p\Delta_{t,\,s} - A_s\Delta_{t,\,p}. \tag{3, b}$$

3.1. If the a's, with the exception of a_r ($r \leqslant n$), are zero, (3, b) gives the identity

$$A_p^{t,\,s}A_r + A_s^{t,\,r}A_p + A_r^{t,\,p}A_s \equiv 0. \tag{3, c}$$

This is a generalisation, for an array of n columns and $n+1$ rows, of an identity which, in the case $n = 3$, is well-known on account of its importance in work on line coordinates* and on cross-ratio.†

3.2. When we apply (3, a) and (3, b) to the array obtained from the original one by interchanging the k-th ($k \leqslant n$) and $(n+u)$-th rows, and then write the identities obtained in notation referring to the original array, we obtain analogues of (3, a) and (3, b); namely,

$$A_p^{ru,\,sk} \equiv -A_s^{ru,\,pk},$$

$$A_p^{tu,\,sk}\Delta_{u,\,k} \equiv A_p^{u,\,k}\Delta_{tu,\,sk} - A_s^{u,\,k}\Delta_{tu,\,pk}.$$

*. Cf. *Analytical Geometry of Three Dimensions*, Salmon (6th ed.), p. 42.

†. Cf. *Introduction to Higher Algebra*, Bôcher (1st ed.), p. 106.

By applying (3, *a*) and (3, *b*) to arrays obtained from the original one by two, three, ... interchanges of rows, we can obtain a series of analogues of these identities.

3 . 3. In dealing with any determinant of the type

$$|A_x \dots A_w A_p^{a,s} \dots A_q^{ab,uv} \dots|$$

it can be shown that, unless the columns may, by the use of (3, *a*) and its analogues, be written with the suffixes all different, the determinant vanishes. Accordingly, we confine our attention to a determinant whose column suffixes are $1 \dots n$ in some order. A renumbering of the rows of the array will enable us to reduce every case to a determinant of the form

$$|A_1 \dots A_r A_{r+1}^{a,s} \dots A_k^{bc,uv} \dots A_n^{de\dots,wx\dots}|.$$

3 . 4. A type giving interesting general results is

$$D \equiv |A_1 \dots A_r A_{r+1}^{a,s} \dots A_n^{e,z}|.$$

DEFINITION.—A set of columns, such that their second affix numbers form the same cycle of numbers as do their suffixes, is called a *cycle set* of columns, *e.g.* $A_\alpha^{x,\beta} A_\beta^{y,\gamma} \dots A_\lambda^{w,\alpha}$.

THEOREM II.—(*a*) *A determinant D of this type, which contains no cycle set of columns, is equal to*

$$(\Delta)^{r-1} \Delta_{a,s} \dots \Delta_{e,z},$$

the suffixes being precisely the affixes of the n—r columns of D of type $A_n^{e,z}$.

(*b*) *If the determinant contains cycle sets of columns, a further factor for each such set is introduced into the result of* (*a*) *: e.g. with the cycle set given above there is an additional factor in the product*

$$1-(\Delta_{x,\alpha} \Delta_{y,\beta} \dots \Delta_{w,\lambda})/(\Delta_{x,\beta} \Delta_{y,\gamma} \dots \Delta_{w,\alpha}).$$

The proof is obtained by using (3, *b*).

A result given by Prof. E. T. Whittaker, *Proc. Edin. Math. Soc.*, Vol. 36, pp. 107–115, Theorem C, is an example, in different notation, of (*a*) above.

4. THEOREM III.—*If* $D \equiv |A_1 \dots A_r A_{r+1}^{a,s} \dots A_n^{de\dots,wx\dots}|$, *then D has a factor* Δ^{r-1} *and the remaining factor is a sum of homogeneous products of degree n—r in the* Δ*'s, the first term of the sum being* $\Delta_{a,s\dots} \Delta_{de\dots,wx\dots}$, *in which the column affixes of D appear as suffixes.*

The following results, derived from (3, b) and its analogues, frequently enable us to evaluate D very shortly.

If D contain columns A_t and $A_p^{rq,\,st}$, the latter may be replaced by

$$\{A_p \Delta_{rq,\,st} - A_s \Delta_{rq,\,pt}\}/\Delta.$$

If D contain columns A_t, A_u, and $A_p^{rqm,\,stu}$, the latter may be replaced by

$$\{A_p \Delta_{rqm,\,stu} - A_s \Delta_{rqm,\,ptu}\}/\Delta,$$

and so on.

5. *Proof of a theorem given by Cayley** (without proof).—Cayley states that, with an array

$$\left\|\begin{array}{cccc} x_1 & a_1 & \ldots & k_1 \\ x_2 & a_2 & \ldots & k_2 \\ \ldots & \ldots & & \ldots \\ x_n & a_n & \ldots & k_n \end{array}\right\| \quad q\,(<n) \text{ quantities } a,\ b,\ \ldots,\ k,$$

if, of the determinants of order $q+1$ which can be formed from the array, any $n-q$ be chosen, the other determinants of order $q+1$ may be expressed as linear functions of these, the coefficients being independent of $x_1\, x_2 \ldots x_n$.

The identities (2, b) and (3, b) lead to a simple proof of this. To bring Cayley's notation into conformity with that of the present paper, we write $q+1 = N$, $n-q-1 = m$. We then show that every determinant of order N formed from the array may be expressed as a linear function, the coefficients not involving the x's, of

$$\Delta,\ \Delta_{1,\,1},\ \Delta_{2,\,1},\ \ldots,\ \Delta_{m,\,1},$$

and so establish Cayley's theorem.

Put $p = 1$ in (3, b). It is seen that any $\Delta_{t,\,s}$ can be expressed in the required form.

Again, the identity (2, b) applied to an array of N columns, whose rows have suffixes $1, \ldots, p-1, p+1, \ldots, N, N+r, N+s$, leads, after the necessary change of notation, to the identity

$$A_p^{r,\,q}\Delta_{s,\,p} - A_p^{s,\,q}\Delta_{r,\,p} \equiv A_p \Delta_{rs,\,pq}. \qquad (5,\ a)$$

It is thus seen that any $\Delta_{rs,\,pq}$ may be expressed in the required form.

* Muir, *History of the Theory of Determinants*, Vol. 2, p. 15.

By using the process of § 3.2, we obtain a series of analogues of (5.a). The first of these, when we write X for A, expresses $\Delta_{rst,\,pqu}$ as a linear function (coefficients not involving x) of $\Delta_{st,\,pu}$ and $\Delta_{rt,\,pu}$, and so proves that any $\Delta_{rst,\,pqu}$ may be expressed in the required form. The second analogue proves the result for any determinant derived from Δ by 4 row changes, and so on for the whole sequence of analogues until every possible type of determinant derived from Δ by row changes has been dealt with.

6. We find, as a consequence of §§ 1–4, that Bazin's theorem deals with one only of a class of determinants, whose elements are determinants of order n, which may be expressed as a product of determinants of type similar to that of the elements.

For convenience of writing we denote $\Delta_{pq,\,rs}$ by $(pq,\,rs)$: $(pp,\,rs)$ is, of course, zero. Consider

$$D \equiv \begin{vmatrix} (1,1) & (2,1) & \dots & (\theta,1) \\ (1,2) & (2,2) & \dots & (\theta,2) \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ (1,r) & (2,r) & \dots & (\theta,r) \\ (p1,\, s\overline{r+1}) & (p2,\, s\overline{r+1}) & \dots & (p\theta,\, s\overline{r+1}) \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ (u1,\, v\theta) & (u2,\, v\theta) & \dots & (u\theta,\, v\theta) \end{vmatrix},$$

where $\theta \leqslant m$, the array having n columns and $n+m$ rows, and the numbers $s \dots v$, chosen from among the numbers $1 \dots \theta$, are such that when we write

$$\left.\begin{matrix} s \dots v \\ r+1 \dots \theta \end{matrix}\right\}, \qquad (6,\,a)$$

no set of numbers in the first row forms the same cycle of numbers as the corresponding members of the second row,

$$D = (\Delta)^{r-1}\,\Delta''\,\Delta_{p,\,s} \dots \Delta_{u,\,v},$$

where Δ'' is the determinant formed by the $(\theta+1)$-th to the $(\theta+n)$-th rows of the array.

Bazin's theorem is given when $r = \theta$.

If there be cycle sets in (6, a) factors are introduced into the expression for D similar to those given in Theorem II (b).

6.1. The method of proof is similar to that indicated in Muir's

*History of the Theory of Determinants.** It also leads to a neat alternative proof of Theorem B of Prof. E. T. Whittaker's paper quoted in § 3.4; for, if $m = n$ and $p < m$,

$$D \equiv \begin{vmatrix} \Delta_{1,1} & \Delta_{2,1} & \dots & \Delta_{p,1} \\ \Delta_{1,2} & \Delta_{2,2} & \dots & \Delta_{p,2} \\ \dots & \dots & \dots & \\ \Delta_{1,p} & \Delta_{2,p} & \dots & \Delta_{p,p} \end{vmatrix},$$

[p letters $a, b, \dots, e$]

$$= \Sigma \begin{vmatrix} A_1 & B_1 & \dots & E_1 \\ A_2 & B_2 & \dots & E_2 \\ \dots & \dots & \dots & \\ A_p & B_p & \dots & E_p \end{vmatrix} \times \begin{vmatrix} a_{n+1} & b_{n+1} & \dots & e_{n+1} \\ a_{n+2} & b_{n+2} & \dots & e_{n+2} \\ \dots & \dots & \dots & \\ a_{n+p} & b_{n+p} & \dots & e_{n+p} \end{vmatrix} \qquad (\alpha)$$

(the obvious intermediate steps of Tanner's method are omitted)

$$= \Delta^{p-1}\Delta'' \text{ (Bazin's theorem)},$$

where $$\Delta'' = |\, a_{p+1}\ a_{p+2}\ \dots\ a_{p+n}\,|\,.$$

Also, if Δ' denote the determinant formed by the last n rows of the array, and F' denote the co-factor of f in Δ', (α) may be written

$$D(\Delta')^{n-p-1} = \Sigma \begin{vmatrix} A_1 & B_1 & \dots & E_1 \\ A_2 & B_2 & \dots & E_2 \\ \dots & \dots & \dots & \\ A_p & B_p & \dots & E_p \end{vmatrix} \times \begin{vmatrix} F'_{n+p+1} & \dots & K'_{n+p+1} \\ \dots & \dots & \dots \\ F'_{2n} & \dots\dots & K'_{2n} \end{vmatrix},$$

i.e. the determinant formed by taking the first p rows of the adjugate of Δ, and the last $n-p$ of the adjugate of Δ', $= D(\Delta')^{n-p-1}$; which is the theorem in question.

7. In conclusion, we outline the proof of (2, b) when n is even. In this case

$$A^s_r \equiv -A^r_s.$$

Consider an array of $2n$ columns and $2n+1$ rows; and let

$$D \equiv |\, A^r_t A^s_1 \dots A^s_{t-1} A^s_{t+1} \dots A^s_{2n+1}\,| \quad \text{(no suffix } s\text{)}.$$

* P. 195, Vol. 3. The proof is due to H. W. Lloyd Tanner. See also *Educational Times Reprint*, Vol. 28 (1877), p. 41.

Expanding D by its first column, we get

(i) if $t < s$, $\quad D = (-)^{t+s}\Delta_r \Delta_s^{2n-2}$,

(ii) if $t > s$, $\quad D = (-)^{t+s+1}\Delta_r \Delta_s^{2n-2}$.

Also, if D have a first column A_k^r instead of A_t^r, where both k and r occur among the suffixes of the remaining $2n-1$ columns, the same method of expansion gives $D = 0$.

Let α_r^s denote the co-factor of A_r^s in $|A_r^s A_1^s \dots A_{r-1}^s A_{r+1}^s \dots A_{2n+1}^s|$ (no suffix s). Then, by the above results, we have $2n$ equations :—

$$\left.\begin{aligned} \Sigma A_r^t \alpha_1^s &= 0 \\ \dots\dots &= 0 \\ \Sigma A_r^t \alpha_r^s &= (-)^{r+s}\Delta_t \Delta_s^{2n-2} \\ \dots\dots &= 0 \\ \Sigma A_r^t \alpha_t^s &= (-)^{s+t}\Delta_r \Delta_s^{2n-2} \\ \dots\dots &= 0 \end{aligned}\right\} \quad \begin{aligned} &[\Sigma \text{ denotes } A_r^t \alpha_1^s + B_r^t \beta_1^s + \dots] \\ &\\ &\text{when } r < s < t. \end{aligned}$$

Multiply these equations respectively by $\pm A_1^s, \dots, \pm A_r^s, \dots, \pm A_t^s, \dots$, and add ; the signs being so chosen that

$$\begin{aligned} \Sigma \pm A_m^s \alpha_m^s &\equiv \pm A_1^s \alpha_1^s \pm A_2^s \alpha_2^s \pm \dots \\ &\equiv |A_{s+1}^s \dots A_{2n+1}^s A_1^s \dots A_{s-1}^s| \equiv D_s \text{ (say).} \end{aligned}$$

Then $\quad \Sigma \pm A_m^s \beta_m^s \equiv 0$, etc.

The alternative signs are important only when $m = r$ and when $m = t$.

Now, $\quad A_r^s \alpha_r^s$ is a term in the expansion of $(-)^{r+s} D_s$,

and $\quad A_t^s \alpha_t^s$ is a term in the expansion of $(-)^{t+s+1} D_s$.

The signs are therefore $(-)^{r+s}$ when $m = r$, and $(-)^{t+s+1}$ when $m = t$. Thus, the addition of the $2n$ equations gives

$$A_r^t D_s = \Delta_s^{2n-2}(A_r^s \Delta_t - A_t^s \Delta_r),$$

i.e. $\quad A_r^t \Delta_s = A_r^s \Delta_t - A_t^s \Delta_r,$

or $\quad A_r^t \Delta_s + A_t^s \Delta_r + A_s^r \Delta_t = 0.$

The form of the equation shows that the condition $r < s < t$ may be removed without affecting the validity of the result.

ON HARMONIC EQUATIONS AND IN PARTICULAR THE EQUATION OF WAVE MOTION

By E. L. INCE.

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PART I.

Harmonic Equations in General.

1. *A General Solution of the Harmonic Equation.*

A harmonic equation* is to be defined as a linear partial differential equation of the form

$$(L_x - L'_y)\, z = 0, \tag{1}$$

in which the operator L_x involves x and $\partial/\partial x$ only, and L'_y involves y and $\partial/\partial y$ only.

Let $z = Z_0$ be a solution of (1) such that the operators L_x and L'_y are permutable with respect to Z_0. Then

$$Z_1 = \tfrac{1}{2}(L_x + L'_y)\, Z_0 = L_x Z_0 = L'_y Z_0$$

is also a solution of (1), for

$$(L_x - L'_y) Z_1 = (L_x^2 - L'_y L_x) Z_0 = (L_x^2 - L_x L'_y) Z_0 = L_x (L_x - L'_y) Z_0 = 0.$$

The solution Z_1 is new, provided Z_0, regarded as a function of x, does not satisfy the ordinary linear differential equation

$$(L_x + a)\, u = 0, \tag{2}$$

for any value of the constant a. Hence Z_0 may not be of the form

$$\Sigma X_i Y_i,$$

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* This use of the term "harmonic equation" is an extension of that of Darboux, *Théorie des Surfaces*, Vol. 2, p. 206.

in which X_i, a function of x only, satisfies (2) and Y_i, a function of y only, satisfies the corresponding equation in y, viz.

$$(L'_y+a)\,v = 0, \tag{2a}$$

for all values of i, the constant a being the same throughout.

A repetition of this process shows that

$$Z_n = (\tfrac{1}{2}L_x+\tfrac{1}{2}L'_y)^n Z_0 = L_x^n Z_0 = L_y'^n Z_0 \quad (n = 1, 2, 3, \ldots) \tag{3}$$

is a solution of (1). To find a condition ensuring that the solutions of this set are linearly independent, consider Z_0 as developed in terms of its normalised fundamental functions,* thus

$$Z_0 = \sum_m \mu_m u_m(x)\, v_m(y), \tag{3}$$

where $u_m(x)$ and $v_m(y)$ satisfy (2) and (2a) respectively when $a = a_m$. Then, in general,

$$Z_r = \sum_m (-1)^r a_m^r \mu_m u_m(x) v_m(y). \tag{3a}$$

Form the sum

$$A_0 Z_0 + A_1 Z_1 + \ldots + A_n Z_n = \sum_{r=0}^{n} \sum_m (-1)^r A_r a_m^r \mu_m u_m(x) v_m(y),$$

$A_0, A_1, \ldots, A_n$ being arbitrary constants; if it be identically zero, then

$$\sum_{r=0}^{n} (-1)^r A_r a_m^r = 0 \quad (m = 1, 2, 3, \ldots),$$

and therefore all determinants of order $n+1$ contained in the matrix

$$\begin{Vmatrix} 1 & 1 & 1 & \ldots \\ a_1 & a_2 & a_3 & \ldots \\ a_1^2 & a_2^2 & a_3^2 & \ldots \\ \ldots & \ldots & \ldots & \\ a_1^n & a_2^n & a_3^n & \ldots \end{Vmatrix}$$

will be zero.

Hence the solutions $Z_0, Z_1, \ldots, Z_n$ are linearly independent when at least $n+1$ of the numbers a are distinct. Thus, if p be the number of distinct constants in the set $a_1, a_2, a_3, \ldots$, the most general solution of

* Schmidt, *Inaugural Dissertation*, Göttingen, 1905; cf. Lalesco, *Théorie des équations intégrales*, p. 96; Goursat, *Cours d'analyse*, Vol. 3, p. 475. See also a paper by the present author, *Proc. Roy. Soc. Edin.*, Vol. 42, pp. 43–53.

(1) obtainable by this method will contain p arbitrary constants. On the other hand, if the number of distinct constants a be unlimited, then

$$z = F(L_x)\, Z_0 \tag{4}$$

is a solution of (1), where F is an arbitrary polynomial or integral function of its argument.

Further, it is clear that the most general solution of the type considered is of the form (4), where $F(L_x)$ is an arbitrary function of the operator L_x subject to the condition of permutability with the operator L'_y, and is such that the development of $F(L_x)Z_0$ is uniformly convergent with respect to x and y.

In what follows, the number of linearly independent elemental functions Z_n will be supposed unlimited.

2. *Boundary Problems.*

The set of elemental functions $Z_0, Z_1, \ldots, Z_n, \ldots$ may be rendered orthogonal in a domain of (x, y) and serve for the expansion of certain particular types of arbitrary functions. This point will not be insisted upon here, but it will be shown that, in certain conditions, the elementary functions render possible the solution of a class of boundary problems.

Form the function

$$Z(x, y\,;\ t) = \sum_n a_n Z_n(x, y)\, t^n, \tag{5}$$

where the constants a_n are such as to render the series uniformly convergent within a given domain D, but are otherwise arbitrary. Then, if any function $\phi(t)$ be taken, such that the integral

$$z = \int_0^1 Z(x, y\,;\ t)\, \phi(t)\, dt \tag{6}$$

exists, z will be a solution of the harmonic equation.

Let a boundary condition be expressed in the form

$$z = f(\theta), \quad x = \xi(\theta), \quad y = \eta(\theta) \quad (0 \leqslant \theta \leqslant 1),$$

where (x, y) is in D. Then the problem reduces to determining $\phi(t)$ such that this condition is satisfied. In other words, $\phi(t)$ is to be obtained as the solution of the Fredholm equation of the first kind

$$f(\theta) = \int_0^1 Z(\xi, \eta\,;\ t)\, \phi(t)\, dt. \tag{7}$$

Let the development of the nucleus of this equation be

$$Z(\xi, \eta; t) = \sum_n \frac{\Theta_n T_n}{\lambda_n},$$

where the functions Θ_n and T_n are normalised orthogonal functions of θ and t respectively, and in the first place let the set Θ_n be assumed to be closed. Then, if the given function $f(\theta)$ be expressed in the form

$$f(\theta) = \sum_n c_n \Theta_n,$$

the equation admits of a unique solution

$$\phi(t) = \sum_n \lambda_n c_n T_n, \tag{8}$$

provided that the series $\sum_n \lambda_n^2 c_n^2$ is convergent.*

The solution satisfying the boundary conditions then is

$$z = \sum_n b_n Z_n(x, y),$$

where $$\cdots \qquad b_n = a_n \int_0^1 t^n \phi(t)\, dt.$$

This solution is unique, *i.e.* independent of the choice of the constants a_n, for if a distinct development be

$$z = \sum_n b'_n Z_n(x, y),$$

then, on the boundary,

$$f(\theta) = \sum_n b_n Z_n(\xi, \eta) = \sum b'_n Z_n(\xi, \eta),$$

and hence $$\sum_n (b'_n - b_n) Z_n(\xi, \eta) = 0,$$

and $$b'_n = b_n,$$

for all values of n.

If the set Θ_n be closed, but the series $\sum_n \lambda_n^2 c_n^2$ be divergent, the problem does not admit of an exact solution; nevertheless if

$$\phi_n(t) = \sum_{r=1}^{n} \lambda_r c_r T_r,$$

$$b_n = a_n \int_0^1 t^n \phi_n(t)\, dt,$$

and $$z_n = \sum_{r\ 1}^{n} b_r Z_r(x, y),$$

* The theorem is due to Picard, *Comptes Rendus*, Vol. 149 (1909); *Rendiconti*, Vol. 29 (1910). The convergence condition is a consequence of the Riesz-Fischer theorem. See Lalesco and Goursat, *loc. cit.*

then, on the boundary, if ϵ be arbitrarily small

$$\int_0^1 [f(\theta)-z_n]^2\, d\theta < \epsilon,$$

for sufficiently large values of n.* Thus a solution may be approached to any desired degree of average approximation.

If, on the other hand, the set Θ_n be not closed, the problem cannot necessarily be solved without modifying the boundary condition.† For, if the function z be formed as above with the coefficients b_n, then equation (7) becomes

$$f(\theta)+g(\theta) = \int_0^1 Z(\xi, \eta;\ t)\, \phi(t)\, dt,$$

where $g(\theta)$ is a definite function orthogonal to the set Θ_n. The boundary condition $y = f(\theta)$ would have to be replaced by $z = f(\theta)+g(\theta)$. If, however, the set Θ_n be *conditionally closed*, that is to say, closed to all functions having a specified property P, and if $f(\theta)$ has the property P, then $g(\theta) = 0$, and the problem is soluble exactly or by approximation according as the Riesz-Fischer condition is or is not satisfied.

PART II.

The Equation of Wave Motion.

3. As an application of the method of solution given in Part I, the equation of two-dimensional wave motion as reduced to the form

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}+k^2 z = 0 \qquad (9)$$

will be taken. The simple solutions

$$z = e^{ikx} \quad \text{and} \quad z = e^{iky}$$

will be utilised as the bases upon which to build up sequences of new solutions. In its present form the equation is unsuited for the application of the method, since nothing new is obtained by operating with such an operator as *e.g.* $\partial^2/\partial x^2+k^2$. In fact, the two solutions taken are of the form excluded in § 1. It is therefore necessary to transform the equation so that the method becomes practicable, and then subsequently to return to the original variables x and y. Three examples will be given; the first, a transformation to parabolic coordinates, does not destroy the

* Goursat, Vol. 3, p. 479.

† *Ibid.*

symmetry of the wave equation, the second, to polar coordinates, renders the equation asymmetrical, the third, to elliptic coordinates, is again symmetrical. It is shown that the two latter are effectively equivalent to the first.

4. *The Equation in Parabolic Coordinates.*

The equation (9) is transformed by the substitution

$$x+iy = (\xi+i\eta)^2,$$

and becomes

$$\frac{\partial^2 z}{\partial \xi^2}+\frac{\partial^2 z}{\partial \eta^2}+4k^2(\xi^2+\eta^2)\,z = 0.$$

For simplicity, and in order to avoid, so far as possible, the use of the imaginary unit i in the following work, the substitution

$$s = \xi\sqrt{(2k)}, \quad t = i\eta\sqrt{(2k)}$$

will be made, giving the equation the form

$$\frac{\partial^2 z}{\partial s^2}-\frac{\partial^2 z}{\partial t^2}+(s^2-t^2)z = 0. \tag{10}$$

The solutions become $z = e^{\frac{1}{2}(s^2+t^2)i}$ and $z = e^{st}$; the former is of the excluded type, the latter will be taken as the initial solution Z_0. The next solution may be taken as

$$Z_1 = \tfrac{1}{2}\left(\frac{\partial^2}{\partial s^2}+s^2\right) Z_0 = \tfrac{1}{2}e^{st}(s^2+t^2),$$

and in general it is convenient to define Z_n by the relation

$$Z_{n+1} = \tfrac{1}{2}\left(\frac{\partial^2}{\partial s^2}+s^2\right) Z_n - \frac{n(2n-1)}{2}\, Z_{n-1}. \tag{11}$$

Z_n is thus evidently of the form $e^{st}W_n$, where W_n is a polynomial of degree $2n$, symmetrical in s and t, satisfying the harmonic equation

$$\frac{\partial^2 w}{\partial s^2}-\frac{\partial^2 w}{\partial t^2}+2t\frac{\partial w}{\partial s}-2s\frac{\partial w}{\partial t} = 0. \tag{12}$$

A considerable simplification is obtained by writing

$$u = \tfrac{1}{2}(s^2+t^2), \quad v = st,$$

so that

$$W_0 = 1, \quad W_1 = u, \quad W_2 = u^2+v,$$
$$W_3 = u^3+3uv, \quad W_4 = u^4+6u^2v+3v(v-1),$$
$$\cdots \quad \cdots \quad \cdots$$

and $$\frac{\partial^2 w}{\partial u^2} - \frac{\partial^2 w}{\partial v^2} - 2\frac{\partial w}{\partial v} = 0. \tag{13}$$

In general, $$W_n = \sum_r u^{n-2r} P_r,$$

where P_r is a polynomial in v, of degree r, satisfying the relation

$$P_r'' + 2P_r = (n-2r+2)(n-2r+1)\, P_{r-1}$$

with $P_0 = 1$. Thus

$$P_1 = \frac{n(n-1)}{2} v, \quad P_2 = \frac{n(n-1)(n-2)(n-3)}{2^2 . 2!} v(v-1),$$

$$P_3 = \frac{n(n-1)\ldots(n-5)}{2^3 . 3!} v(v^2-3v+3),$$

.....

and, in general,

$$P_r = \frac{n(n-1)\ldots(n-2r+1)}{2^r . r!} \sum_{p=0}^{r-1} (-1)^p \frac{(r+p-1)(r+p-2)\ldots(r-p)}{2^p . p!} v^{r-p}.$$

The general form of Z_n can thus be written down.

The transformation back to x and y is now immediate, for $u = kx$, $v = iky$. A second and distinct set of solutions is obtained by changing the sign of y, η, t and v throughout. Furthermore, x and y may evidently be permuted.

The general solution of (10) may thus be written in the form

$$z = f\left(\frac{\partial^2}{\partial s^2} + s^2\right) e^{st} + g\left(\frac{\partial^2}{\partial s^2} + s^2\right) e^{-st},$$

which is particularly applicable to problems in which the boundary is of the form $t = \text{const.}$, *i.e.* of the form $y^2 = 4a(x+a)$.

5. *The Solution as a Definite Integral.*

As in § 2, consider the series

$$Z(s, t;\ \xi) = Z_0 + Z_1\xi + Z_2\frac{\xi^2}{2!} + \ldots + Z_n\frac{\xi^n}{n!} + \ldots$$

$$= e^v \left\{1 + u\xi + (u^2+v)\frac{\xi^2}{2!} + (u^3+3uv)\frac{\xi^3}{3!} + \ldots\right\}.$$

This may be written as

$$e^{u\zeta+v}\left\{1+v\frac{\zeta^2}{2}+v(v-1)\frac{\zeta^4}{2^2.2!}+\ldots+V_n\frac{\zeta^{2n}}{2^n.n!}+\ldots\right\},$$

where $$V_n=\sum_{r=0}^{n-1}(-1)^r(n+r-1)(n+r-2)\ldots(n-r)\,v^{n-r}.$$

The coefficient of $e^{u\zeta+v}$ is independent of u; let it be denoted by V. By substitution in (13) it is found that

$$V''+2V'-\zeta^2V=0,$$

whence it follows that $V=\exp\{-v+v\sqrt{(1+\zeta^2)}\}$.

Hence Z_n as defined in § 4 is the coefficient of $\zeta^n/n!$ in the expansion of $\exp\{u\zeta+v\sqrt{(1+\zeta^2)}\}$, that is to say

$$Z_n=\left[\frac{d^n}{d\zeta^n}e^{u\zeta+v\sqrt{(1+\zeta^2)}}\right]_{\zeta=0}, \qquad (14)$$

and the corresponding member of the second set of solutions is obtained by changing v into $-v$.

A general class of solutions of the wave equation is therefore included in the expression

$$z=\int_\gamma\exp\{u\zeta+v\sqrt{(1+\zeta^2)}\}\,\phi(\zeta)\,d\zeta. \qquad (15)$$

If rectangular coordinates be re-introduced and $\zeta=i\cos\theta$, there appears the known solution

$$z=\int_0^{2\pi}e^{ik(x\cos\theta+y\sin\theta)}\,\psi(\theta)\,d\theta.$$

The boundary problem for the parabola may now be taken up; since the contour γ is not necessarily such as to render $\sqrt{(1+\zeta^2)}$ uniform, the problem will, in general, require the two sets of functions $Z_n(s,\,t)$ and $Z_n(s,\,-t)$ for its solution. There is every reason to believe that these two sets taken together form a closed system, but a direct proof would be a matter of some difficulty.

6. *The Equation in Polar Coordinates.*

The transformation $$x+iy=re^{i\theta},$$

transforms (9) into

$$r^2\frac{\partial^2z}{\partial r^2}+r\frac{\partial z}{\partial r}+\frac{\partial^2z}{\partial\theta^2}+k^2r^2z=0. \qquad (16)$$

This equation is particularly easy to deal with, for the operator $\partial/\partial\theta$ is

permutable with the operators of the equation. Thus, if

$$K_0 = e^{iky} = e^{ikr\sin\theta},$$

then $K_1 = \dfrac{\partial}{\partial\theta} K_0 = ikr\cos\theta\, e^{ikr\sin\theta} = ikx\, e^{iky},$

$$K_2 = \frac{\partial}{\partial\theta} K_1 = -(k^2r^2\cos^2\theta + ikr\sin\theta)\, e^{ikr\sin\theta} = -(k^2x^2 + iky)\, e^{iky},$$

not differing essentially from the Z_1 and Z_2 of § 4. In fact, if

$$K_n = \frac{\partial^n}{\partial\theta^n} K_0,$$

K_n is a linear combination of the functions $Z_0, Z_1, \ldots, Z_n$ of § 4. Consequently nothing new is gained from the transformation to polar coordinates, except a simple form of general solution, viz.

$$z = f\left(\frac{\partial}{\partial\theta}\right) e^{ikr\sin\theta} + g\left(\frac{\partial}{\partial\theta}\right) e^{ikr\cos\theta}. \tag{17}$$

7. *The Equation in Elliptic Coordinates.*

In coordinates ξ, η defined by

$$x + iy = \cosh(\xi + i\eta),$$

equation (9) becomes

$$\frac{\partial^2 z}{\partial\xi^2} + \frac{\partial^2 z}{\partial\eta^2} + k^2(\cosh^2\xi - \cos^2\eta)\, z = 0. \tag{18}$$

Here it is found that if $K_0 = \exp(ik\sinh\xi\sin\eta)$,

then $K_2 = \left(\dfrac{\partial^2}{\partial\xi^2} + k^2\cosh^2\xi\right) K_0 = (k^2\cosh^2\xi\cos^2\eta + ik\sinh\xi\sin\eta)\, K_0$

$= (k^2x^2 + iky)\, e^{iky},$

and, in general, $K_{2n} = \left(\dfrac{\partial^2}{\partial\xi^2} + k^2\cosh^2\xi\right)^n K_0$

does not differ essentially from Z_{2n} of § 4. To find K_{2n+1} to correspond to Z_{2n+1}, let

$$K_1 = kx\, e^{iky} = k\cosh\xi\cos\eta\exp(ik\sinh\xi\sin\eta).$$

then $K_{2n+1} = \left(\dfrac{\partial^2}{\partial\xi^2} + k^2\cosh^2\xi\right)^n K_1,$

and the correspondence is complete.

SOME PROBLEMS OF "PARTITIO NUMERORUM" (V): A FURTHER CONTRIBUTION TO THE STUDY OF GOLDBACH'S PROBLEM

By G. H. Hardy *and* J. E. Littlewood.

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1. *Introduction.*

1.1. This paper is a sequel to the third of the series.* We proved there that if a certain hypothesis, which we called Hypothesis R*, and which is a natural generalisation of the hypothesis of Riemann concerning the zeros of $\zeta(s)$, is true, then *every large odd number is the sum of three primes*; and we determined an asymptotic formula for the number of such representations. But our analysis broke down when the number of primes considered is less than three, and we were unable to make any rigorous contribution to the study of Goldbach's Problem itself.

In the present paper we prove (still, naturally, on the assumption of Hypothesis R*) that *almost all even numbers are sums of two primes*, that is to say that the number of numbers less than n, for which "Goldbach's Theorem" is false, is $o(n)$ when n is large. We prove, in fact, considerably more, but this is the essential result.

1.2. We recall the terminology of P. N. 3, in so far as it is relevant here.† Our fundamental hypothesis‡ is

Hypothesis R*.—*Every zero of every Dirichlet's function*

$$L(s) = \Sigma \frac{\chi(m)}{m^s},$$

* G. H. Hardy and J. E. Littlewood, "Some problems of 'Partitio Numerorum' (III): On the expression of a number as a sum of primes", *Acta Mathematica*, Vol. 44 (1922), pp. 1–70. We refer to this memoir as P. N. 3. The analysis which it contains has been considerably simplified by Landau ["Zur Additive Primzahltheorie", *Rend. di Palermo* Vol. 46 (1922), pp. 349–356].

† We have modified the notation of P. N. 3, since we prefer now to denote a typical prime, as is usual, by p. What were there p, q are now h, k; what was there h is now $\phi(k)$; and what was there ϕ is now ψ.

‡ In P. N. 3 we did not actually use Hypothesis R*, but a slightly less drastic hypothesis (Hypothesis R) which proved sufficient for our purpose.

where $\chi(m)$ is a character to modulus k, has a real part less than or equal to $\frac{1}{2}$.

We write p for a prime,

$$f(x) = \Sigma \log p\, x^p \quad (|x| < 1),$$

$$\big(f(x)\big)^2 = \Sigma \nu_2(m) x^m,$$

$$\nu(m) = \nu_2(m) = \sum_{p+p'=m} \log p \log p',$$

$$F(x) = \Sigma \Omega(m) x^m = \Sigma m S(m) x^m,$$

where

$$S(m) = \sum_k A_k(m) = \sum_k \left(\frac{\mu(k)}{\phi(k)}\right)^2 c_k(-m),$$

$$c_k(-m) = \sum_h e^{-2hm\pi i/k} = \sum_h e_k(-hm).$$

Here $m = 1, 2, 3, \ldots$, $k = 1, 2, 3, \ldots$, and h is positive, less than k, and prime to k, except when $k = 1$, when $h = 0$, $c_1(-m) = 1$ and $A_1 = 1$. $S(m)$ is the "singular series", and its sum is given by*

$$S(m) = 0 \quad (m \text{ odd}),$$

$$S(m) = 2C \Pi \left(\frac{P-1}{P-2}\right) \quad (m \text{ even}),$$

where P is an odd prime divisor of m, and

$$C = \prod_{p \geqslant 3} \left(1 - \frac{1}{(p-1)^2}\right).$$

We use the machinery of the "Farey dissection", explained in our memoirs on Waring's Problem. The circle Γ to which it is applied is defined by

$$|x| = e^{-H} = e^{-1/n},$$

and the "Farey arcs" are denoted by $\xi_{h,k}$ or ξ. The dissection is of order

$$N = [\sqrt{n}],$$

and there is, in this problem, no distinction of "major" and "minor" arcs.

When we are studying the arc $\xi_{h,k}$, we write

$$x = e_k(h) e^{-Y}, \quad Y = \frac{1}{n} + i\theta, \quad -\theta_0' \leqslant \theta \leqslant \theta_0,$$

* For the formal summation of the singular series see P. N. 3, pp. 26–29.

where θ_0 and θ_0' lie between $2\pi/kN$ and π/kN. The function which affords an approximation to $f(x)$ on $\xi_{h,k}$ is

$$\psi = \psi_{h,k} = \frac{\mu(k)}{\phi(k)} \frac{1}{Y}.$$

We write further

$$F_{h,k} = \left(\frac{\mu(k)}{\phi(k)}\right)^2 \Sigma m\Big(xe_k(-h)\Big)^m = \left(\frac{\mu(k)}{\phi(k)}\right)^2 g\Big(xe_k(-h)\Big),$$

where

$$g(z) = \Sigma m z^m = \frac{z}{(1-z)^2},$$

so that

$$F = F(x) = \sum_m mx^m \sum_k A_k(m) = \sum_m mx^m \sum_k \left(\frac{\mu(k)}{\phi(k)}\right)^2 \sum_h e_k(-hm)$$

$$= \sum_{k,h} \left(\frac{\mu(k)}{\phi(k)}\right)^2 \sum_m m\left(xe_k(-h)\right)^m = \sum_{k,h} F_{h,k}.$$

Finally, we write

$$S(m) = S_1(m) + S_2(m),$$

$$F = \Sigma mS(m)\, x^m = \Sigma mS_1(m)\, x^m + \Sigma mS_2(m)\, x^m,$$

where the suffix 1 limits k to the range $k \leqslant \nu$, and the suffix 2 to the range $k > \nu$.* Then

$$F_1 = \sum_{k \leqslant \nu,\ h} F_{h,k}, \qquad F_2 = \sum_{k > \nu,\ h} F_{h,k}.$$

2. *Preliminary lemmas.*

2.1. In all that follows A denotes an absolute constant, and O's and o's are uniform in all parameters that occur, except ϵ.

LEMMA 1.—*On* $\xi_{h,k}$,

$$\Psi = f - \psi = f(x) - \psi_{h,k} = O\left(n^{\frac{3}{4}}(\log n)^A\right).$$

See P. N. 3, p. 23, equation (3.124), and take $\Theta = \frac{1}{2}$.

LEMMA 2.—*If* $x = |x|e^{i\omega} = e^{-H+i\omega}$, *so that* x *lies on* Γ, *then*

$$\int_0^{2\pi} |f(x)|^2\, d\omega = O(n \log n).$$

See P. N. 3, p. 24.

* ν is an integer as yet undetermined. We ultimately take $\nu = [n^{\frac{1}{4}}]$.

LEMMA 3.—*We have*

$$\Sigma \int_{\xi} |\psi|^2 \, d\theta = O\left(n (\log n)^A\right).$$

From P. N. 3, p. 23 (bottom), with $r = 3$, we have

$$\int_{\xi} |\psi|^2 \, d\theta < A \frac{n}{\left(\phi(k)\right)^2}.$$

Hence

$$\Sigma \int_{\xi} |\psi|^2 \, d\theta < An \sum_{k,\, h} \frac{1}{\left(\phi(k)\right)^2} < An \sum_{k} \frac{1}{\phi(k)} < An \sum_{k} \frac{(\log k)^A}{k} < An (\log n)^A.$$

LEMMA 4.—*We have*

$$\Sigma \int_{\xi} |f^2 - \psi^2|^2 \, d\theta = O(n^{\frac{5}{2}+\epsilon})$$

for every positive ϵ.

For

$$\Sigma \int_{\xi} |f^2 - \psi^2|^2 \, d\theta < A \Sigma \int_{\xi} |f - \psi|^2 |f|^2 \, d\theta + A \Sigma \int_{\xi} |f - \psi|^2 |\psi|^2 \, d\theta$$
$$= A\Sigma_1 + A\Sigma_2,$$

say. But

$$\Sigma_1 < An^3 (\log n)^A \Sigma \int_{\xi} |f|^2 \, d\theta = O\left(n^{\frac{5}{2}} (\log n)^A\right) = O(n^{\frac{5}{2}+\epsilon}),$$

by Lemmas 1 and 2; and

$$\Sigma_2 < An^3 (\log n)^A \Sigma \int_{\xi} |\psi|^2 \, d\theta = O\left(n^{\frac{5}{2}} (\log n)^A\right) = O(n^{\frac{5}{2}+\epsilon}),$$

by Lemmas 1 and 3; whence the conclusion.

2.2. LEMMA 5. — *If* $k > 1$, $|\mu(k)| > 0$,* *and* $k = k_1 k_2$, *where* $(k_2, m) = 1$, *then*

$$|A_k(m)| < \frac{A (\log k)^A}{k_1 k_2^2}.$$

For $$|A_k(m)| = \frac{|c_k(-m)|}{\left(\phi(k)\right)^2} < \frac{A (\log k)^A}{k^2} |c_k(-m)|.$$

* So that k has no repeated factor (is *quadratfrei*). Otherwise $A_k(m) = 0$.

But, if $k = \Pi p_s$, where the p's are primes, then*

$$|c_k(-m)| = \prod_{p_s | m} (p_s - 1) < \prod_{p_s | m} p_s = k_1,$$

whence the result of the lemma.

LEMMA 6.—*If* $m > 1$, $\nu > 1$, *then*

$$|S_2(m)| < A (\log m)^A \, d(m) \, \frac{(\log \nu)^A}{\nu},$$

where $d(m)$ *is the number of divisors of* m.

By Lemma 5,

$$|S_2(m)| \leqslant \sum_{k > \nu} |A_k(m)| < A \Sigma \frac{(\log k)^A}{k_1 k_2^2},$$

the summation extending over those values of k for which $k > \nu$, $|\mu(k)| > 0$.

Suppose that $\pi_1, \pi_2, \ldots, \pi_t$ are the prime divisors of m, and that $\delta = \Pi \pi_i$ is a typical *quadratfrei* divisor of m; and consider those terms of Σ for which

$$k > \nu, \quad |\mu(k)| > 0, \quad k = \delta k_2, \quad (k_2, m) = 1,$$

δ is fixed and k_2 varies. The contribution of these terms is less than

$$\sum_{\delta k_2 > \nu} \frac{(\log \delta k_2)^A}{\delta k_2^2} < \frac{A (\log m)^A}{\delta} \sum_{k_2 > \nu/\delta} \frac{(\log k_2)^A}{k_2^2}$$

$$< \frac{A (\log m)^A}{\delta} \, \frac{A\delta}{\nu} \left(\log \left(\frac{\nu}{\delta} + A \right) \right)^A$$

$$< A (\log m)^A \, \frac{(\log \nu)^A}{\nu}.$$

Hence

$$\Sigma \frac{(\log k)^A}{k_1 k_2^2} < A (\log m)^A \, \frac{(\log \nu)^A}{\nu} \sum_{\delta | m} 1,$$

which proves the lemma.

2.3. LEMMA 7.—*If* $x = |x| e^{i\omega}$, *as in Lemma* 2, *then*

$$\int_0^{2\pi} |F_2(x)|^2 \, d\omega = O \left(n^3 (\log n)^A \, \frac{(\log \nu)^A}{\nu^2} \right).$$

* See P. N. 3, p. 28 (bottom).

For $F_2(x) = \Sigma m S_2(m) x^m$, and so

$$(2.31) \quad \int_0^{2\pi} |F_2(x)|^2 \, d\omega = A \Sigma \left(m S_2(m) \right)^2 |x|^{2m}$$

$$= O\left(\frac{(\log \nu)^A}{\nu^2} \Sigma m^2 (\log m)^A \left(d(m) \right)^2 |x|^{2m} \right),$$

by Lemma 6. But

$$\sum_{m<n} \left(d(m) \right)^2 = O\left(n (\log n)^A \right),*$$

$$\sum_{m<n} m^2 (\log m)^A \left(d(m) \right)^2 = O\left(n^3 (\log n)^A \right),$$

and so $\left(\text{since } |x| < 1 - \frac{A}{n} \right)$

$$(2.32) \quad \Sigma m^2 (\log m)^A \left(d(m) \right)^2 |x|^{2m} = O\left(n^3 (\log n)^A \right).$$

The lemma follows from (2.31) and (2.32).

LEMMA 8.—*We have*

$$\Sigma \int_\xi |F_{h,k} - \psi_{h,k}^2|^2 \, d\theta = O(1).$$

This is trivial, since

$$F_{h,k} - \psi_{h,k}^2 = \left(\frac{\mu(k)}{\phi(k)} \right)^2 \left(g\left(x e_k(-h) \right) - \frac{1}{Y^2} \right)$$

$$= \left(\frac{\mu(k)}{\phi(k)} \right)^2 \left\{ \frac{x e_k(-h)}{\left(1 - x e_k(-h) \right)^2} - \frac{1}{\left(\log \frac{1}{x e_k(-h)} \right)^2} \right\} = O(1).$$

LEMMA 9.—*If ξ_2 denotes a typical Farey arc for which $k > \nu$, then*

$$\Sigma \int_{\xi_2} |\psi|^4 \, d\theta = O\left(n^3 \frac{(\log \nu)^A}{\nu^2} \right).$$

We have

$$\int_\xi |\psi|^4 \, d\theta \leqslant \frac{1}{\left(\phi(k) \right)^4} \int_\xi \frac{d\theta}{|Y|^4} = \frac{1}{\left(\phi(k) \right)^4} \int_\xi \frac{d\theta}{(n^{-2} + \theta^2)^2}$$

* In fact the sum is $O\left(n (\log n)^3 \right)$. See S. Ramanujan, "Some formulæ in the analytic theory of numbers", *Messenger of Mathematics*, Vol. 45 (1916), pp. 81–84.

$$< \frac{A(\log k)^A}{k^4} \int_{-\infty}^{\infty} \frac{d\theta}{(n^{-2}+\theta^2)^2} < An^3 \frac{(\log k)^A}{k^4}.$$

Hence $$\Sigma \int_{\xi_2} |\psi|^4 \, d\theta < An^3 \sum_{k>\nu} \frac{(\log k)^A}{k^3} < An^3 \frac{(\log \nu)^A}{\nu^2}.$$

2.4. LEMMA 10.—*Suppose that x lies on $\xi_{h,k}$ and that $\xi_{H,K}$ is a Farey arc different from $\xi_{h,k}$. Then*

$$|F_{H,K}| < \frac{An(\log K)^A}{(hK-kH)^2}.$$

We have $$x = |x| e_k(h) e^{-i\theta} = |x| e_K(H) e^{-i\omega},$$

where $$\omega = 2\pi\left(\frac{H}{K} - \frac{h}{k}\right) + \theta,$$

and

$$|F_{H,K}| = \left(\frac{\mu(K)}{\phi(K)}\right)^2 |g\left(xe_K(-H)\right)| = \left(\frac{\mu(K)}{\phi(K)}\right)^2 \left| g(|x| e^{-i\omega}) \right| < \frac{A(\log K)^A}{K^2 |\omega|^2}.$$

Suppose first that the arcs $\xi_{h,k}$, $\xi_{H,K}$ are not adjacent in the dissection. Then

$$\left| 2\pi\left(\frac{h}{k} - \frac{H}{K}\right) \right| \geqslant \frac{4\pi}{kK} \geqslant 2\frac{2\pi}{kN} \geqslant 2|\theta|,$$

and so $$|\omega| > A\left|\frac{h}{k} - \frac{H}{K}\right| = A\frac{|hK-kH|}{kK},$$

$$|F_{H,K}| < \frac{A(\log K)^A}{K^2} \frac{k^2K^2}{(hK-kH)^2} < \frac{An(\log K)^A}{(hK-kH)^2},$$

since $k^2 \leqslant N^2 \leqslant n$. The argument fails when the arcs are adjacent. In this case $|hK-kH| = 1$. As x is outside the arc $\xi_{H,K}$, $|\omega| > A/KN$; and so

$$|F_{H,K}| < A\frac{(\log K)^A}{K^2} K^2N^2 < An(\log K)^A = \frac{An(\log K)^A}{(hK-kH)^2}.$$

The result of the lemma is therefore true in any case.

2.5. LEMMA 11.—*Let*

$$G = \bar{\Sigma} |F_{H,K}|,$$

where the sign $\bar{\Sigma}$ implies a summation over those pairs of values (H, K),

* So that G is a function of h and k, as well as of x and ν.

distinct from (h, k), *for which* $K \leqslant \nu$.* *Then*

$$\Sigma \int_{\xi_{h,k}} G^2 d\theta = O\left(n^{\frac{3}{2}} (\log n)^A \nu^4\right).$$

Since G is a sum of less than ν^2 terms, we have

$$G^2 < A\nu^2 \bar{\Sigma} |F_{H,K}|^2 < An^2\nu^2 \bar{\Sigma} \frac{(\log K)^A}{(hK-kH)^4},$$

by Lemma 10. Hence

$$\begin{aligned}\Sigma \int_{\xi_{h,k}} G^2 d\theta &< An^2\nu^2 \sum_{h,k} \frac{1}{kN} \bar{\sum_{H,K}} \frac{(\log K)^A}{(hK-kH)^4} \\ &< An^{\frac{3}{2}} (\log n)^A \nu^2 \sum_{h,k} \frac{1}{k} \bar{\sum_{H,K}} \frac{1}{(hK-kH)^4} \\ &< An^{\frac{3}{2}} (\log n)^A \nu^2 \sum_{H,K} \sum_{h,k} \frac{1}{k(hK-kH)^4},\end{aligned}$$

where now the inner summation is defined by

$$k \leqslant N, \quad (h, k) \neq (H, K),$$

and the outer summation by $K \leqslant \nu$. But

$$\begin{aligned}\sum_{h,k} \frac{1}{k(hK-kH)^4} &= \frac{1}{K^4} \sum_k \frac{1}{k} \sum_h \frac{1}{\left(h-\frac{kH}{K}\right)^4} \\ &< \frac{1}{K^4} \sum_k \frac{A}{k} \left(\frac{1}{\left(\frac{1}{K}\right)^4} + \frac{1}{\left(1+\frac{1}{K}\right)^4} + \frac{1}{\left(2+\frac{1}{K}\right)^4} + \dots \right) \\ &< \frac{A}{K^4} \sum_k \frac{K^4}{k} < A \sum_k \frac{1}{k} < A \log n;\end{aligned}$$

and so

$$\sum_{H,K} \sum_{h,k} \frac{1}{k(hK-kH)^4} < A\nu^2 \log n,$$

$$\Sigma \int_{\xi_{h,k}} G^2 d\theta < An^{\frac{3}{2}} (\log n)^A \nu^2 . A\nu^2 \log n < An^{\frac{3}{2}} (\log n)^A \nu^4,$$

the result of the lemma.

3. *The main theorems.*

3.1. THEOREM A.—*If Hypothesis* R* *is true, then*

$$\sum_1^n \left(\nu(m) - \Omega(m)\right)^2 = O(n^{\frac{3}{2}+\epsilon})$$

for every positive ϵ.

It is sufficient* to prove that

$$\Sigma \left(\nu(m)-\Omega(m)\right)^2 |x|^{2m} = O(n^{\frac{5}{3}+\epsilon}),$$

when $|x| = e^{-H}$, or that

$$\int_0^{2\pi} |f^2-F|^2\, d\omega = O(n^{\frac{5}{3}+\epsilon}),$$

when $$x = |x|\, e^{i\omega} = e^{-H+i\omega}.$$

Now $$\int_0^{2\pi} |f^2-F|^2\, d\omega = \Sigma \int_\xi |f^2-F|^2\, d\theta$$

$$< A\Sigma \int_\xi |f^2-\psi^2|^2\, d\theta + A\Sigma \int_\xi |\psi^2-F|^2\, d\theta;$$

and $$\Sigma \int_\xi |f^2-\psi^2|^2\, d\theta = O(n^{\frac{5}{3}+\epsilon}),$$

by Lemma 4. It is therefore sufficient to prove that

$$\Sigma \int_\xi |\psi^2-F|^2\, d\theta = O(n^{\frac{5}{3}+\epsilon}). \tag{3.11}$$

3.2. Since $F = F_1+F_2$, we have

$$\Sigma \int_\xi |\psi^2-F|^2\, d\theta < A\Sigma \int_\xi |\psi^2-F_1|^2\, d\theta + A\Sigma \int_\xi |F_2|^2\, d\theta; \tag{3.21}$$

and

$$\Sigma \int_\xi |F_2|^2\, d\theta < An^3(\log n)^A \frac{(\log \nu)^A}{\nu^2}, \tag{3.22}$$

by Lemma 7. Next

$$F_1 = \sum_{H,K \leqslant \nu} F_{H,K} = F_{h,k} + \bar{\Sigma} F_{H,K}$$

or $$F_1 = \bar{\Sigma} F_{H,K},$$

according as $k \leqslant \nu$ or $k > \nu$, the range of summation indicated by $\bar{\Sigma}$ being defined as in Lemma 11. If $k \leqslant \nu$, then, we have

$$|\psi^2-F_1|^2 < A\,|\psi^2-F_{h,k}|^2 + A\,|\bar{\Sigma} F_{H,K}|^2 < A\,|\psi^2-F_{h,k}|^2 + AG^2;$$

---

* Since $$\sum_1^n a_m^2 \leqslant e^{2nH} \sum_1^n a_m^2 e^{-2mH} \leqslant A \sum_1^\infty a_m^2 |x|^{2m}.$$

while if $k > \nu$ we have

$$|\psi^2 - F_1|^2 < A|\psi|^4 + A|F_1|^2 < A|\psi|^4 + A|\bar{\Sigma} F_{H,K}|^2 < A|\psi|^4 + AG^2.$$

Hence

$$\Sigma\int_\xi |\psi^2 - F_1|^2\,d\theta < A\sum_{k\leqslant\nu}\int_\xi |\psi^2 - F_{h,k}|^2\,d\theta + A\sum_{k>\nu}\int_\xi |\psi|^4\,d\theta + A\Sigma\int_\xi G^2\,d\theta$$

$$< A + A\frac{n^3(\log\nu)^A}{\nu^2} + An^3(\log n)^A\nu^4, \tag{3.23}$$

by Lemmas 8, 9, and 11. From (3.21), (3.22), and (3.23), we conclude that

$$\Sigma\int_\xi |\psi^2 - F|^2\,d\theta < An^3(\log n)^A\frac{(\log\nu)^A}{\nu^2} + An^3(\log n)^A\nu^4.$$

Taking $\nu = [n^{\frac{1}{4}}]$, we obtain (3.11), and complete the proof of the theorem.

3.2. THEOREM B.—*If Hypothesis* R* *is true, then the number of even numbers less than* n, *for which Goldbach's Theorem is false, is* $O(n^{\frac{1}{2}+\epsilon})$ *for every positive* ϵ.

We have $\Omega(m) = 0$, if m is odd, and

$$\Omega(m) = Am\,\Pi\left(\frac{P-1}{P-2}\right) > Am,$$

if m is even; and so

$$\big(\nu(m) - \Omega(m)\big)^2 = \big(\Omega(m)\big)^2 > Am^2,$$

for every even number m for which Goldbach's Theorem is false. The number of such numbers between $\frac{1}{2}n$ and n (inclusive of the limits) is therefore less than

$$B(\epsilon)n^{\frac{1}{2}+\epsilon},$$

where $B(\epsilon)$ is a function of ϵ only; and the number of such numbers less than n is less than

$$B(\epsilon)n^{\frac{1}{2}+\epsilon}\left(1 + (\tfrac{1}{2})^{\frac{1}{2}+\epsilon} + (\tfrac{1}{4})^{\frac{1}{2}+\epsilon} + \ldots\right) < B(\epsilon)n^{\frac{1}{2}+\epsilon}.$$

4. *Conclusion.*

4.1. It may be observed that the main conclusion of P. N. 3, viz. that *every large odd number is* (on Hypothesis R*) *the sum of three primes*, is

an immediate corollary of Theorem B. For, if n is a large odd number which is not the sum of three primes, then none of the numbers

$$n-p \quad (2 < p < n)$$

can be the sum of two, which obviously contradicts Theorem B.

The method which we have followed here has other important applications. It enables us, for example, to prove that almost all numbers are sums of

five cubes,

sixteen biquadrates,

two squares and a cube (or any odd power),

two squares and a prime,

—the last theorem being, naturally, subject to Hypothesis R*. We shall prove the first two of these assertions in the next memoir of the series, in which we return to Waring's Problem.

ASYMPTOTIC EXPANSIONS OF GENERALIZED HYPERGEOMETRIC FUNCTIONS

By W. R. BURWELL.

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PART I.

1. This paper contains a discussion of the integrals

$$F(v) = \int e^{i(x^k - vx)}\,dx, \tag{1}$$

$$C(\lambda) = \int_0^\infty e^{-x^k} \cos \lambda x\,dx, \tag{2}$$

$$S(\lambda) = \int_0^\infty e^{-x^k} \sin \lambda x\,dx. \tag{3}$$

When $k = 3$, $F(v)$ is the Airy integral which was first discussed by Airy in connection with the theory of the diffraction of light.* Later Stokes† obtained its asymptotic expansion for real values of v and derived an expression for its large zeros. $F(v)$ for $k = 4$ also occurs in the theory of the diffraction of light, and in that connection was studied by Lord Rayleigh.‡ L. Brillouin§ has obtained the asymptotic expansion for both of these integrals when v is real by means of the method of steepest descents.‖

* *Camb. Phil. Trans.*, Vol. 6, p. 379; Vol. 8, p. 595.

† *Math. and Phys. Papers*, Vol. 2, pp. 329–349.

‡ *Phil. Mag.* (5), Vol. 8 (1879), p. 403.

§ *Ann. Ec. Norm.* (3), Vol. 33 (1916), pp. 17–69.

‖ See also for examples of this method: Riemann, *Werke* (1876), pp. 402–406, and Debye, *Math. Ann.*, Vol. 67 (1909), p. 535; *Sitzungsber. Bayr. Akad. Wiss.*, Feb. 1910.

The same method has been used in this paper to obtain the expansions for the integral for general positive integral values of k and for real and complex values of v.

From these results, in section 5, the expansions for $C(\lambda)$ and $S(\lambda)$ have been deduced.

In Part II, the expressions for the large zeros of $F(v)$ and $C(\lambda)$ have been derived.

2. Consider $F(v)$ first for infinite limits of integration. Let

$$w = i(x^k - vx), \quad v = Pe^{i\Phi}, \quad \text{and} \quad x = \rho e^{i\theta}.$$

The *cols*, given by $dw/dx = 0$, are

$$x_n = (P/k)^{1/(k-1)} \exp\left[i(\Phi + 2n\pi)/(k-1)\right] \quad (n = 0, 1, 2, \ldots k-2).$$

Complete sets of figures for $0 \leqslant \Phi < 2\pi$ when $k = 3$ and $k = 4$ showing the lines of steepest descent and the level curves for $R(w)$ through the *cols* can be easily generalized to give respectively the figures for k even and for k odd.

In Figs. 1 and 2, drawn for the case $\Phi = (k-1)\pi/2k$, the shaded part represents low ground relative to the *col* x_0.

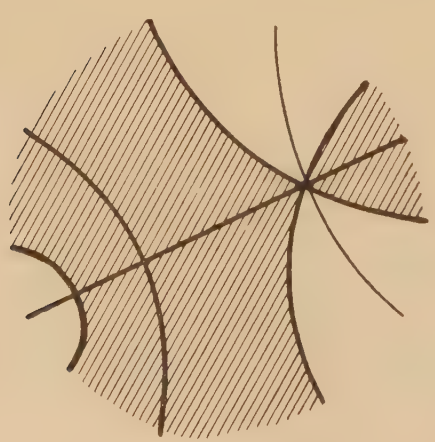

FIG. 1. $k = 3$.

FIG. 2. $k = 4$.

Later we shall obtain in detail the asymptotic expansion for this value of Φ, but let us turn first to the question of the convergency of the integral.

The integral (1), taken between the limits $x = -\infty$ and $x = +\infty$ on the real axis, is not convergent for complex values of v. For

$$R\{i(x^k - vx)\} = Px \sin \Phi,$$

and therefore the modulus of the integrand approaches infinity exponentially as $x \to \infty$ if $\sin \Phi > 0$ and as $x \to -\infty$ if $\sin \Phi < 0$.

The integral is convergent, however, for $\Phi = 0$. We may write

$$F(v) = 2\int_0^\infty \cos(x^k - vx)\, dx = 2\int_0^\infty \cos y\, dx,$$

where $$y = x^k - vx, \quad dy/dx = kx^{k-1} - v.$$

It is evident that x and y increase together after

$$x = (v/k)^{1/(k-1)}.$$

Therefore $$F(v) = 2\left[\int_0^a + \int_a^\infty \frac{\cos y\, dy}{\phi(y)}\right],$$

where $$\phi(y) = kx^{k-1} - v \quad \text{and} \quad a = -(k-1)(v/k)^{k/(k-1)}.$$

It is sufficient to consider the integral between ρ and ∞, where ρ exceeds the critical value of x. Since

$$\frac{d\phi(y)}{dy} = k(k-1)x^{k-2}\frac{dx}{dy},$$

it is evident that $1/\phi(y)$ tends steadily to zero as $y \to \infty$. Also

$$\int^y \cos y\, dy$$

is bounded as $y \to \infty$; and therefore the integral under consideration is convergent.

We must therefore find another path of integration along which the integral is convergent when $\Phi \neq 0$, and which can be deformed into the real axis for real values of v. Consider the path C which ultimately coincides with the lines

$$x = r \exp(i\delta), \qquad x = r \exp[i(\pi + (-)^k \delta)].$$

Since

$$|\exp[i(x^k - vx)]| = \exp[-\rho^k \sin k\theta + P\rho \sin(\theta + \Phi)],$$

for convergence along C we must have $\sin k\theta > 0$ when $\theta = \delta, \pi + (-)^k\delta$, which is the case when $0 < \delta < \pi/k$.

Suppose K is the upper bound of $P \sin(\theta+\Phi)$ when v varies in any finite region V. Then

$$|\exp[i(x^k-vx)]| < \exp[K\rho-\rho^k \sin k\theta]$$

and $F(v)$ is uniformly convergent by comparison with

$$\int_0^\infty \exp[K\rho-\rho^k \sin k\theta]\,d\rho.$$

Therefore $F'(v)$, the integral taken along the path C, is an analytic function regular for any finite range of variation of v.

Further, the path C can be deformed into the real axis for real values of v. By Cauchy's theorem

$$\int_{x=-r}^{x=r} + \int_{\delta=0}^{\delta=\pi/k-\epsilon} + \int_{-(C)} + \int_{\delta=\pi/k-\epsilon'}^{\delta=0} = 0.$$

The second and fourth integrals are evanescent with δ. Take, for example,

$$\int_{\delta=0}^{\delta=\pi/k-\epsilon} \exp[i(x^k-vx)]\,dx = I_1.$$

Then $$|I_1| \leqslant \int_0^{\delta_0} r \exp[-r^k \sin k\delta + vr \sin\delta]\,d\delta = \int_0^\eta + \int_\eta^{\delta_0}.$$

Now

$$r\int_\eta^{\delta_0} \exp[-r^k \sin k\delta + vr \sin\delta]\,d\delta < r\delta_0 \exp[-r^k \sin k\eta + vr \sin\delta_0],$$

which approaches zero as r increases indefinitely.

For the other part, we have

$$\int_0^\eta < r\int_0^\eta \exp[-\tfrac{1}{2}kr^k\delta + vr\delta]\,d\delta = \frac{1-\exp[-\frac{1}{2}kr^k\eta + vr\eta]}{\frac{1}{2}kr^{k-1}-v},$$

which also approaches zero for any finite η as r increases indefinitely.

So the fourth integral too is evanescent and $F'(v)$ is the analytic continuation of the function represented by $F(v)$ when v is real.

3. The path C can be chosen so as to be composed of one or more of the lines of steepest descent passing through the *cols*. The path to be chosen through any *col* passes into low ground relative to that *col*, and it can be proved that the integral near the *col* only need be considered.

In particular, then, consider the integral near x_0 when

$$\Phi = \tfrac{1}{2}(k-1)\pi/k.$$

Take $x = x_0 + re^{\frac{1}{2}\pi i/k} = [(P/k)^{1/(k-1)} + r]e^{\frac{1}{2}\pi i/k} = (p+r)e^{\frac{1}{2}\pi i/k}$.

The integral becomes

$$I = \int_{-\eta}^{\eta} \exp[-(p+r)^k + (p+r)k \,.\, p^{k-1} + \tfrac{1}{2}\pi i/k]\, dr$$

$$= \int_{-\eta}^{0} + \int_{0}^{\eta} = I_1 + I_2,$$

$$I_1 = \exp[(k-1)p^k + \tfrac{1}{2}\pi i/k] \int_0^{\eta} \exp[-(a_2 r^2 + \ldots + a_k r^k)]\, dr$$

where a_j is the coefficient of r^j in $(p-r)^k$.

If we let $a_2 r^2 = y$, I_1 takes the form

say. Then

$$I_1 = \frac{a_2^{-\frac{1}{2}}}{2} \exp[(k-1)p^k + \tfrac{1}{2}\pi i/k] \int_0^{a_2\eta^2} e^{-y} y^{-\frac{1}{2}} \left\{ 1 - \left[a_3 \left(\frac{y}{a_2}\right)^{\frac{3}{2}} + \ldots \right.\right.$$

$$\left.\left. + a_k \left(\frac{y}{a_2}\right)^{k/2} \right] + \frac{1}{2!}[s]^2 - \frac{1}{3!}[s]^3 + \ldots + O\left\{ \frac{(-)^n}{n!}[s]^n \right\} \right\} dy$$

$$= l[I_\alpha + I_\beta],$$

say, where s is the expression in the first square brackets under the integral sign. We may write

$$I_\beta = O\left\{ \int_0^{\infty} \frac{e^{-y}}{n!} \frac{a_3^n y^{(3n-1)/2}}{a_2^{3n/2}}\, dy \right\} = O\left\{ \frac{\Gamma\left(\frac{3n+1}{2}\right)}{p^{\frac{1}{2}nk-1}} \right\}$$

and $$I_\alpha = \int_0^{\infty} - \int_{a_2\eta^2}^{\infty} = A + B.$$

Then

$$A = \int_0^{\infty} \sum_{\nu=0}^{n-1} \frac{(-)^\nu}{\nu!}[s]^\nu e^{-y} y^{-\frac{1}{2}} dy$$

$$= \left\{ \Gamma\left(\frac{1}{2}\right) - \left[\frac{a_3}{a_2^{3/2}} \Gamma(2) + \frac{a_4}{a_2^2} \Gamma\left(\frac{5}{2}\right) + \frac{a_5}{a_2^{5/2}} \Gamma(3) + \ldots + \frac{a_k}{a_2^{k/2}} \Gamma\left(\frac{k+1}{2}\right) \right] \right.$$

$$\left. + \frac{1}{2!}\left[\frac{a_3^2}{a_2^3} \Gamma\left(\frac{7}{2}\right) + \frac{a_4^2}{a_2^4} \Gamma\left(\frac{9}{2}\right) + \ldots \right] + \ldots \right\}$$

and $$-B = \sum_{\nu=0}^{n-1} j_\nu = \sum_{\nu=0}^{n-1} \int_{a_2\eta^2}^{\infty} \frac{(-)^{\nu-1}}{\nu!}[s]^\nu y^{-\frac{1}{2}} e^{-y} dy.$$

But $$|j_\nu| \leqslant \left| \frac{(-)^{\nu-1}}{\nu!} e^{-\frac{1}{2}a_2\eta^2} \int_{a_2\eta^2}^{\infty} e^{-\frac{1}{2}y} y^{-\frac{1}{2}} [s]^\nu dy \right|$$

which is less than the same expression with the lower limit of the integral changed to zero. It is therefore less than

$$\left| \frac{(-)^{\nu-1}}{\nu!} e^{-\frac{1}{2}a_2\eta^2} 2^{\frac{1}{2}} a_3^{\nu} \left(\frac{2}{a_2}\right)^{3\nu/2} O\left\{\Gamma\left(\frac{3\nu+1}{2}\right)\right\} \right|.$$

Thus the part A only is of importance.

So for the integral on the other side of the *col* we obtain as the significant part an expression A' which is the same as A except that the signs of the a's with odd subscripts are changed.

Thus $I(0)$, the integral near x_0, equals $A+A'$. If we now substitute for the coefficients a_j their values as binomial coefficients and replace p by its value

$$(v/k)^{1/(k-1)} e^{-\frac{1}{2}\pi i/k},$$

we deduce that

$$I(0) = \frac{(2\pi)^{\frac{1}{2}} \exp\left[-i\left\{(k-1)(v/k)^{k/(k-1)} - \pi/4\right\}\right]}{(v/k)^{k-2/\lfloor 2(k-1)\rfloor}\left[k(k-1)\right]^{\frac{1}{2}}} \left\{1 - \frac{(k-2)(k-3)3i}{4!\,k(k-1)(v/k)^{k/(k-1)}} + \dots \right.$$

$$+ \frac{1}{2!}\left[\frac{(k-2)^2.5.3.i}{(3!)^2 k(k-1)(v/k)^{k/(k-1)}} - \frac{(k-2)^2(k-3)^2.7.5.3}{(4!)^2\left[k(k-1)\right]^2 (v/k)^{2k/(k-1)}} + \dots\right]$$

$$- \frac{1}{3!}\left[-\frac{3(k-3).9.7.5.3}{(3!)^2.4!\left[k(k-1)\right]^2 (v/k)^{2k/(k-1)}} + \dots\right]$$

$$\left. + \frac{1}{4!}\left[-\frac{(k-2)^4.11.9.7.5.3}{(3!)^4\left[k(k-1)\right]^2 (v/k)^{2k/(k-1)}} + \dots\right] + \dots \right\}$$

where v has always the value kx_0^{k-1}.

So near the *col* x_n the expansion $I(n)$ is obtained from the above by substituting x_n for x_0. The following are the expansions for $F'(v)$ for all values of Φ:—

k even.

$$0 \leqslant \Phi < \frac{(k-\frac{1}{2})\pi}{k}, \qquad F'(v) \sim I(0),$$

$$\Phi = \frac{(k-\frac{1}{2})\pi}{k}, \qquad \sim I(0) + (-)^{\frac{1}{2}(k+2)} I\left(\frac{k-2}{2}\right),$$

$$\frac{(k-\frac{1}{2})\pi}{k} < \Phi < \frac{(2k-\frac{1}{2})\pi}{k}, \qquad \sim (-)^{\frac{1}{2}(k+2)} I\left(\frac{k-2}{2}\right),$$

$$\Phi = \frac{(2k-\frac{1}{2})\pi}{k}, \qquad \sim (-)^{\frac{1}{2}(k+2)} I\left(\frac{k-2}{2}\right) + I(k-2),$$

$$\frac{(2k-\frac{1}{2})}{k} < \Phi < 2\pi, \qquad \sim I(k-2).$$

k odd.

$$\Phi = 0, \qquad F'(v) \sim I(o) + (-)^{\frac{1}{2}(k+1)} I\left(\frac{k-1}{2}\right),$$

$$0 < \Phi < \pi, \qquad \sim I(0),$$

$$\Phi = \pi, \qquad \sim I(0) + (-)^{\frac{1}{2}(k+1)} I\left(\frac{k-3}{2}\right),$$

$$\pi < \Phi < 2\pi, \qquad \sim (-)^{\frac{1}{2}(k+1)} I\left(\frac{k-3}{2}\right).$$

4. Next consider

$$\int_r^s \exp\left[i(x^k - vx)\right] dx.$$

It will now be necessary to obtain the expansion near the limits of this integral. Take

$$x = s + \rho e^{\frac{1}{2}\pi i/k}.$$

Then

$$i(x^k - vx) = i(s^k + ks^{k-1}\rho e^{\frac{1}{2}\pi i/k} + \ldots + \rho^k e^{\frac{1}{2}\pi i} - vs - v\rho e^{\frac{1}{2}\pi i/k}) = if(\rho).$$

and

$$I_s = e^{\pi i/2k} \int_0^\eta e^{if(-\rho)} d\rho.$$

Let

$$i(-ks^{k-1}\rho e^{\frac{1}{2}\pi i/k} + v\rho e^{\frac{1}{2}\pi i/k}) = -i\lambda\rho e^{\frac{1}{2}\pi i/k} = -y.$$

Then

$$I_s = \frac{-i \exp\left[i(s^k - vs)\right]}{\lambda} \int_0^\infty \exp\left[-y + i(c_2 y^2 + c_3 y^3 + \ldots + c_k y^k)\right] dy,$$

where c_j is the coefficient of y^j in $(s + iy/\lambda)^k$,

$$= \frac{-i \exp\left[i(s^k - vs)\right]}{\lambda} \left\{\Gamma(1) + ic_2\Gamma(3) + ic_3\Gamma(4) + \left(ic_4 - \frac{c_2^2}{2!}\right)\Gamma(5)\right.$$

$$+ (ic_5 - c_2c_3)\Gamma(6) + \left(ic_6 - c_2c_4 - \frac{c_3^2}{2!} - \frac{ic_2^3}{3!}\right)\Gamma(7)$$

$$+ \left(ic_7 - c_2c_5 - c_3c_4 - \frac{ic_2^2c_3}{2!}\right)\Gamma(8)$$

$$\left. + \left(ic_8 - c_2c_6 - c_3c_5 - \frac{c_4^2}{2!} - \frac{ic_2c_3^2}{2!} - \frac{ic_2^2c_4}{2!} + \frac{c_2^4}{4!}\right)\Gamma(9) + \ldots \right\}.$$

The expansion near r can be obtained from I_s by changing the sign and replacing s by r.

The results for s and r finite or one infinite are:

k even.

$$\Phi = 0, \quad F(v, r, s) \sim I_r + I_s, \qquad s > r > x_0, \quad x_0 > s > r,$$

$$\sim I_r + I(0) + I_s, \qquad s > x_0 > r,$$

$$0 < \Phi < \pi, \quad \sim I_s$$

$$\Phi = \pi, \quad \sim I_r + I_s, \qquad s > r > x_{\frac{1}{2}(k-2)}, \quad x_{\frac{1}{2}(k-2)} > s > r,$$

$$\sim I_r + (-)^{\frac{1}{2}(k+2)} I\left(\frac{k-2}{2}\right) + I_s, \quad s > x_{\frac{1}{2}(k-2)} > r,$$

$$\pi < \Phi < 2\pi, \quad \sim I_r.$$

k odd.

$$\Phi = 0, \quad F(v, r, s) \sim I_r + I_s, \qquad s > r > x_0, \quad x_0 > s > r > x_{\frac{1}{2}(k-1)},$$

$$x_{\frac{1}{2}(k-1)} > s > r,$$

$$\sim I_r + I(0) + I_s, \qquad s > x_0 > r > x_{\frac{1}{2}(k-1)},$$

$$\sim I_r + (-)^{\frac{1}{2}(k+1)} I\left(\frac{k-1}{2}\right) + I(0) + I_s, \quad s > x_0 > x_{\frac{1}{2}(k-1)} > r,$$

$$\sim I_r + (-)^{\frac{1}{2}(k+1)} I\left(\frac{k-1}{2}\right) + I_s, \quad x_0 > s > x_{\frac{1}{2}(k-1)} > r,$$

$$0 < \Phi < \pi, \quad \sim I_s,$$

$$\Phi = \pi, \quad \sim I_r + I_s,$$

$$\pi < \Phi < 2\pi, \quad \sim I_r.$$

k even.

$$\Phi = 0, \quad F(v, r, \infty) \sim I_r, \qquad r > x_0,$$

$$\sim I_r + I(0), \qquad r < x_0,$$

$$0 < \Phi < \frac{(k-\frac{1}{2})\pi}{k}, \quad \sim I_r, \qquad r > \rho_0,$$

$$\sim I_r + I(0), \qquad r = \rho_0,$$

$$\sim I(0), \qquad r < \rho_0,$$

$$\Phi = \frac{(k-\frac{1}{2})\pi}{k} \quad \sim I_r, \qquad r > \rho_0 = \rho_{\frac{1}{2}(k-2)},$$

$$\sim I_r + (-)^{\frac{1}{2}(k+2)} I\left(\frac{k-2}{2}\right) + I(0), \quad r = \rho_0 = \rho_{\frac{1}{2}(k-2)},$$

$$\sim(-)^{\frac{1}{2}(k+2)}I\left(\frac{k-2}{2}\right)+I(0), \qquad r<\rho_0=\rho_{\frac{1}{2}(k-2)},$$

$$\frac{(k-\frac{1}{2})\pi}{k}<\Phi<\pi, \qquad \sim I_r, \qquad r>\rho_{\frac{1}{2}(k-2)},$$

$$\sim I_r+(-)^{\frac{1}{2}(k+2)}I\left(\frac{k-2}{2}\right), \qquad r=\rho_{\frac{1}{2}(k-2)},$$

$$\sim(-)^{\frac{1}{2}(k+2)}I\left(\frac{k-2}{2}\right), \qquad r<\rho_{\frac{1}{2}(k-2)},$$

$$\Phi=\pi, \qquad \sim I_r, \qquad r>\rho_{\frac{1}{2}(k-2)}=x_{\frac{1}{2}(k-2)},$$

$$\sim I_r+(-)^{\frac{1}{2}(k+2)}I\left(\frac{k-2}{2}\right), \qquad r<\rho_{\frac{1}{2}(k-2)}=x_{\frac{1}{2}(k-2)},$$

$$\pi<\Phi<2\pi, \qquad \sim I_r.$$

k odd.

$$\Phi=0, \quad F(v,r,\infty)\sim I_r, \qquad r>x_0,$$

$$\sim I_r+I(0), \qquad x_0>r>x_{\frac{1}{2}(k-1)},$$

$$\sim I_r+(-)^{\frac{1}{2}(k+1)}I\left(\frac{k-1}{2}\right)+I(0), \qquad r<x_{\frac{1}{2}(k-1)}<x_0,$$

$$0<\Phi<\pi, \qquad \sim I_r, \qquad r>\rho_0,$$

$$\sim I_r+I(0), \qquad r=\rho_0,$$

$$\sim I(0), \qquad r<\rho_0,$$

$$\pi\leqslant\Phi<2\pi, \qquad \sim I_r.$$

k even.

$$\Phi=0, \quad F(v,-\infty,s)\sim I(0)+I_s, \qquad s>x_0,$$

$$\sim I_s, \qquad s<x_0,$$

$$0<\Phi<\pi, \qquad \sim I_s,$$

$$\Phi=\pi, \qquad \sim(-)^{\frac{1}{2}(k+2)}I\left(\frac{k-2}{2}\right)+I_s, \qquad s>x_{\frac{1}{2}(k-2)},$$

$$\sim I_s, \qquad s<x_{\frac{1}{2}(k-2)},$$

$$\pi<\Phi<\frac{(2k-\frac{1}{2})\pi}{k}, \qquad \sim(-)^{\frac{1}{2}(k+2)}I\left(\frac{k-2}{2}\right), \qquad s>\rho_{\frac{1}{2}(k-2)},$$

$$\sim(-)^{\frac{1}{2}(k+2)}I\left(\frac{k-2}{2}\right)+I_s, \qquad s=\rho_{\frac{1}{2}(k-2)},$$

$$\sim I_s, \qquad s < \rho_{\frac{1}{2}(k-2)},$$

$$\Phi = \frac{(2k-\frac{1}{2})\pi}{k} \qquad \sim(-)^{\frac{1}{2}(k+2)} I\left(\frac{k-2}{2}\right) + I(k-2), \qquad s > \rho_{\frac{1}{2}(k-2)} = \rho_{k-2},$$

$$\sim(-)^{\frac{1}{2}(k+2)} I\left(\frac{k-2}{2}\right) + I(k-2) + I_s, \qquad s = \rho_{\frac{1}{2}(k-2)} = \rho_{k-2},$$

$$\sim I_s, \qquad s < \rho_{\frac{1}{2}(k-2)} = \rho_{k-2},$$

$$\frac{(2k-\frac{1}{2})\pi}{k} < \Phi < 2\pi, \qquad \sim I(k-2), \qquad s > \rho_{k-2},$$

$$\sim I(k-2) + I_s, \qquad s = \rho_{k-2},$$

$$\sim I_s, \qquad s < \rho_{k-2},$$

k odd.

$$\Phi = 0, \qquad F(v, -\infty, s) \sim (-)^{\frac{1}{2}(k+1)} I\left(\frac{k-1}{2}\right) + I(0) + I_s, \qquad s > x_0,$$

$$\sim(-)^{\frac{1}{2}(k+1)} I\left(\frac{k-1}{2}\right) + I_s, \qquad x_0 > s > x_{\frac{1}{2}(k-1)}, \qquad s < x_{\frac{1}{2}(k-1)} < x_0,$$

$$0 < \Phi \leqslant \pi, \qquad \sim I_s,$$

$$\pi < \Phi < 2\pi, \qquad \sim(-)^{\frac{1}{2}(k+1)} I\left(\frac{k-3}{2}\right), \qquad s > \rho_{\frac{1}{2}(k-3)},$$

$$\sim(-)^{\frac{1}{2}(k+1)} I\left(\frac{k-3}{2}\right) + I_s, \qquad s = \rho_{\frac{1}{2}(k-3)},$$

$$\sim I_s, \qquad s < \rho_{\frac{1}{2}(k-3)}.$$

Throughout ρ_n has the value

$$\left(\frac{k-1}{k}\right)\left(\frac{P}{k}\right)^{1/(k-1)} \sin\left(\frac{k\Phi+2n\pi}{k-1}\right) \Big/ \sin\Phi.$$

5. The foregoing results can be readily used to obtain the asymptotic expansions of the integrals $C(\lambda)$ and $S(\lambda)$.

Consider first $$C(\lambda) = \int_0^\infty e^{-x^k} \cos\lambda x \, dx.$$

This can be written

$$\tfrac{1}{2}\left[\int_0^\infty \exp[-x^k + i\lambda x]\,dx + \int_0^\infty \exp[-x^k - i\lambda x]\,dx\right].$$

If we apply the transformation $x = x'e^{-\frac{1}{2}\pi i/k}$, it becomes

$$\frac{e^{-\pi i/2k}}{2}\left[\int_{C_1}\exp\left[i(x^k - v'x)\right]dx + \int_{C_1}\exp\left[i(x^k - vx)\right]dx\right],$$

where $v = \lambda e^{-\frac{1}{2}\pi i/k} = -v'$ and the path C_1 starts at the origin and ultimately coincides with the line $x = re^{\frac{1}{2}\pi i\, k}$.

In the following summary obtained from the results of section 4, the dash means that v has been replaced by $-v$ and the parenthesis denotes that the expression enclosed is of lower order than the rest of the expansion given for that particular value of Φ.

When k is even

$$2e^{\pi i/2k}C(\lambda) = \int_C \exp\left[i(x^k - vx)\right]dx,$$

and section 3 gives:—

$$0 \leqslant \Phi < \frac{(k-\frac{1}{2})\pi}{k}, \qquad 2e^{\frac{1}{2}\pi i/k}C(\lambda) \sim I(0),$$

$$\Phi = \frac{(k-\frac{1}{2})\pi}{k}, \qquad \sim I(0) + (-)^{\frac{1}{2}(k+2)}I\left(\frac{k-2}{2}\right),$$

$$\frac{(k-\frac{1}{2})\pi}{k} < \Phi < \frac{(2k-\frac{1}{2})\pi}{k}, \qquad \sim (-)^{\frac{1}{2}(k+2)}I\left(\frac{k-2}{2}\right),$$

$$\Phi = \frac{(2k-\frac{1}{2})\pi}{k}, \qquad \sim I(k-2) + (-)^{\frac{1}{2}(k+2)}I\left(\frac{k-2}{2}\right),$$

$$\frac{(2k-\frac{1}{2})\pi}{k} < \Phi < 2\pi, \qquad \sim I(k-2).$$

k odd.

$$\Phi = 0, \qquad 2e^{\frac{1}{2}\pi i/k}C(\lambda) \sim I_0 + I(0) + I_0',$$

$$0 < \Phi < \frac{(k-1)\pi}{k}, \qquad \sim I(0) + (I_0'),$$

$$\Phi = \frac{(k-1)\pi}{k}, \qquad \sim I_0 + I(0) + I_0',$$

$$\frac{(k-1)\pi}{k} < \Phi < \pi, \qquad \sim I_0 + I_0',$$

$$\Phi = \pi, \qquad \sim I_0 + I_0' + I'(0),$$

$$\pi < \Phi < \frac{(2k-1)\pi}{k}, \qquad \sim(I_0)+I'(0),$$

$$\Phi = \frac{(2k-1)\pi}{k}, \qquad \sim I_0+I_0'-I'(0),$$

$$\frac{(2k-1)\pi}{k} < \Phi < 2\pi, \qquad \sim I_0+I_0'.$$

Next consider $S(\lambda)$. Using the same transformation as above, we have

$$\frac{e^{-\pi i/2k}}{2i}\left[\int_{C_1} \exp[i(x^k-v'x)]\,dx - \int_{C_1} \exp[i(x^k-vx)]\,dx\right].$$

The results of section 4 give:—

k even or odd.

$$\Phi = 0, \qquad 2ie^{\frac{1}{2}\pi i/k} S(\lambda) \sim I_0'-I_0-I(0),$$

$$0 < \Phi < \frac{(k-1)\pi}{k}, \qquad \sim(I_0')-I(0),$$

$$\Phi = \frac{(k-1)\pi}{k}, \qquad \sim I_0'-I_0-I(0),$$

$$\frac{(k-1)\pi}{k} < \Phi < \pi, \qquad \sim I_0'-I_0,$$

$$\Phi = \pi, \qquad \sim I_0'+I'(0)-I_0,$$

$$\pi < \Phi < \frac{(2k-1)\pi}{k}, \qquad \sim I'(0)-(I_0),$$

$$\Phi = \frac{(2k-1)\pi}{k}, \qquad \sim I_0'+I'(0)-I_0,$$

$$\frac{(2k-1)\pi}{k} < \Phi < 2\pi, \qquad \sim I_0'-I_0.$$

6. It should be noted here that the functions $F(v)$, $C(\lambda)$, and $S(\lambda)$ can be expressed in terms of generalized hypergeometric functions. The expansions just obtained could, therefore, be obtained by the method of contour integration employed by Dr. Barnes*, since, theoretically, the expansion for any ${}_pF_q$ can be obtained by that method. In practice, how-

* *Proc. London Math. Soc.*, Vol. 5 (1906–7), p. 59.

ever, the amount of labour involved would make one turn to a more direct method such as that used in this paper.

An expansion for ${}_0F_n$ has been obtained recently by Dr. Wrinch.* This, also, could be used, together with that for ${}_1F_n$, to get the above expansions.

Part II.

7. Using the results of the previous sections, we shall now examine the functions $F(v)$ and $C(\lambda)$ for the existence of zeros. The following results will be obtained:—

(1) The zeros of $F(v)$ are given asymptotically by

k odd.

$$v_n = k\left[\frac{4n-1}{4(k-1)}\pi\right]^{(k-1)/k}$$

and

$$v_n = k\left[\frac{4n(k-1)-k-1}{4(k-1)^2\cos\pi/(k-1)}\pi\right]^{(k-1)/k} e^{\pi i}.$$

k even.

$$v_n = k\left[\frac{4n(k-1)-k}{4(k-1)^2\cos\pi/[2(k-1)]}\pi\right]^{(k-1)/k} e^{[(k-\frac{1}{2})\pi i]/k}$$

and

$$v_n = k\left[\frac{4n(k-1)-k}{4(k-1)^2\cos\pi/[2(k-1)]}\pi\right]^{(k-1)/k} e^{[(2k-\frac{1}{2})\pi i]/k}.$$

(2) The zeros of $C(\lambda)$ are given asymptotically when k is even by

$$\lambda_n = \pm k\left[\frac{4n(k-1)-k}{4(k-1)^2\cos\pi/[2(k-1)]}\pi\right]^{(k-1./k}.$$

When k is odd $C(\lambda)$ has no real zeros.

The expressions for v_n and λ_n are kept in the above form, since they are correct to a higher degree of approximation than $O(n^{[(k-1)/k]-1})$.

8. Plainly the large zeros of $F(v)$ and $C(\lambda)$ can only occur in those directions in which the simultaneous consideration of two or more asymptotic expansions is necessary.

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* *Phil. Mag.* (6), Vol. 41 (1921), pp. 161–186.

Referring to the summaries given in section 3, we find that the following cases for $F(v)$ satisfy that condition :—

(1) *k odd* $\quad F(v)\sim I(0)+(-)^{\frac{1}{2}(k+1)}I\left(\frac{k-1}{2}\right), \qquad \Phi = 0,$

(2) *k odd* $\quad F(v)\sim I(0)+(-)^{\frac{1}{2}(k+1)}I\left(\frac{k-3}{2}\right), \qquad \Phi = \pi,$

(3) *k even* $\quad F(v)\sim I(0)+(-)^{\frac{1}{2}(k+2)}I\left(\frac{k-2}{2}\right), \qquad \Phi = \frac{(k-\frac{1}{2})\pi}{k},$

(4) *k even* $\quad F(v)\sim I(k-2)+(-)^{\frac{1}{2}(k+2)}I\left(\frac{k-2}{2}\right), \qquad \Phi = \frac{(2k-\frac{1}{2})\pi}{k},$

For the consideration of these cases, let $(P/k)^{1/(k-1)} = y$. We obtain then

Case 1.

$$F\sim\frac{2(2\pi)^{\frac{1}{2}}e^{\pi i}}{[k(k-1)]^{\frac{1}{2}}y^{\frac{1}{2}(k-2)}}\left\{\sum_{n=0}^{\infty}\frac{a_n}{y^{2kn}}\cos\left[(k-1)y^k+\frac{3\pi}{4}\right]\right.$$
$$\left.+\sum_{n=0}^{\infty}\frac{b_n}{y^{(2n+1)k}}\sin\left[(k-1)y^k+\frac{3\pi}{4}\right]\right\}.$$

Cases 2–4.

$$F\sim\frac{2(2\pi)^{\frac{1}{2}}e^{-(k-1)y^k\sin\pi/[(s+1)(k-1)]}e^{s\pi i/2k}}{[k(k-1)]^{\frac{1}{2}}y^{\frac{1}{2}(k-2)}}\left\{\sum_{n=0}^{\infty}\frac{a_ne^{(n+1)\pi i}}{y^{2kn}}\right.$$
$$\times\sin\left[(k-1)y^k\cos\frac{\pi}{(s+1)(k-1)}+\frac{4n(k+s-3)+k+1-s}{4(k-1)}\pi\right]$$
$$+\sum_{n=0}^{\infty}\frac{b_ne^{-\frac{1}{2}(k+1-s)\pi i}e^{(n+1)\pi i}}{y^{(2n+1)k}}\sin\left[(k-1)y^k\cos\frac{\pi}{(s+1)(k-1)}\right.$$
$$\left.\left.+\frac{4n(k+s-3)+k+1-s}{4(k-1)}\pi+\frac{k(k+s-3)}{2(k-1)}\pi\right]\right\} = C\,.\,F',\text{ say,}$$

where $s = 0$ for Case 2 and $s = 1$ for Cases 3 and 4.

Case 1 is distinguished by the fact that the two *cols* are symmetrically situated with respect to the origin, and as a result we get a simpler form for F.

9. We want, then, to find the zeros of the part in brackets, F'. The form of this expansion suggests that there may be a 1-1 correspondence

between the large zeros of F' and those of

$$\sin\left[(k-1)y^k\cos\frac{\pi}{(s+1)(k-1)}+\frac{k+1-s}{4(k-1)}\pi\right]=\sin Y.$$

To find the latter, let $Y=n\pi$, and we shall get

$$y_n=\left[\frac{4n(k-1)-k-1+s}{4(k-1)^2\cos\pi/[(s+1)(k-1)]}\pi\right]^{1/k}. \tag{4}$$

Now $y_{n+1}-y_n=O(n^{(1-k)/k})$, so we may surround each of the points y_n by a circular contour C_n of radius $O(n^{(1-k)/k})=An^{(1-k)/k}$, say, which will contain only one of these points.

(A) Within C_n, F' has one and only one zero.

Let $F'=f+\phi$, where $f=-\sin Y$.

Then to prove (A), we must show that on C_n $|f|>|\phi|$. This necessitates our finding a lower bound for $|f|$ on C_n, that is, on a contour which decreases in size as n increases, since the radius is $O(n^{(1-k)/k})$.

We therefore find that, instead of the contours C_n, it is more convenient to consider the contours Γ_n around the points $Y_n=n\pi$ in the Y-plane which correspond to the points y_n in the y-plane. If C_n is given by $y=y_n+\rho e^{i\theta}$, then Γ_n is given by

$$\begin{aligned}Y&=(k-1)\cos\frac{\pi}{(s+1)(k-1)}[y_n^k+ky_n^{k-1}\rho e^{i\theta}+O(y_n^{k-2}\rho^2)]+\frac{k+1-s}{4(k-1)}\pi\\&=Y_n+Re^{i\theta}+O(1/n),\end{aligned}$$

and the curves Γ_n are seen to be roughly circles of fixed radius R.

On these curves $|\sin Y|$ is greater than some function of R, *i.e.* greater than some constant K. Therefore on C_n, $|f|>K$.

Next consider $|\phi|$.

$$|\phi|=O\left\{\frac{b_0}{y^k}\sin\left[Y+\frac{k(k+s-3)}{2(k-1)}\pi\right]\right\}=O(1/y^k)=O(1/n).$$

Therefore on C_n, $|f|>K>|\phi|$ for sufficiently large n, and F' and f have the same number of zeros within C_n, that is, one and only one.

Furthermore, at y_n

$$\begin{aligned}F'&=\frac{b_0e^{-\frac{1}{2}(k-1-s)\pi i}}{y_n^k}\sin\left[Y_n+\frac{k(k+s-3)}{2(k-1)}\pi\right]+O\left(\frac{1}{y_n^{2k}}\right)\\&=\frac{(-)^{n+1}b_0e^{-\frac{1}{2}(k+1-s)\pi i}}{y_n^k}\sin\frac{k(k+s-3)}{2(k-1)}\pi+O\left(\frac{1}{y_n^{2k}}\right).\end{aligned}$$

so that F' has a change of sign between y_n and y_{n+1}, showing that the zero within C_n is real, since F' is a real function.

The zeros of F' are given asymptotically, then, by those of $\sin Y$, (4), or since $v = Pe^{i\Phi}$, the zeros of $F(v)$ are given asymptotically by

$$v_n = Pe^{i\Phi} = k\left[\frac{4n(k-1)-k-1+s}{4(k-1)^2\cos\pi/[(s+1)(k-1)]}\right]^{(k-1)/k} e^{i\Phi}.$$

From this the results for Cases 2–4 can be derived by giving to Φ and s the proper values.

Also, since when k is even

$$F(v) = 2e^{\pi i/2k}\int_0^\infty e^{-x^k}\cos\lambda x\,dx = 2e^{\pi i/2k}C(\lambda),$$

where $\lambda = ve^{\frac{1}{2}\pi i/k}$, the expression for the zeros of $C(\lambda)$, derived from Cases 3–4, is readily seen to be

$$\lambda_n = v_n e^{\pi i/2k} = \pm k\left[\frac{4n(k-1)-k}{4(k-1)^2\cos\pi/[2(k-1)]}\pi\right]^{(k-1)/k},$$

and we are able to draw the conclusion that all but a finite number of the zeros of $C(\lambda)$ lie on the real axis. This is of interest in connection with a paper by F. Bernstein* in which he has proved by rather elaborate methods the existence of an infinity of zeros of $C(\lambda)$ when k is even and greater than 2.

Case 1 presents no new difficulties and can be dealt with in a similar way. The zeros are given by those of $\cos[(k-1)y^k+(3\pi/4)]$ which are

$$y_n = \left[\frac{4n-1}{4(k-1)}\pi\right]^{1/k} \quad \text{or} \quad v_n = k\left[\frac{4n-1}{4(k-1)}\pi\right]^{(k-1)/k}.$$

10. When k is odd $C(\lambda)$ has no real zeros, for the expansions are either algebraic or are made up of an algebraic part and $I(0)$ or $I'(0)$.

Zeros can only occur, then, in those directions in which the exponential factor of $I(0)$ or of $I'(0)$ remains more or less bounded. These directions are given by $\Phi = m(k-1)\pi/k$ and $\Phi = [m(k-1)-k]\pi/k$, none of which makes λ real.

* *Math. Annalen*, Vol. 79, (3) (1918), p. 265.

ON A TABLE FOR CALCULATING EULERIAN NUMBERS BASED ON A NEW METHOD

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Introduction.

In a paper on "Some Properties of Eulerian and Prepared Bernoullian Numbers,"* we gave a formula for E_n, viz.†

$$E_n = \Sigma 1^2 \Sigma 2^2 \Sigma 3^2 \dots \Sigma n^2,$$

and showed how a simple table can be constructed from the formula. We had there stated that we were calculating forty Eulerian numbers with the help of the table. We have completed the table as far as E_{30}, though it is only printed here as far as E_8. The principal object of this paper is to prove that the numbers in the even columns of the rows other than the first are also coefficients in trigonometric expansions. It is this latter fact that has induced us to do the calculation. Owing to pressure of other work, we shall not be able to complete the table as far as E_{40} for some time. We hope to continue it later on. We here remark that, though the method is simple, the calculation is by no means so. From E_{20} onwards, the calculation of one number takes from two to three hours. Great care has to be taken at each stage, since a small error in the earlier stages shows itself as a large one in the final result. The numbers were calculated one after another. Since thirty-two Eulerian numbers‡ are

* Presented to the third Conference of the Indian Mathematical Society, Lahore (March 1921), and communicated to the London Mathematical Society (April 1921).

† The significance of the Σ notation is explained in the note at the end of the present paper.

‡ See Glaisher, *Quarterly Journal*, Vol. 45; Joffe, *ibid.*, Vol. 47.

tabulated, the verification was done immediately after a number was obtained. When the Eulerian numbers are known the table may even be worked backwards, as is shown in the last article of this paper. This has the advantage that the division by primes or numbers containing primes as 3, 7, ... afford a proof of the correctness of the previous work.

1. We have proved in the paper referred to above that if

$$y = \sec x, \qquad t = \tan x,$$

then

$$\frac{d^n y}{dx^n} = y\{n!\,t^n + (n-2)!\,t^{n-2}\,\Sigma(n-1)^2 + (n-4)!\,t^{n-4}\,\Sigma(n-3)^2\,\Sigma(n-2)^2 + \ldots + (n-2r)!\,t^{n-2r}\,\Sigma(n-2r+1)^2\,\Sigma(n-2r+2)^2 \ldots \Sigma(n-r)^2 + \ldots\}, \quad (1)$$

the last term being

$$\Sigma 2^2\,\Sigma 3^2 \ldots \Sigma\left(\frac{n+1}{2}\right)^2 t \text{ if } n \text{ is odd}$$

and

$$\Sigma 1^2\,\Sigma 2^2 \ldots \Sigma\left(\frac{n}{2}\right)^2 \quad \text{if } n \text{ is even.}$$

Now from the way in which the table is constructed, we have the values of

$\Sigma(n-1)^2$ in the $(n-2)$-th row of the second column,

$\Sigma(n-3)^2\,\Sigma(n-2)^2$ in the $(n-4)$-th row of the fourth column,

$\Sigma(n-5)^2\,\Sigma(n-4)^2\,\Sigma(n-3)^2$ in the $(n-6)$-th row of the sixth column,

and so on. Thus, given the table, we can immediately write down the n-th differential coefficient of $\sec x$ as the product of $\sec x$ and a power series in $\tan x$. The coefficients in the power series are to be obtained from the even columns of the table. Thus, from the table,

$$\frac{d^6 y}{dx^6} = y\{6!\,t^6 + 4!\,t^4 . 55 + 2!\,t^2 . 331 + 61\}.$$

2. We next consider the expansion

$$f(x) = \frac{1}{\cos x - a \sin x}$$

$$= \sec x + a \sec x \tan x + a^2 \sec x \tan^2 x + \ldots + a^n \sec x \tan^n x + \ldots . \quad (2)$$

If $f(x)$ is expanded in ascending powers of x, we have shown in the paper referred to above that the expansion may be written

$$\frac{1}{\cos x - a \sin x} = 1 + \Sigma A_n(a) \frac{x^n}{n!},$$

where $A_n(a)$ is a function of a of degree n, odd or even according as n is odd or even. Also (if $a = \tan\theta$)

$$A_n(a) = \left[\frac{d^n}{dx^n}\left(\frac{1}{\cos x - a \sin x}\right)\right]_{x=0}$$

$$= \cos\theta \left[\frac{d^n}{dx^n} \sec(x+\theta)\right]_{x=0}$$

$$= \cos\theta \frac{d^n}{d\theta^n} (\sec\theta),$$

so that, from equation (1) above, we may write*

$$A_n(a) = n!\, a^n + (n-2)!\, a^{n-2} \Sigma(n-1)^2 + (n-4)!\, a^{n-4} \Sigma(n-3)^2 \Sigma(n-2)^2 + \ldots$$
$$+ (n-2r)!\, a^{n-2r} \Sigma(n-2r+1)^2 \Sigma(n-2r+2)^2 \ldots \Sigma(n-r)^2 + \ldots . \quad (3)$$

Hence by rearranging the above equation in powers of a, and comparing it with equation (2), we obtain the formula

$$\sec x \tan^n x = n! \left\{ \frac{x^n}{n!} + \frac{x^{n+2}}{(n+2)!} \Sigma(n+1)^2 + \frac{x^{n+4}}{(n+4)!} \Sigma(n+1)^2 \Sigma(n+2)^2 + \ldots \right.$$
$$\left. + \frac{x^{n+2r}}{(n+2r)!} \Sigma(n+1)^2 \Sigma(n+2)^2 \ldots \Sigma(n+r)^2 + \ldots \right\}. \quad (4)$$

As before we have the values of

$\Sigma(n+1)^2$ in the n-th row of the second column,

$\Sigma(n+1)^2 \Sigma(n+2)^2$ in the n-th row of the fourth column,

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* We may here remark that, if $a = 1$, $f(x) = \frac{1}{\cos x - \sin x}$, and $A_n(1) = A_n$, a coefficient discussed by Mr. C. Krishnamachary in his paper "On the Summation of certain Series," presented to the third Conference of the Indian Mathematical Society, March 1921. A_n is Dr. Glaisher's P_n or Q_n according to the integral character of n. (See Dr. Glaisher, *Quarterly Journal*, Vols. 29, 45.) Thus we may write Mr. Krishnamachary's A_n or Dr. Glaisher's P_n and Q_n in the form

$$A_n = n! + (n-2)!\, \Sigma(n-1)^2 + (n-4)!\, \Sigma(n-3)^2 \Sigma(n-2)^2 + \ldots$$
$$+ (n-2r)!\, \Sigma(n-2r+1)^2 \ldots \Sigma(n-r)^2 + \ldots .$$

A_n may therefore be calculated from the table given here.

and generally the value of

$\Sigma(n+1)^2\,\Sigma(n+2)^2 \ldots \Sigma(n+r)^2$ in the n-th row of the $(2r)$-th column.

Thus the numbers in the even columns of the n-th row are all coefficients in the trigonometric expansion $\sec x \tan^n x$, the factorials being taken along with x.

Or, if we wish, we may write

$$\sec x \tan^n x = x^n + \Sigma(n+1)^2 \frac{x^{n+2}}{\Pi(n+2)} + \Sigma(n+1)^2\,\Sigma(n+2)^2 \frac{x^{n+4}}{\Pi(n+4)} + \ldots$$

$$+ \Sigma(n+1)^2\,\Sigma(n+2)^2 \ldots \Sigma(n+r)^2 \frac{x^{n+2r}}{\Pi(n+2r)} + \ldots, \quad (5)$$

where $\Pi(n+2r)$ is defined by the equation

$$\Pi(n+2r) = (n+1)(n+2) \ldots (n+2r).$$

3. We may show, as in § 9 of our paper referred to above, that if $\sec x \tan^n x$ is expanded in ascending powers of x, the coefficients can be expressed in terms of Euler's numbers. Proof by induction (along the lines of § 9) is probably the simplest. In fact, the result is

$$n! \sec x \tan^n x = \Sigma \{ E_{r+k} - \Sigma(n-1)^2 E_{r+k-1} + \Sigma(n-1)^2\,\Sigma(n-3)^2 E_{r+k-2} - \ldots$$

$$+ (-1)^k\,\Sigma(n-1)^2\,\Sigma(n-2)^2\,\Sigma(n-5)^2 \ldots E_r \} \frac{x^l}{l!},$$

where $$k = \frac{n}{2} \text{ or } \frac{n-1}{2} \text{ and } l = 2r \text{ or } 2r-1, \quad (6)$$

according as n is even or odd.

The descending* Σ notation needs a word of explanation. Its meaning resembles the meaning we have attached to the ascending Σ notation. We give two simple examples :

$$\Sigma 3^2\,\Sigma 1^2 = 3^2 . 1^2,$$

$$\Sigma 5^2\,\Sigma 3^2\,\Sigma 1^2 = 5^2 . 3^2 . 1^2.$$

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* See the note at the end of the paper.

And we have, in fact, generally

$$\Sigma(2n-1)^2\,\Sigma(2n-3)^2\ldots\Sigma 1^2=(2n-1)^2\,(2n-3)^2\ldots 1^2,$$

$$\Sigma 4^2\,\Sigma 2^2=4^2.\,\Sigma 2^2+\Sigma 3^2\,\Sigma 1^2$$
$$=4^2\,(2^2+1^2)+3^2.1^2,$$

$$\Sigma 6^2\,\Sigma 4^2=6^2.\Sigma 4^2+\Sigma 5^2\,\Sigma 3^2$$
$$=6^2.\Sigma 4^2+5^2.\Sigma 3^2+\Sigma 4^2\,\Sigma 2^2.$$

Generally $\Sigma n^2\,\Sigma(n-2)^2=n^2\Sigma(n-2)^2+\Sigma(n-1)^2\,\Sigma(n-3)^2,$

and so on. Reverting to equation (6), we see that we may write from (4),

$$E_{r+\frac{1}{2}n}-\Sigma(n-1)^2\,E_{r+\frac{1}{2}n-1}+\Sigma(n-1)^2\,\Sigma(n-3)^2\,E_{r+\frac{1}{2}n-2}-\ldots$$
$$+(-1)^{\frac{1}{2}n}\,\Sigma(n-1)^2\,\Sigma(n-3)^2\ldots\Sigma 2^2.E_r$$
$$=(n!)^2\,\Sigma(n+1)^2\,\Sigma(n+2)^2\ldots\Sigma(\tfrac{1}{2}n+r)^2, \qquad (7)$$

or $$E_{r+\frac{1}{2}(n-1)}-\Sigma(n-1)^2\,E_{r+\frac{1}{2}(n-1)-1}+\Sigma(n-1)^2\,\Sigma(n-3)^2\,E_{r+\frac{1}{2}(n-1)-2}-\ldots$$
$$+(-1)^{\frac{1}{2}(n-1)}\,\Sigma(n-1)^2\,\Sigma(n-3)^2\ldots\Sigma 1^2\,E_r$$
$$=(n!)^2\,\Sigma(n+1)^2\,\Sigma(n+2)^2\ldots\Sigma[\tfrac{1}{2}(n-1)+r]^2, \quad (8)$$

according as n is even or odd.

We may write (7) and (8) more conveniently in the form

$$E_n-\Sigma r^2\,E_{n-1}+\Sigma r^2\,\Sigma(r-2)^2\,E_{n-2}-\ldots+(-1)^k\,\Sigma r^2\,\Sigma(r-2)^2\ldots E_{n-k}$$
$$=[(r+1)!]^2\,\Sigma(r+2)^2\,\Sigma(r+3)^2\ldots\Sigma n^2, \quad (9)$$

where $k=\frac{1}{2}(r+1)$ or $\frac{1}{2}r$ according as r is odd or even.

4. This formula (9) may also be directly obtained from the fundamental formula for E_n.

For $$E_n=\Sigma 1^2\,\Sigma 2^2\ldots\Sigma n^2$$
$$=1^2\,\Sigma 2^2\,\Sigma 3^2\ldots\Sigma n^2$$
$$=1^2.2^2\,\Sigma 3^2\,\Sigma 4^2\ldots\Sigma n^2+1^2\,\Sigma 1^2\,\Sigma 2^2\ldots\Sigma(n-1)^2;$$

therefore $$E_n-1^2.E_{n-1}=(2!)^2\,\Sigma 3^2\,\Sigma 4^2\ldots\Sigma n^2$$
$$=(2!)^2.3^2.\Sigma 4^2\ldots\Sigma n^2+(2!)^2\,\Sigma 2^2\,\Sigma 3^2\ldots\Sigma(n-1)^2$$
$$=(3!)^2\,\Sigma 4^2\,\Sigma 5^2\ldots\Sigma n^2+(2!)^2\,\frac{E_{n-1}}{1^2};$$

therefore

$$E_n-(1^2+2^2)E_{n-1} = (3!)^2\,\Sigma 4^2\,\Sigma 5^2 \ldots \Sigma n^2$$

$$= (3!)^2\,4^2\,\Sigma 5^2\,\Sigma 6^2 \ldots \Sigma n^2+(3!)^2\,\Sigma 3^2\,\Sigma 4^2 \ldots \Sigma(n-1)^2$$

$$= (4!)^2\,\Sigma 5^2\,\Sigma 6^2 \ldots \Sigma n^2+(3!)^2\,\frac{E_{n-1}-1^2.E_{n-2}}{(2!)^2};$$

therefore $E_n-\Sigma 3^2.E_{n-1}+3^2.1^2.E_{n-2} = (4!)^2\,\Sigma 5^2\,\Sigma 6^2 \ldots \Sigma n^2$.

And so we get (9) by a process of induction.

5. The method of working the table backwards from E_n is given here for E_7. The figures given are all to be found in the table, and the rationale will be obvious by a comparison of the table and the work given here.

E_7	199,360,981
Subtract E_6	2,702,765
	196,658,216
Divide by 2^2	49,164,554
Subtract E_6	2,702,765
	46,461,789
Divide by 3^2	5,162,421
Subtract	663,061
	4,499,360
Divide by 4^2	281,210
Subtract	68,060
	213,150
Divide by 5^2	8,526
Subtract	3,486
	5,040
Divide by 6^2	140
Subtract	91
	$7^2 = 49$

From this it is obvious that the part of the figures relating to E_7 can be at once written down in the table.

We remark that, as proved in § 2, we can prove that the numbers occurring in the even columns of the m-th row in the table formed for $\sec^n x$ (see our paper referred to above) can be proved to be coefficients in the expansion of $\sec^n x \tan^m x$, the factorials being taken with x.

Construction of Table.

The numbers in the first column are the squares of the natural numbers. Each even column is formed by taking the sums to 2, 3, 4, ... terms of the numbers in the preceding column. Each odd column after the first is formed by multiplying the numbers in the preceding column by $1^2, 2^2, 3^2, 4^2, \ldots$ respectively. The number at the head of the $(2n-2)$-th and $(2n-1)$-th columns is E_n.

Note.—The meaning of the summations in the formula

$$E_n = \Sigma 1^2 \Sigma 2^2 \ldots \Sigma n^2$$

should be explained, since it may not be quite obvious. The arithmetical values of the numbers occurring in this notation increase as we go to the right, and in reducing a sum of this type, say

$$\Sigma(a+1)^2\, \Sigma(a+2)^2 \ldots \Sigma b^2,$$

we proceed onwards till the first digit becomes zero. Thus

$$\Sigma 3^2\, \Sigma 4^2 = 3^2\, \Sigma 4^2 + 2^2\, \Sigma 3^2 + 1^2\, \Sigma 2^2,$$

when the sums on the right denote 4^2, 3^2+4^2, $2^2+3^2+4^2$, respectively. We stop at the term $1^2\, \Sigma 2^2$ because the next will be $0^2\, \Sigma 1^2$. This we have called the ascending Σ notation. In the other, viz. what we called the descending Σ notation, the reduction proceeds till the last on the right becomes zero, and the arithmetical values of the numbers diminish as we proceed to the right. Thus

$$\Sigma 4^2\, \Sigma 2^2 = 4^2\, \Sigma 2^2 + 3^2\, \Sigma 1^2.$$

We stop because the next will be $2^2\, \Sigma 0^2$. This will make clear the distinction between ascending and descending Σ notation.

We conclude by expressing our thanks to Mr. S. Muthuswami, B.A., a student of the College of Engineering, Bangalore, for the help he rendered in the calculation of the table, and for getting a neat fair copy of the table for submission to the London Mathematical Society.

1	2	3	4	5	6	7	8	9	10	11	12	13	14
E_1	E_2		E_3		E_4		E_5		E_6		E_7		E_8
1^2	5	5	61	61	1,385	1,385	50,521	50,521	2,702,765	2,702,765	199,360,981	199,360,981	19,391,512,145
2^2	14	56	331	1,324	12,284	49,136	663,061	2,652,244	49,164,554	196,658,216	4,798,037,791	19,192,151,164	
3^2	30	270	1,211	10,899	68,060	612,540	5,162,421	46,461,789	510,964,090	4,598,676,810			
4^2	55	880	3,486	55,776	281,210	4,499,360	28,862,471	461,799,536					
5^2	91	2,275	8,526	213,150	948,002	23,700,050							
6^2	140	5,040	18,522	666,792									
7^2	204	9,996											
8^2													
r	$\Sigma(r+1)^2$	$r^2\Sigma(r+1)^2$	$\Sigma(r+1)^2$ $\Sigma(r+2)^2$	$r^2\Sigma(r+1)^2$ $\Sigma(r+2)^2$	$\Sigma(r+1)^2$ $\Sigma(r+2)^2$ $\Sigma(r+3)^2$	$r^2\Sigma(r+1)^2$ $\Sigma(r+2)^2$ $\Sigma(r+3)^2$	$\Sigma(r+1)^2$ $\Sigma(r+2)^2$ $\Sigma(r+3)^2$ $\Sigma(r+4)^2$	... etc. ...					

N.B.—The last row indicates the value of the general term in the columns. The general term of all is thence obvious. The factorials being taken with x, the numbers in the even columns of the r-th row are coefficients in the algebraic expansion of $\sec x \tan^r x$ in ascending powers of x.

ON THE FUNDAMENTAL THEOREM OF INTEGRATION FOR MULTIPLE INTEGRALS

By W. H. YOUNG.

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1. It was inevitable that the question as to the possibility of generalising for more than a single variable the fundamental theorem connecting a primitive and its derivates should engage the attention of mathematicians, as soon as the theory could be announced to be complete in the simpler case. And in 1910, scarcely ten years after his earliest essays in the theory of generalised integration, Lebesgue undertook, in a paper of some hundred quarto pages,* the task of analysing, criticising, and extending, what had hitherto been attempted in the direction in question. Almost the first positive result of real value, however, was obtained by Leonida Tonelli in 1916, and completed with the help of Fubini. At the time when their correspondence on this subject was published,† even the existence of a mixed differential coefficient $\frac{d}{dx}\frac{dF}{dy}$ or $\frac{d}{dy}\frac{dF}{dx}$ of a double integral $F(x, y) = \iint f(x, y)\,dx\,dy$ was uncertain, and their theorem that a double integral possesses finite mixed derivates‡ almost everywhere, that is, except at points forming a set of plane content zero, and that these derivates are equal, except for a similar set, must have come as a surprise to earlier workers. Soon after,§ I established the corresponding theorem for a more general type of function, including

* H. Lebesgue, *Ann. sci. de l'École norm. sup.*, Sér. 3., t. 40 (1910), pp. 361–450.

† G. Fubini and L. Tonelli. "Sulla derivata seconda mista di un integrale doppio," *Rend. di Palermo*, Vol. 40 (1916), pp. 295–299.

‡ See foot-note below.

§ At the time Fubini and Tonelli published their result I was out of Europe, and was not aware of their work until my attention was called to it by Lebesgue at the time (April, 1917) when my own note was presented to the Académie des Sciences (*Comptes Rendus*, Vol. 164, p. 622). The fact is interesting as showing the inevitableness of the theorem, as well as of the type of method employed in its demonstration.

the double integral as a special case, namely, what I, with certain writers, call functions of bounded variation.*

It will be noted that these results refer to repeated derivates and not to double derivates,† such as it has been attempted to introduce, but without conspicuous success. It does not seem certain that double derivates can be usefully employed; and, if we bear in mind that, at any rate in the case of absolutely convergent integrals, there is no distinction between double and repeated integrals, while for non-absolutely convergent integrals no theory of double integration as distinct from repeated integration exists, it does not seem certain that the study of double derivates is likely to render services commensurate with the magnitude of the effort required for surmounting the difficulties involved.

In the present communication I confine myself to repeated derivates and their absolutely convergent integrals. On the other hand, this limitation does not prevent the extension of the theory of what may be called *repeated differentiation with respect to a function*,‡ and our results are immediately generalisable for any number of dimensions.

The theorems proved are as follows:—

THEOREM I.—*If for every x, except at most for a set of values of x of content zero, the same for all y's, $f(x,y)$ satisfies the following conditions:—*

(1) *of possessing an absolutely convergent integral with respect to y;*

(2) *of being finite except at most for a countable set of values of y (which may depend on the choice of the fixed x);*

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* Hobson uses the term "bounded variation" as characterising a type of functions so called by Arzelà, forming a wider class including that referred to in the text. It remains to be proved whether the properties of Arzelà's functions will be found to justify the denomination proposed. It should be remarked that the account given by Hobson of the properties of these functions needs revision.

† The double derivates of a function $f(x, y)$ are the limits of the double incrementary ratio

$$\frac{[F(x+h, y+k)-F(x+h, y)-F(x, y+k)+F(x, y)]}{hk},$$

when the pair (h, k) tend to the pair $(0, 0)$ in any and every manner; neither h nor k being actually zero during the process. When, on the other hand, we make first k and then h tend to zero, we have as limits the repeated derivates in this order, which, for brevity, we may characterise as that of $\frac{d}{dx}\frac{d}{dy}$, the reverse order being that of $\frac{d}{dy}\frac{d}{dx}$. The repeated derivates are equal to certain of the double derivates. When $f(x, y)$ is a continuous function of (x, y), the derivates in question always fill up the interval between the greatest and least of them.

‡ See below.

(3) *of being a derivate with respect to y of a function $F(x, y)$ continuous with respect to y (save for the excepted set of values of x of zero content);*

and if further $F(x, y)$ possesses an absolutely convergent integral with respect to x, is finite except for a countable set of values of x, the same for all y's, and is a derivate with respect to x of a function $G(x, y)$ continuous with respect to x; then $G(x, y)$ is the indefinite repeated integral of $f(x, y)$.

THEOREM II.—*If both the upper and the lower mixed derivates of a continuous function $F(x, y)$ possess absolutely convergent double integrals and are finite except at most for the points of a set whose projection on the axis of x is countable* $\left(\text{supposing the order of derivation to be that of } \frac{d}{dy}\frac{d}{dx}\right)$, *then $F(x, y)$ is an absolutely continuous function, and is accordingly the integral of any one of its mixed derivates, whether in the order in question or in the reverse order.*

The theorem remains true if, without knowing that the projection of the exceptional set is countable, we know that it contains no perfect subset, *i.e.* that the derivates of $F(x, y)$ are finite at some point of every perfect set of straight lines parallel to one of the two axes of coordinates.

2. The proof of Theorem I involves nothing more than the repeated application of the theorem* that, if $f(y)$ is finite except for a countable set of values of y, possesses an absolutely convergent integral, and is a derivate† of a continuous function $F(y)$, then this primitive function $F(y)$ is the indefinite integral of $f(y)$.

In fact, in the statement of our present theorem, $f(x, y)$ and $F(x, y)$ regarded as functions of y, satisfy the conditions required of $f(y)$ and $F(y)$ respectively, for every fixed x not belonging to the excepted set of zero content, the same for all y's. Therefore, except for this set,

$$\int f(x, y)\, dy = F(x, y),$$

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* W. H. Young, "On Derivates and their Primitive Functions," *Proc. London Math. Soc.*, Ser. 2, Vol. 12 (1912), p. 216. "On Integrals and Derivates with respect to a Function," *Proc. London Math. Soc.*, Vol. 15 (1914), p. 55.

† That is any function lying between the lower and the upper derivates, that is the lower and the upper limits of $\frac{F(y+k)-F(y)}{k}$, when $k \to 0$ by means of both positive and negative values.

and therefore we have everywhere

$$\iint f(x, y)\, dx\, dy = \int F(x, y)\, dx,$$

since $G(x, y)$ and $F(x, y)$, regarded as functions of x, satisfy the conditions required in the theorem quoted of $F(x)$ and $f(x)$ respectively, so that the integral on the right exists and we have

$$G(x, y) = \int F(x, y)\, dx.$$

Combining the two results, we get as required

$$G(x, y) = \iint f(x, y)\, dx\, dy.$$

3. If we are to proceed further we require more than what we have virtually employed in the preceding proof, namely, the use of Scheeffer's theorem twice; we need a properly two-dimensional Scheeffer's theorem. As, however, the proof of Scheeffer's theorem requires the notion of one dimensional order, it does not seem possible to adapt it for higher space. That is, for the rest, consistent with the fact that in Theorem I a condition is imposed on the intermediate derivate;* and we should naturally expect some equivalent condition to be necessary, when all reference to this intermediate derivate is omitted.

The form of Scheeffer's theorem which I have obtained is that given in the next article. For the proof we require the theorem on inner limiting sets of my early paper in the *Leipziger Bericht.*† The result there enunciated is that the potency of an ordinary inner limiting set is that of the continuum, if it is not countable, and that the former is the case whenever there is an increasing succession of positive integers

$$r_1 < r_2 < r_3 < \ldots < r_n < r_{n+1} < \ldots,$$

such that there is an interval of the r_1-th defining set of intervals which contains at least two of the r_2-th set, each of which contains at least two of the r_3-th set, and so on for ever—this condition is moreover necessary as well as sufficient for the potency to be c. It was then proved that another form of the same condition is that the inner limiting set should contain a component dense in itself. This statement includes as a special case the theorem that a countable inner limiting set is reducible, and in a subsequent paper I showed that every reducible set is an ordinary inner limiting set. I should like to point out here that, in virtue of the remark

* The derivate with respect to x in § 2.

† "Zur Lehre der nicht abgeschlossenen Punktmengen," *Leipz. Ber. Math.-phys. Klasse* (1903), pp. 287–293.

made in § 9 of the Leipzig paper, to the effect that, if the intervals defining the set are at each stage (i) non-overlapping, and (ii) lie inside the intervals at the preceding stage—in which case they are what is there called *normal intervals*—and are (iii) finite in number, the inner limiting set is a closed set, *the sub-set of potency c of the whole limiting set found by means of pairs of intervals at the* r_1-th, r_2-th, ... *stages in accordance with my condition is a perfect set.* We shall use this result in the present paper.*

4. AUXILIARY THEOREM.—*If the net double increment of a continuous function* $F(x, y)$ *over the rectangle* $(a, b; A, B)$ *is negative,* (< 0), *then the set of points inside the rectangle at which the lower mixed derivate of* $F(x, y)$, *in the order* $\frac{d}{dy}\frac{d}{dx}$, *is negative inside the rectangle, projects on to the axis of x into a set of potency c, containing a perfect set.*

By the hypothesis made, we may write

$$F(A, B) - F(A, b) - F(a, B) + F(a, b) = -\gamma \quad (\gamma > 0).$$

If the fundamental rectangle is divided up by means of parallels to the axes in any way into a finite number of sub-rectangles, the sum of the net increments over these is equal to $-\gamma$; therefore there must be at least one of these sub-rectangles such that the net increment over it is negative; and the sum of the negative increments over all such sub-rectangles does not exceed $-\gamma$. Hence, dividing up into a series of sets of sub-rectangles in successive stages according to any law, omitting at each stage those sub-rectangles which yield increments $\geqslant 0$, and only continuing the subdivision at the next stage in the remaining sub-rectangles, we define, as the set of all the points belonging at each stage to the retained sub-rectangles, an inner limiting set (i-set) E.

This set E cannot contain an isolated point.

For consider any point M of this set. By the definition of the inner limiting set E we can choose out a succession of closed or open rectangles, one belonging to each of the defining sets, each lying inside the preceding in the succession, and having M for common point. If this point is isolated, there is a circle with M as centre containing no other point of the set E, and, sooner or later all the rectangles of our succession must come to lie inside this circle, since otherwise the succession would cer-

* I fancy that this result, in common with other results of this early paper of mine, has been the subject of investigation since by other mathematicians, but I cannot recall the facts exactly or give references.

tainly contain a continuous sub-set of E. We may therefore affirm that from and after a definite stage no rectangle of this succession contains another point of the i-set besides M. Hence we easily prove that, from and after another stage, any rectangle of our succession contains no rectangle of the original series of rectangles, except those of our succession determining M. From and after a definite stage, then, each rectangle of our succession defining M contains one and only one rectangle of negative double increment belonging to the next stage of subdivision. Therefore, as pointed out in subdividing the original rectangle, the (negative) increment over this single rectangle does not increase from stage to stage, and cannot therefore tend to zero as limit. This is in contradiction with the supposed continuity of the function $F(x, y)$ at the point M to which the rectangles shrink up. Therefore the assumption is false that M is an isolated point of the set E.

This proves that the i-set E is dense in itself, and therefore contains a perfect sub-set of potency c.

We may so subdivide our original rectangle as to get as inner limiting set a set of potency c of parallels to one of the axes, say the axis of x, by drawing the dividing lines parallel to that axis alone. The same argument as above shows that no straight line of this inner limiting set can be isolated. We could otherwise find a succession of rectangles with negative increments which do not increase as the rectangles diminish and yet tend to a stretch as limit, whereas, by continuity, the limit of the increments must be zero. Hence the section of the limiting lines by the perpendicular axis is an i-set dense in itself, and therefore contains a perfect sub-set, which proves the set of parallels to be of potency c.

Now consider the succession of long rectangles defining some particular one of the limiting stretches. By lines perpendicular to the limiting segment I divide it into a finite number of portions, and, corresponding to each of these, each of the long rectangles yields a subrectangle, these subrectangles forming a succession having that portion as limiting segment. If we retain only those of the subrectangles as give negative double increments for $F(x, y)$, we shall have reduced my infinite succession of long rectangles to a finite number of successions of subrectangles, one at least of which must be infinite and have for limit the corresponding portion of the original limiting segment. If there be more than one such, we may take the highest portion having this property and the corresponding succession of subrectangles, so as to have a definite one defined on the chosen limiting segment. This process can now be continued, and we thus define a succession of stretches (portions) on the limiting segment chosen, and corresponding to these a succession of sets of rectangles, the

rectangles of each set lying one inside the other and having the corresponding stretch as limiting segment, while the successive stretches lie each inside its predecessor and determine a unique point belonging to them all, supposing, as we shall, the division of the fundamental rectangle to be such that the span of the greatest rectangle at any stage tends to zero as the stages advance. This unique point on the chosen limiting segment is the sole point common to all the subrectangles of our succession of sets of subrectangles.

We have thus obtained a set of points of potency c which projects on to one of the axes (axis of x) into an inner limiting set of potency c, containing therefore a perfect component. I affirm that at each point of this set the lowest of the repeated derivates of the function $F(x, y)$ is negative or zero.

To prove this, let (x, y) be any one of the points we have constructed, and consider the succession of rectangles by means of which we demonstrated its existence; this was a double succession of rectangles, having (x, y) for sole common point.

First let us suppose (x, y) to be a common corner of all the rectangles. Denoting the left-hand bottom corner of any one of these by (z, t), and the height and breadth of the rectangle by $\bar{h}$ and $\bar{k}$, we have by construction

$$F(z+\bar{h}, t+\bar{k})-F(z+\bar{h}, t)-F(z, t+\bar{k})+F(z, t) < 0;$$

and therefore also

$$\{F(z+\bar{h}, t+\bar{k})-F(z+\bar{h}, t)-F(z, t+\bar{k})+F(z, t)\}/h\bar{k} < 0,$$

or, which is the same thing,

$$\{F(x+h, y+k)-F(x+h, y)-F(x, y+k)+F(x, y)\}/hk < 0,$$

where $(x+h, y+k)$ denotes the corner opposite to (x, y).

Now our mode of construction is such that we first make h tend to zero, k being fixed, by which means our rectangles shrink up to a portion of height k of the ordinate of the point (x, y), with that point as bottom extremity. The subsequent process causes k to approach zero, and these portions of the ordinate to shrink up to the point (x, y), while the rectangles do the same, each lying inside the preceding. The first progression to the limit tells us that all the limits of the incrementary ratio in the last inequality are negative or zero, when k is constant, that is

$$\operatorname*{Llt}_{h \to 0} \{F(x+h, y+k)-F(x+h, y)-F(x, y+k)+F(x, y)\}/hk \leqslant 0.$$

Since this is true for each value of k, as we proceed towards the

second limit, it remains true ultimately, and we have

$$\operatorname*{Llt}_{k\to 0}\{\operatorname*{Llt}_{h\to 0}[F(x+h, y+k)-F(x+h, y)-F(x, y+k)+F(x, y)]\} \leqslant 0.$$

That is, all the limits when $k\to 0$ of any one of the limits when $h\to 0$ of the incrementary ratio are negative or zero, when $k\to 0$ and $h\to 0$ in the appropriate manner, determined by the rectangles of the double succession defining the point (x, y) as above prescribed.

A fortiori, if we let h and k assume all possible values during their progression to the limit, the lowest limit of the incrementary ratio at the point (x, y) will be $\leqslant 0$, in other words *the lowest mixed derivate of* $F(x, y)$ *is* $\leqslant 0$; *we mean here derivation first with respect to* x *and then with respect to* y.

Having thus proved what we require when the point (x, y) is the corner of all the defining rectangles, we can easily prove the same without this supposition. I draw through the point (x, y) two straight lines parallel to the axes. These divide up each rectangle of my succession into two or four rectangles, one at least of which must give a negative increment. I retain only these rectangles, and thus reduce my succession to two or four successions of rectangles having the point (x, y) as corner. One at least of these must be infinite and serves to reduce this case to that already discussed.

Thus the set of points at which the lowest of these repeated derivates is $\leqslant 0$ contains an inner limiting set of potency c, lies on a set of ordinates of potency c, and projects on the axis of x into an inner limiting set of potency c, which accordingly contains a perfect set.

Taking the axis of y in place of the axis of x, we have a similar result for the repeated derivates in the contrary order.

5. To prove Theorem II, we require certain known results, which, for convenience of reference, we enunciate here without proof as Lemmas.

Lemma i.—*If* $f(x, y)$ *possesses an absolutely convergent double integral, there are an upper semi-continuous function* $u(x, y)$ *and a lower semi-continuous function* $l(x, y)$ *satisfying the conditions that*

$$u(x, y) \leqslant f(x, y) \leqslant l(x, y).$$

wherever $f(x, y)$ *is finite and that their double integrals over any convenient area are absolutely convergent and differ by less than an assigned quantity* e *as small as we please.**

* Cf. in particular "On Integrals and Derivates with respect to a Function", p. 38, *loc. cit.*, § 2.

As I have from time to time pointed out, this theorem holds for any number of dimensions.

LEMMA ii.—*All the double derivates (and therefore all the repeated derivates) of the double integral of an upper (lower) semi-continuous function are less (greater) than or equal to the value of that function at the point considered.*

LEMMA iii.—*If* $F(x, y) = \iint f(x, y)dx\,dy$

*is an absolutely convergent double integral, its repeated derivates agree and are equal to the integrand, except at the points of a plane set of content zero.**

6. We have now all the material required for the proof of the final theorem.

THEOREM II.—*If both the upper and the lower mixed derivates of a continuous function* $F(x, y)$ *possess absolutely convergent double integrals, and are finite except at most for the points of a set whose projection on the axis of* x *is countable (supposing the order of derivation to be first with respect to* x *and then with respect to* y*), then* $F(x, y)$ *is an absolutely continuous function, and is accordingly the integral of any one of its repeated derivates, whether in the order in question or the reverse order.*

The theorem remains true if, without knowing that the projection of the exceptional set is countable, we know that it contains no perfect subset, i.e. that $F(x, y)$ *is finite at some point of every perfect set of straight lines parallel to one of the axes.*

Let $f_1(x, y)$ denote the lower mixed derivate of $F(x, y)$ with the given order of derivation, namely, first with respect to x, and then with respect to y. By Lemma i, there is an upper continuous function $u(x, y)$ nowhere greater than $f_1(x, y)$—exception being made of the set specified in the enunciation, say the set E—and possessing an absolutely convergent double integral $U(x, y)$.

Consider the function $F-U$; its lower mixed derivate, being equal to a certain derivate of F minus a certain derivate of U, is not less than $f-u$, since f is the lower repeated derivate of F, and, by Lemma ii, all the derivates of U are less than or equal to u. Thus the lower repeated derivate of the function $F-U$ is $\geqslant 0$, except possibly at the points of the set E.

* "Sulla derivata mista di un integrale doppio," G. Fubini and L. Tonelli, *Rend. di Palermo*, Vol. 40 (1915), pp. 295–298. Cf. also a note by the present author in the *Comptes Rendus*, Vol. 164 (1917) p. 624.

It at once follows, by our Theorem I, that the double increment of $F-U$ is always $\geqslant 0$. For if it were negative, over any rectangle, the lower repeated derivate of $F-U$ would be negative at a set which could not coincide with the set E or be contained in it, since its projection on the axis of x would contain a perfect component.

Therefore adding, if necessary, to $F-U$ a function of x alone and a function of y alone,* we have a monotonely monotone increasing function of (x, y). Therefore our function $F(x, y)$ is an upper semi-integral.†

Similarly, using the upper instead of the lower mixed derivate, and a lower semi-continuous function which is nowhere less than it, we see that $F(x, y)$ is a lower semi-integral.

As it is thus both an upper and a lower semi-integral, $F(x, y)$ is a double integral, that is an absolutely continuous function. This proves the theorem.

7. We have for simplicity used ordinary derivates with two independent variables in what precedes; but in virtue of the results of my paper on integrals and derivates with respect to a function, we can extend our results so as to apply for derivates with respect to a monotonely monotone function $g(x, y)$. I do not propose to give the details of this development here. The reader will be able to make the necessary alterations in the

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* Denoting $F-U$ by G, $G(x, y)-G(a, y)-G(x, b)+G(a, b)$ has its increments with respect to (x, y), x and y all $\geqslant 0$, and is therefore what I call a monotonely monotone increasing function. Absorbing the terms $G(a, y)+G(x, b)-G(a, b)$ in the integral U, we see that F is the sum of an integral U and a monotonely increasing function, and is therefore by definition what I call an upper semi-integral. An upper (lower) semi-integral may be said to be absolutely upper (lower) semi-continuous, in that its variation over any set of points of content zero is zero or positive (negative). A function which is both upper and lower absolutely continuous is accordingly absolutely continuous, and therefore an integral, by the known property.

Lebesgue had been compelled to devise a somewhat complicated conception of little practical use to take the place of repeated differentiation.

† These latter functions can be scarcely defined as they are by Arzelà, and do not appear to possess bounded variation in a strictly two-dimensional sense, as they are the difference of two factors which possess only one-dimensional properties. Researches in which I have been for some time engaged have led me to conclude in the possibility and convenience of defining a relationship between the parts of two-ples of a nature generalising that between two lines embodied in the notion of a function of bounded variation of a single variable. If I am right Arzelà's function with perhaps an added edition and not a function referred above in the text would be the one indicated for such a purpose, x and y the coordinates of the one-ple would be respectively functions $x(u, v)$, $y(u, v)$ of the coordinates (u, v) in the other -ple. Perhaps I may add that I was many years since led to study some of the properties of these functions of Arzelà, and my subsequent adoption of the term function of bounded variation in a two-dimensional sense was on my part deliberate.

reasoning given above with the aid of my paper last quoted. The definition of a mixed derivate of $f(x, y)$ with respect to $g(x, y)$ is naturally based on the relative double increment

$$\frac{f(x+h, y+k)-f(x+h, y)-f(x, y+k)+f(x, y)}{g(x+h, y+k)-g(x+h, y)-g(x, y+k)+g(x, y)},$$

the mixed derivates in the order $\frac{d}{dy}\,\frac{d}{dx}$ being obtained as the limits of this expression when first h and then k tend to zero in any and every manner.

On the other hand, the method of monotone sequences gives us a satisfactory theory of repeated integration with respect to a monotonely monotone function of (x, y).

ON THE GENERATION OF THE GROUND FORM OF THE STATICAL GRAVITATION FIELD IN EINSTEIN'S THEORY OF GRAVITATION FROM THE GROUND FORM OF ANY SURFACE IN ORDINARY SPACE

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IF we take the ground form

$$f^2dt^2-a_{ik}dx_idx_k,$$

where f and a_{ik} are functions of x_1, x_2, x_3 only, and if it satisfies Einstein's gravitation equations it is said to be the ground form of a statical field.

The object of this paper is to show how such a ground form is to be obtained in the most general way possible.

It is shown how the form can be built up from the ordinary expression for the square of the element of length on a surface

$$a_{11}dx_1^2+2a_{12}dx_1dx_2+a_{22}dx_2^2.$$

The part of the ground form

$$a_{ik}dx_idx_k \quad (i, k = 1, 2, 3),$$

which refers to space rather than time satisfies the tensor equations belonging to this three-way form

$$\Delta_2(f) = 0, \qquad \{ittk\}+\frac{f_{ik}}{f} = 0,$$

where the four index symbols are the contracted tensors, and f_{ik} is the tensor derived by covariant differentiation from f, while $\Delta_2(f)$ is the second differential parameter.

It is shown that in proceeding from the two-way form we arrive at a three-way space which is not Euclidean; and simple expressions are

found for Riemann's measure of curvature, corresponding to the orientation given by any two elementary lines on an equipotential surface in this space, and corresponding to the orientation given by two elementary lines on any surface cutting the equipotential surface orthogonally.

I have freely used the method of tensor analysis which very much lessens the calculations necessary to establish the results given in the paper, but I have given, I hope, sufficient explanation of the method to render it intelligible.

1. Let us consider the expression

$$a_{ik}dx_i dx_k,$$

which, in the tensor notation, is briefly written for the sum of n^2 such terms obtained by giving to i, k independently the values 1, 2, ..., n.

The coefficients $a_{ik} = a_{ki}$ are at present arbitrarily assigned functions of the variables $x_1 \ldots x_n$.

Let a^{ik} denote the minor of a_{ik} in the determinant

$$a \equiv \begin{vmatrix} a_{11} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots \\ a_{n1} & \ldots & a_{nn} \end{vmatrix},$$

divided by the determinant itself.

The symbol e^i_k denotes zero if i and k are unequal, and unity if they are equal. We have

$$a^{it}a_{tk} = e^i_k,$$

the law of the tensor notation being that, wherever a suffix, which occurs in one factor of a product, is repeated in another factor, the sum of n^2 such terms is to be taken.

We have as the definition of Christoffel's symbols

$$(ikt) \equiv \tfrac{1}{2}\left(\frac{\partial a_{it}}{\partial x_k} + \frac{\partial a_{kt}}{\partial x_i} - \frac{\partial a_{ik}}{\partial x_t}\right),$$

$$\{ikj\} \equiv a^{jt}(ikt),$$

$$(rkih) \equiv \frac{\partial}{\partial x_h}(rik) - \frac{\partial}{\partial x_i}(rhk) + (ikt)\{rht\} - (hkt)\{rit\},$$

$$\{rkih\} \equiv a^{kt}(rtih),$$

and therefore

$$(ikj) = a_{jt}(ikt), \qquad (rkih) = a_{kt}\{rtih\}.$$

The three index symbols are not tensors, but the four index symbols are. We denote $\{rkih\}$ by A^k_{rih} and $(rkih)$ by A_{rkih}.

Let us now consider the operator

$$\bar{r} \equiv \frac{\partial}{\partial x_r} + \{tr\lambda\}\begin{pmatrix} t \\ \lambda \end{pmatrix} - \{\mu rt\}\begin{pmatrix} t \\ \mu \end{pmatrix},$$

where $\begin{pmatrix} t \\ \lambda \end{pmatrix}$ denotes that t is to be substituted for λ, λ being any integer in the upper row of the tensor component on which $\bar{r}$ operates, and $\begin{pmatrix} t \\ \mu \end{pmatrix}$ has a similar meaning with regard to the lower row of the tensor component.

Any tensor when operated on by $\bar{r}$ remains a tensor. Thus

$$\bar{r}A^k_{pih} = A^k_{pih*r},$$

where the star placed before any integer in the lower row of the component tensor indicates that the integers following denote operators.

We shall generally omit the star, unless for the moment we wish to emphasize the special way in which the tensor component is built up from the integers which precede it.

When we have an expression which we recognise as a tensor component and have reason to suppose it is zero, we may quickly test the correctness or otherwise of our supposition by the use of what are called Galilean coordinates at any given point.

We can always so choose our coordinates that at any given point a_{ik} is zero if i and k are unequal, and unity if they are equal, and that the first derivatives of a_{ik} vanish at the point. Such a system of coordinates are the Galilean coordinates at the point.

2. We now limit our consideration to the form

$$a_{ik}\,dx_i\,dx_k,$$

for the case where $n = 2$.

Let Ω_{ik} be a tensor which satisfies the differential equations

$$\Omega_{ik*r} = \Omega_{ir*r};$$

that is the two equations $\Omega_{11*2} = \Omega_{12*1}$,

$$\Omega_{22*1} = \Omega_{12*2}.$$

It follows that the tensor component

$$\Omega_{ikj}$$

is such that any of the integers may be interchanged: that is, the tensor is quite symmetrical.

Let ϕ be a function of the variables defined by

$$\tfrac{1}{2}a\phi^{-2} \equiv (1212) - \Omega_{11}\Omega_{22} + \Omega_{12}\Omega_{12},$$

where

$$a = a_{11}a_{22} - a_{12}a_{12}.$$

The coefficients a_{ik} are any assigned functions of x_1 and x_2, so that the expression

$$a_{ik}dx_i dx_k$$

is the square of the element of length on any two-way surface manifold.

We now wish to consider how the expression changes when we apply the infinitesimal transformations

$$a'_{ik} = a_{ik} - 2u\Omega_{ik}\phi,$$

$$\Omega'_{ik} = \Omega_{ik} + uV_{ik},$$

where V_{ik} is a tensor given by

$$V_{ik} \equiv \phi_{ik} - \frac{1}{2\phi}a_{ik} - a^{\lambda\mu}\Omega_{i\lambda}\Omega_{k\mu}\phi,$$

and u is a small constant.

Here ϕ_{ik} denotes ϕ_{*ik}, and, as we said, we are omitting the star.

We may denote the increment rate of any function with which we may be concerned by $\partial/\partial u$ placed before the function: thus

$$\frac{\partial}{\partial u}a_{ik} = -2\Omega_{ik}\phi,$$

$$\frac{\partial}{\partial u}\Omega_{ik} = V_{ik}.$$

The fundamental formula in the transformation theory of the variables is

$$\frac{\partial^2 x_k}{\partial x'_r \partial x'_s} + \{pqk\}\frac{\partial x_p}{\partial x'_r}\frac{\partial x_q}{\partial x'_s} = \{rst\}'\frac{\partial x_k}{\partial x'_t},$$

so that the symbol $\{ikj\}$ is not a tensor. But we see immediately from this formula that, although $\{ikj\}$ is not a tensor, $\partial/\partial u\ \{ikj\}$ is a tensor of the form T^j_{ik}.

Returning for a moment to the general formulæ of tensor operators,

we see that

$$\bar{p}\bar{q}-\bar{q}\bar{p} = \{t\lambda qp\}\begin{pmatrix} t \\ \lambda \end{pmatrix} + \{\mu tpq\}\begin{pmatrix} t \\ \mu \end{pmatrix},$$

where λ is any figure in the upper row, and μ any figure in the lower row of a tensor component.

Thus, ϕ being any function of the variables,

$$\phi_{*irk}-\phi_{*ikr} = (\bar{k}\bar{r}-\bar{r}\bar{k})\,\phi_i = \{itkr\}\,\phi_t.$$

In the case of the infinitesimal transformation we have been considering, we also see that

$$\frac{\partial}{\partial u}\bar{r} = \bar{r}\frac{\partial}{\partial u} + \left(\frac{\partial}{\partial u}\{tr\lambda\}\right)\begin{pmatrix} t \\ \lambda \end{pmatrix} - \left(\frac{\partial}{\partial u}\{\mu rt\}\right)\begin{pmatrix} t \\ \mu \end{pmatrix}.$$

We notice, coming back to the problem we are considering, that

$$\frac{\partial}{\partial u}\{rht\} + a^{pt}(\phi\Omega_{rhp}+\phi_r\Omega_{hp}+\phi_h\Omega_{rp}-\phi_p\Omega_{rh})$$

is a tensor component. Referring to Galilean coordinates at a given point, we see at once that it vanishes at the given point, and therefore vanishes identically.

3. We are now in the position to be able to prove that the equations

$$\Omega_{ir*k} = \Omega_{ik*r}$$

admit the given infinitesimal transformation. What we have to prove is that

$$\frac{\partial}{\partial u}(\Omega_{ir*k}-\Omega_{ik*r}) = 0,$$

that is, we have to prove that

$$V_{ir*k}-V_{ik*r}-\left(\frac{\partial}{\partial u}\{rkt\}\right)\Omega_{it}-\left(\frac{\partial}{\partial u}\{ikt\}\right)\Omega_{rt} + \left(\frac{\partial}{\partial u}\{irt\}\right)\Omega_{kt}+\left(\frac{\partial}{\partial u}\{krt\}\right)\Omega_{it} = 0.$$

It can be at once verified that every operator $\bar{r}$ annihilates a_{ik}, and therefore every function of the coefficients, though not of their derivatives.

From the definition of V_{ir}, we see that

$$V_{irk} = \phi_{irk}+\tfrac{1}{2}\phi^{-2}\,a_{ir}\phi_k-a^{pt}(\Omega_{ipk}\Omega_{rt}\phi+\Omega_{ip}\Omega_{rtk}\phi+\Omega_{ip}\Omega_{rt}\phi_k),$$

$$V_{ikr} = \phi_{ikr}+\tfrac{1}{2}\phi^{-2}\,a_{ik}\phi_r-a^{pt}(\Omega_{ipr}\Omega_{kt}\phi+\Omega_{ip}\Omega_{ktr}\phi+\Omega_{ip}\Omega_{kt}\phi_r)\,;$$

and therefore

$$V_{irk}-V_{ikr}=\{itkr\}\,\phi_t+\tfrac{1}{2}\phi^{-2}(a_{ir}\,\phi_k-a_{ik}\,\phi_r)$$
$$+a^{pt}\left(\Omega_{kt}(\Omega_{ipr}\phi+\Omega_{ip}\,\phi_r)-\Omega_{rt}(\Omega_{ipk}\phi+\Omega_{ip}\,\phi_k)\right).$$

We have now only to prove that

$$a^{pt}(ipkr)\,\phi_t+\tfrac{1}{2}\phi^{-2}(a_{ir}\,\phi_k-a_{ik}\,\phi_r)$$
$$+\Omega_{kt}\left(\frac{\partial}{\partial u}\{irt\}+a^{pt}(\Omega_{ipr}\,\phi+\Omega_{ip}\,\phi_r)\right)$$
$$-\Omega_{rt}\left(\frac{\partial}{\partial u}\{ikt\}+a^{pt}(\Omega_{ipk}\phi+\Omega_{ip}\,\phi_k)\right)=0.$$

Making use of the expressions we have found for $\frac{\partial}{\partial u}\{irt\}$ and $\frac{\partial}{\partial u}\{ikt\}$, we have to prove that

$$a^{pt}\left((ipkr)\,\phi_t+\phi_t(\Omega_{kp}\Omega_{ir}-\Omega_{rp}\Omega_{ik})\right)+\tfrac{1}{2}\phi^{-2}(a_{ir}\,\phi_k-a_{ik}\,\phi_r)=0;$$

and, since the integers can only be 1 or 2, we see that this expression vanishes, from the value given for ϕ.

The equations $$\Omega_{irk}=\Omega_{ikr},$$

therefore admit the given infinitesimal transformation.

4. We now notice that

$$\frac{\partial}{\partial u}(1212)+\bar{1}\bar{2}(\Omega_{12}\,\phi)+\bar{2}\bar{1}(\Omega_{12}\phi)-\bar{2}\bar{2}(\Omega_{11}\phi)-\bar{1}\bar{1}(\Omega_{22}\phi)$$
$$+\{1t12\}\,\phi\Omega_{t2}+\{2t21\}\,\phi\Omega_{t1}$$

is a tensor component.

We see this most easily on expanding it into the form

$$\frac{\partial}{\partial u}(1212)+\bar{1}(\Omega_{122}\,\phi+\Omega_{12}\phi_2-\Omega_{221}\phi-\Omega_{22}\phi_1)$$
$$+\bar{2}(\Omega_{121}\,\phi+\Omega_{12}\phi_1-\Omega_{112}\,\phi-\Omega_{11}\phi_2)$$
$$+a^{tp}(1p12)\,\phi\Omega_{t2}+a^{tp}(2p21)\,\phi\Omega_{t1}$$
$$=\frac{\partial}{\partial u}(1212)+\bar{1}(\Omega_{12}\phi_2-\Omega_{22}\phi_1)+\bar{2}(\Omega_{12}\phi_1-\Omega_{11}\phi_2)$$
$$+(a^{t2}\Omega_{t2}+a^{t1}\Omega_{t1})\,\phi(1212)$$
$$=\frac{\partial}{\partial u}(1212)+\Omega_{12}\phi_{21}+\Omega_{12}\phi_{12}-\Omega_{22}\phi_{11}-\Omega_{11}\phi_{22}$$
$$+\phi(1212)\,a^{\lambda\mu}\Omega_{\lambda\mu}.$$

For Galilean coordinates we see that the first form vanishes at a given point, and therefore, being a tensor, vanishes everywhere.

Again we see that

$$\frac{\partial}{\partial u}(\Omega_{11}\Omega_{22}-\Omega_{12}\Omega_{12}) = \Omega_{11}\phi_{22}+\Omega_{22}\phi_{11}-2\Omega_{12}\phi_{12}$$
$$-\tfrac{1}{2}\phi^{-1}(a_{22}\Omega_{11}+a_{11}\Omega_{22}-2a_{12}\Omega_{12})$$
$$-(\Omega_{11}\Omega_{22}-\Omega_{12}\Omega_{12})\,\phi a^{\lambda\mu}\Omega_{\lambda\mu};$$

and therefore
$$\frac{\partial}{\partial u}\left(\Omega_{11}\Omega_{22}-\Omega_{12}\Omega_{12}-(1212)\right)$$
$$= -\tfrac{1}{2}\phi^{-1}(a_{22}\Omega_{11}+a_{11}\Omega_{22}-2a_{12}\Omega_{12})$$
$$+\phi a^{\lambda\mu}\Omega_{\lambda\mu}\left((1212)-\Omega_{11}\Omega_{22}+\Omega_{12}\Omega_{12}\right)$$
$$= \phi a^{\lambda\mu}\Omega_{\lambda\mu}\left((1212)-\Omega_{11}\Omega_{22}+\Omega_{12}\Omega_{12}-\tfrac{1}{2}\phi^{-2}(a_{11}a_{22}-a_{12}a_{12})\right),$$

and this is zero by the definition of ϕ.

We thus see that the given infinitesimal transformation leaves unaltered the equations
$$\Omega_{irk} = \Omega_{ikr},$$

and the expression $\Omega_{11}\Omega_{22}-\Omega_{12}\Omega_{12}-(1212)$,

and therefore also $a\phi^{-2}$.

5. The transformation generated by such infinitesimal transformations will therefore leave these forms unaltered in magnitude, and we arrive at functions a'_{2k}, Ω'_{2k}, of the variables x_1, x_2, and a new variable u, which occurs as a parameter in

$$a'_{11},\quad a'_{12},\quad a'_{22},\quad \Omega'_{11},\quad \Omega'_{12},\quad \Omega'_{22},$$

but does not occur in $\Omega'_{11}\Omega'_{22}-\Omega'_{12}\Omega'_{12}-(1212)'$.

At the same time we arrive at a function ϕ' defined by

$$a'(\phi')^{-2} = a(\phi)^{-2},$$

where
$$\tfrac{1}{2}a\phi^{-2} = (1212)-\Omega_{11}\Omega_{22}+\Omega_{12}\Omega_{12};$$

and we have
$$\Omega'_{11*2} = \Omega'_{12*1},$$
$$\Omega'_{22*1} = \Omega'_{12*2}.$$

If we now omit accents and call the new variable u, x_3, we have

arrived at the form

$$\phi^2 dx_3^2 + a_{ik} dx_i dx_k \quad (i, k = 1, 2),$$

with the properties

$$\frac{\partial}{\partial x_3} a_{ik} + 2\Omega_{ik}\phi = 0,$$

$$\Omega_{112} = \Omega_{121},$$

$$\Omega_{221} = \Omega_{122},$$

$$\frac{a_{11}a_{22} - a_{12}a_{12}}{2\phi^2} = (1212) - \Omega_{11}\Omega_{22} + \Omega_{12}\Omega_{12}.$$

6. The form $\phi^2 dx_3^2 + a_{ik} dx_i dx_k \quad (i, k = 1, 2)$

will therefore have the following properties

$$(2323) = -\tfrac{1}{2}a_{22}, \quad (3131) = -\tfrac{1}{2}a_{11}, \quad (3123) = \tfrac{1}{2}a_{12},$$

$$(2312) = 0, \qquad (3112) = 0, \qquad (1212) = \frac{a_{11}a_{22} - a_{12}a_{12}}{2\phi^2}.$$

We see this by noticing that

$$(rkih) = (rkih)' + \Omega_{rh}\Omega_{ik} - \Omega_{ri}\Omega_{kh},$$

$$(rki3) = \phi(\Omega_{rik} - \Omega_{ikr}),$$

$$(r3i3) = \phi\left(\frac{\partial}{\partial x_3}\Omega_{ir} + a^{\lambda\mu}\Omega_{i\lambda}\Omega_{r\mu}\phi - \phi_{ir}\right),$$

where r, k, i, h, λ, μ can only have the values 1 or 2, and the dash in $(rkih)'$ shows that the symbol has only reference to the form

$$a_{ik} dx_i dx_k.$$

For the form $\phi^2 dx_3^2 + a_{ik} dx_i dx_k,$

we also have* $\Delta_2(x_3) = -a^{\lambda\mu}\{\lambda\mu 3\}$

$$= -a^{\lambda\mu}a^{3p}(\lambda\mu p)$$

$$= -a^{\lambda\mu}a^{33}(\lambda\mu 3);$$

and we at once see that $a^{\lambda\mu}(\lambda\mu 3) = 0,$

since $\frac{\partial}{\partial x_3}(a\phi^{-2}) = 0.$

It follows that for the form $\Delta_2(x_3) = 0.$

Expressed in terms of Christoffel's second kind of four index symbols, we

* Here λ and μ are to be taken for the values 1, 2, and 3.

have $$\{2tt3\} = 0, \quad \{3tt1\} = 0, \quad \{1tt2\} = 0,$$
$$\{1tt1\} = 0, \quad \{2tt2\} = 0, \quad \{3tt3\} = 1.$$

The form at which we have arrived

$$\phi^2 dx_3^2 + a_{ik} dx_i dx_k \quad (i, k = 1, 2),$$

satisfies then the conditions that all the contracted Christoffel symbols vanish, except $\{3tt3\}$ which is unity, and that $\Delta_2(x_3) = 0$.

We can see conversely that every such form can be generated in the manner we have explained.

7. If we now transform to any variables we get the general result that the three way form

$$a_{ik} dx_i dx_k \quad (i, k = 1, 2, 3),$$

which defines the three way space has the properties that

$$\Delta_2(V) = 0,$$
$$\{2tt3\} = V_2 V_3, \quad \{3tt1\} = V_3 V_1, \quad \{1tt2\} = V_1 V_2,$$
$$\{1tt1\} = V_1 V_1, \quad \{2tt2\} = V_2 V_2, \quad \{3tt3\} = V_3 V_3.$$

Here V is a potential function for this space. The form of this space can be at once reduced to the form

$$\phi^2 dx_3^2 + a_{ik} dx_i dx_k \quad (i, k = 1, 2),$$

by taking $$x_3 = V, \quad x_1 = U, \quad x_2 = W,$$

where $$\nabla(V, U) = 0, \quad \nabla(V, W) = 0,$$

the symbol ∇ being used for the mixed differential parameter.

8. If K is the measure of curvature of an equipotential surface according to Gauss; that is, when we regard the expression for the square of the element of length on the surface as the square of the element of length on a surface lying in Euclidean space; and if K_0 is the reciprocal of the product of the principal radii of curvature when we regard the equipotential surface as lying, not in Euclidean space, but in the space we have arrived at, then Riemann's measure of curvature corresponding to the orientation given by any two line elements on the equipotential surface is

$$K - K_0,$$

and his measure of curvature corresponding to the orientation given by

any two line elements on any surface cutting the equipotential surface orthogonally is*

$$K_0 - K.$$

9. We may take the form for the three way space we are now considering as

$$b_{ik}\,dx_i\,dx_k,$$

where the coefficients are such that

$$\Delta_2(V) = 0,$$

$$\{ittk\} + 2V_i\,V_k = 0.$$

This form is deduced from the one previously used by taking as the new V, $\iota/\sqrt{2}$ times the old one.

Let $$f = e^{-V}, \quad a_{ik} = e^{2V} b_{ik},$$

and consider now the expression

$$a_{ik}\,dx_i\,dx_k.$$

We easily verify that

$$\{ikj\}_a = \{ikj\}_b + a^{jt}(a_{it}\,V_k + a_{kt}\,V_i - a_{ik}\,V_t),$$

where the suffix b refers to the form

$$b_{ik}\,dx_i\,dx_k,$$

and the suffix a refers to the form

$$a_{ik}\,dx_i\,dx_k.$$

We notice that $$a^{ik} = e^{-2V} b^{ik},$$

and that the expression

$$a^{jt}\,(a_{it}\,V_k + a_{kt}\,V_i - a_{ik}\,V_t)$$

is the same for either of the forms.

Let $$\theta^k \equiv a^{tk}\theta_t,\dagger$$

and therefore $$\theta_k = a_{tk}\,\theta^t.$$

The expression $$a^{jt}(a_{it}\,V_k + a_{kt}\,V_i - a_{ik}\,V_t)$$

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* As regards the connexion between radii of principal curvature, Gauss's measure of curvature of a surface and Riemann's measure of curvature of space, see Bianchi, *Differential Geometry*, German Translation, § 334.

† The suffix here does not, of course, indicate a power, but is used in accordance with tensor notation.

is equal to $$e^j_i V_k + e^j_k V_i - a_{ik} V^j,$$

and we will denote it by V^j_{ik}.

Similarly, θ being any function, the corresponding function

$$e^j_i \theta_k + e^j_k \theta_i - a_{ik} \theta^j,$$

would be denoted by θ^j_{ik}.

We see that $$\Delta(\theta) = \theta^t \theta_t,$$

$$\Delta_2(\theta) = \bar{t}\, \theta^t.$$

It is easy to verify the following relations:

$$\theta^t_{rh} \theta^i_{ti} = 6\theta_r \theta_h - 3a_{rh} \Delta(\theta),$$

$$\theta^i_{th} \theta^t_{ri} = 5\theta_r \theta_h - 2a_{rh} \Delta(\theta),$$

$$\theta^i_{ri*h} - \theta^i_{rh*i} = \theta_{rh} + a_{rh} \Delta_2(\theta).$$

We also see that

$$\big(\Delta_2(V)\big)_a = e^{-2V} \big(\Delta_2(V)\big)_b + \big(\Delta(V)\big)_a,$$

and that $$\big(\Delta_2(e^{-V})\big)_a = -e^{-V} \Big(\big(\Delta_2(V)\big)_a - \big(\Delta(V)\big)_a\Big);$$

and therefore that, since $\big(\Delta_2(V)\big)_b = 0$, we have

$$\big(\Delta_2(e^{-V})\big)_a = 0.$$

It follows that

$$V^i_{rih} - V^i_{rhi} + V^t_{rh} V^i_{ti} - V^i_{th} V^t_{ri} = (V_{rh})_a + V_r V_h.$$

Now $$\{ikj\}_a = \{ikj\}_b + V^j_{ik},$$

and $$\{rkih\} = \frac{\partial}{\partial x_h} \{rik\} - \frac{\partial}{\partial x_i} \{rhk\} + \{rit\}\{thk\} - \{rht\}\{tik\};$$

and therefore

$$\begin{aligned} \{rkih\}_a = \{rkih\}_b &+ \frac{\partial}{\partial x_h} V^k_{ri} - \frac{\partial}{\partial x_i} V^k_{rh} \\ &+ \{rit\}_a V^k_{th} + V^t_{ri} (\{thk\}_a - V^k_{th}) \\ &- \{rht\}_a V^k_{ti} - V^t_{rh} (\{tik\}_a - V^k_{ti}) \\ = \{rkih\}_b &+ V^k_{rih} - V^k_{rhi} + V^k_{ti} V^t_{rh} - V^k_{th} V^t_{ri}. \end{aligned}$$

We thus obtain the formula

$$\{rtth\}_a = \{rtth\}_b + V^i_{rih} - V^i_{rhi} + V^i_{ti} V^t_{rh} - V^i_{th} V^t_{ri}$$
$$= \{rtth\}_b + (V_{rh})_a + V_r V_h ;$$

and we see that $$\{rtth\}_a + V_r V_h - (V_{rh})_a = 0,$$

$$\left(\Delta_2(e^{-V})\right)_a = 0.$$

Now $$f_{rh} = -f(V_{rh}) - V_r V_h ;$$

and therefore, for the form $a_{ik} dx_i dx_k$,

$$\{rtth\} + \frac{f_{rh}}{f} = 0,$$

where $$\Delta_2 (f) = 0.$$

We thus see how to obtain the three-way form with these properties.

10. From this form we can immediately obtain the four-way form

$$f^2 dt^2 - a_{ik} dx_i dx_k \quad (i, k = 1, 2, 3)$$

(t being a new variable which does not occur in f or a_{ik}), with the properties

$$\left(\Delta_2(f)\right)_a = 0, \qquad \{ittk\}_a + \left(\frac{f_{ik}}{f}\right)_a = 0,$$

where the suffix a indicates the part of the form to which the symbols relate.

If we use the symbol g to refer to the four-way form, we now see at once that

$$\{ittk\}_g = 0,$$

and the form is that necessary for a stationary gravitation field.

PONCELET'S PORISTIC POLYGONS

By T. W. CHAUNDY.

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1. *Preamble.*

The condition (arising from Poncelet's theorem), which is necessary in order that a single infinitude of n-sided polygons can be inscribed in a conic S and at the same time circumscribed to a conic S', has been investigated by many writers,* and is completely determinate in the sense that the condition can be stated in terms of elliptic functions or can be inductively established in purely algebraic terms by means of suitable recurrence-formulæ. For increasing values of n, however, the formulæ rapidly become unmanageable, and there still seems need of algebraic simplification of the condition. Certain possibilities of such simplification appear, when the two conics (by homographic transformation or otherwise) are replaced by two circles, and when appropriate use is made of radical signs, *i.e.* when irrational factors are admitted. The conditions then assume, in a number of cases, a comparatively simple form.

As usual R denotes the radius of the circle circumscribed to the polygons, r the radius of the circle inscribed in them, and d the distance between the centres of the circles. It will also at times be convenient to denote by $P[n]$ the condition or "porism" for two conics or two circles, the polygons being n-sided.

The condition for the triangle due to Euler is

$$R^2-d^2 = 2Rr, \tag{1}$$

and is classical.

More strictly I should write $R^2-d^2 = \pm 2Rr$, but I shall everywhere

* Reference may be specially made to Darboux, *Principes de Géométrie Analytique*, Livre 3 (1917), Chaps. 2, 3; Halphen, *Traité de Fonctions Elliptiques*, tome 2, p. 370 on; and Cayley, *Phil. Trans.*, 1861, p. 225.

avoid the confusion of double signs by supposing the ambiguity of sign absorbed into R, r, or d, as the case may be. This is possible, since these lengths enter the analysis defined only by their squares. In the final formulæ, then, we may take R, r, d of arbitrary but consistent signs. I shall treat the ambiguity of radical signs in similar fashion.

Two forms of condition for the quadrilateral are known :*

$$R = d, \tag{2}$$

$$(R+d)^{-2}+(R-d)^{-2} = r^{-2}. \tag{3}$$

They are evidently distinct conditions.

In a note in *Crelle's Journal*,† Steiner states, in addition to (1) above, the following conditions :

For the quadrilateral

$$(R^2-d^2)^2 = 2r^2(R^2+d^2),$$

and $$r^4 = (R+r+d)(R+r-d)(R-r+d)(R-r-d),$$

each equivalent to (3).

For the pentagon,

$$r(R-d) = (R+d)\sqrt{\{(R-r+d)(R-r-d)\}}+(R+d)\sqrt{\{(R-r-d)2R\}}. \tag{4}$$

This, as I shall show later, contains an irrelevant factor appropriate to the triangle.

For the hexagon,

$$3(R^2-d^2)^4 = 4r^2(R^2+d^2)(R^2-d^2)^2+16r^4d^2R^2. \tag{5}$$

For the octagon,

$$\begin{aligned} 8r^2\{(R^2-d^2)^2-r^2(R^2+d^2)\}\{(R^2+d^2)[(R^2-d^2)^4+4r^4d^2R^2] \\ -8r^2d^2R^2(R^2-d^2)^2\} \\ = \{(R^2-d^2)^4-4r^4d^2R^2\}^2. \end{aligned} \tag{6}$$

This condition is incorrect, as is pointed out in the *Encyclopädie*.

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* I find these in a paper by Rosanes and Pasch, *Crelle*, Vol. 64, § 9, pp. 163, 164. I do not know if there are earlier references.

† *Crelle*, Vol. 2 (1827), p. 289.

Richelot in *Crelle*, Vol. 38, p. 353, gives conditions for the pentagon, heptagon, and enneagon. They are expressed, not too simply, in rational form, and are, I think, not very relevant to my point of view.

Mr. E. C. Titchmarsh has stated* a condition for the hexagon

$$R^2-d^2 = 2rd, \tag{8}$$†

and a condition for the octagon

$$R^2-d^2 = 2r\sqrt{(Rd)}. \tag{9}$$

I propose in this paper to discuss certain of these conditions and to add to them.

2. *Synopsis of the General Theory.*

A poristic relation between two conics is essentially an invariant relation, and so is very naturally to be discussed in terms of Δ, Θ, Θ', Δ', the invariants of the two conics. A still more natural and convenient expression is in terms of k_1, k_2, k_3, the invariant roots of the invariant cubic

$$\Delta k^3+\Theta k^2+\Theta' k+\Delta' = 0.$$

If the polygon is even-sided, a further possibility of simplification arises from the known fact that the median diagonals (*i.e.* those joining *diametrically* opposite vertices) pass all and always through one vertex of the common self-polar triangle of the two conics. To select one such vertex is to select one out of k_1, k_2, k_3.

The porism for a $2n$-agon is thus the symmetrical product of three factors, each asymmetrical but of similar form, and hence indistinguishable, if the two conics are unspecialised. Thus for the quadrilateral we have, as is well-known,

$$k_2k_3 = k_1(k_2+k_3) \quad \text{or} \quad k_3k_1 = k_2(k_3+k_1) \quad \text{or} \quad k_1k_2 = k_3(k_1+k_2).$$

Now the common self-polar triangle of two circles is formed by K, the point at infinity on the radical axis, and L, L', the two limiting points. The point K is separable from L, L', but L, L' are themselves insepar-

* *Messenger of Mathematics*, July 1922, Vol. 52, p. 42.

† The equivalent form $r/(R+d)-r/(R-d) = 1$ is given in the paper by Rosanes and Pasch already cited.

able, being, for instance, conjugate imaginaries, if the circles intersect. Thus according as the point of concurrence of the median diagonals of the $2n$-agon is at K or at one of L, L', so we must expect *two* distinct factors of the porism, indicating two distinct configurations of the pair of circles, which I will call respectively the "separable case" and the "inseparable cases." It is to be presumed (and experience confirms) that the factor of the porism representing the separable case will be much simpler than the composite factor representing the inseparable cases, and it is only with such "separable" factor of $P[2n]$ that I shall, in general, be concerned.

The analysis, however presented, must follow, essentially, the algebra of the elliptic functions. We may define a pencil of conics by the equations

$$S \equiv x^2+y^2+z^2 = 0, \quad S_\lambda \equiv S'+\lambda S \equiv a_\lambda^2 x^2+b_\lambda^2 y^2+c_\lambda^2 z^2 = 0,$$

where $a_\lambda^2-b_\lambda^2$, $b_\lambda^2-c_\lambda^2$, $c_\lambda^2-a_\lambda^2$ are equal to some a^2-b^2, b^2-c^2, c^2-a^2, independent of λ, and we may correlate the variable conic S_λ of the pencil with an elliptic argument u_λ by either of the equivalent substitutions

$$a_\lambda = i \operatorname{sn} u_\lambda, \quad b_\lambda = \operatorname{cn} u_\lambda, \quad c_\lambda = k^{-1} \operatorname{dn} u_\lambda,$$

where

$$k^2 = (a^2-b^2)/(a^2-c^2),$$

or

$$a_\lambda^2 = \wp(u_\lambda)-e_1, \quad b_\lambda^2 = \wp(u_\lambda)-e_2, \quad c_\lambda^2 = \wp(u_\lambda)-e_3,$$

where

$$a^2+e_1 = b^2+e_2 = c^2+e_3 = \tfrac{1}{3}(a^2+b^2+c^2).$$

The conic of the pencil, so defined by an elliptic argument u, may be known simply as "the conic $\{u\}$." Poncelet's theorem then essentially states that, if a variable triangle be always inscribed in S and two of its sides envelop the conics $\{u_1\}$, $\{u_2\}$, then the third side envelops one of the conics $\{\pm u_1 \pm u_2\}$. Hence, too, if a variable n-agon be inscribed in S, and $n-1$ of its sides envelop the conics $\{u_1\}$, $\{u_2\}$, ..., $\{u_{n-1}\}$, then the n-th side envelops one of the conics $\{\pm u_1 \pm u_2 \pm \ldots \pm u_{n-1}\}$.

The ambiguity of sign can be accounted for,* if we attempt to construct the open polygon, choosing A_0 an arbitrary point of S as first vertex. A tangent from A_0 to $\{u_1\}$ cuts S again in A_1; a tangent from A_1 to $\{u_2\}$ cuts S again in A_2, and so forth. But since from any A_r two tangents can always be drawn to $\{u_{r+1}\}$, we have, given A_r, two possible positions of A_{r+1}, and just so much ambiguity enters as is represented by the double signs in $\{\pm u_1 \pm u_2 \pm \ldots \pm u_{n-1}\}$.

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* Rohn discusses this question of sign at some length in a paper in *Leipz. Bericht.*, Vol. 60 (1908), p. 94.

If, now, we make all the conics $\{u_1\}, \ldots, \{u_{n-1}\}$ identical, we have inscribed in S a variable n-agon $A_0A_1 \ldots A_{n-1}$ whose $n-1$ sides $A_0A_1, A_1A_2, \ldots, A_{n-2}A_{n-1}$ all touch $\{u\}$, say. The n-th side therefore touches one of the conics $\{\pm u \pm u \pm \ldots \pm u\}$. But the two tangents from A_r to $\{u\}$ are now just A_rA_{r+1} and A_rA_{r-1}. If we choose A_{r-1} instead of A_{r+1}, we are retracing our way along the polygon, subtracting instead of adding a side, and so subtracting instead of adding u. The envelope of $A_{n-1}A_0$ is accordingly the unique conic $\{(n-1)u\}$.

If, in addition, this last side $A_{n-1}A_0$ itself envelops $\{u\}$, then we have the closed polygon $A_0A_1 \ldots A_{n-1}A_0$, inscribed in S and circumscribed to $\{u\}$, *i.e.* the poristic condition is obtained by identifying $\{u\}$, $\{(n-1)u\}$. But now that the polygon is closed, we might equally well commence with A_r and describe firstly the $(r+1)$-agon $A_rA_{r-1} \ldots A_0$, whose last side A_rA_0 envelops $\{ru\}$, and then, in the opposite sense, the $(n-r+1)$-agon $A_rA_{r+1} \ldots A_{n-1}A_0$ whose last side A_rA_0 envelops $\{(n-r)u\}$. The poristic condition is then also to be obtained by identifying $\{ru\}$ and $\{(n-r)u\}$. Now $\{u\}$ and $\{v\}$ are identical, if and only if, one of $u \pm v$ is a period of the elliptic functions a_λ^2, b_λ^2, c_λ^2. Thus we can identify the pairs of conics $\{ru\}$, $\{(n-r)u\}$ for every r, only if nu is a period. Mere identification of $\{(n-r)u\}$ and $\{ru\}$ for one value of r alone might have deceived us with the porism $P[n-2r]$. We must therefore be careful that our porism for n is not entangled with factors appropriate to $n-2$, $n-4$, etc. Such presumably is the explanation of the irrelevant factor (poristic for the triangle) which occurs in Steiner's condition (4) for the pentagon. We avoid this danger, if we take r and $n-r$ as nearly equal as possible in the identification of $\{ru\}$, $\{(n-r)u\}$.

3. *Fundamental Formulæ.*

Proceeding now to the analysis I take $a^2x^2+b^2y^2+c^2z^2$ as the conic $\{u\}$, and $a_n^2x^2+b_n^2y^2+c_n^2z^2$ as the conic $\{nu\}$, where still

$$a_n^2-a^2 = b_n^2-b^2 = c_n^2-c^2 = \lambda_n. \tag{10}$$

The elliptic addition-formula is then conveniently quoted in the shapes

$$\left\{\begin{aligned} a_{m+n} &= \frac{b_nc_na_m-b_mc_ma_n}{\lambda_n-\lambda_m}, \\ a_{m-n} &= \frac{b_nc_na_m+b_mc_ma_n}{\lambda_n-\lambda_m}. \end{aligned}\right. \tag{11}$$

These formulæ impose conventions on the signs of a_n, b_n, c_n which otherwise enter only through their squares; it can be shown that these implicit conventions are self-consistent. Write now

$$A^4 \equiv (a^2-b^2)(a^2-c^2) = (a_r^2-b_r^2)(a_r^2-c_r^2) \tag{12}$$

for any r, and

$$a_n \equiv a_n/A. \tag{13}$$

From (11) by use of (10), (12), (13) and reduction, we obtain

$$a_{m+n}a_{m-n} = \frac{a_m^2 a_n^2-1}{a_n^2-a_m^2}. \tag{14}$$

From (11) we have also

$$\operatorname*{Lt}_{m\to n} a_{m-n}(\lambda_n-\lambda_m) = 2a_n b_n c_n.$$

Hence, from (14),

$$a_{2n} = \frac{(a_n^4-1)A^4}{2a_n b_n c_n}. \tag{15}$$

Now $P[2n]$ requires that $a_r = -a_{2n-r}$ for every r, and hence, in particular, that $a_n = 0$.* Thus $P[2m+2n]$ requires that $a_{m+n} = 0$, *i.e.* from (14) $a_m a_n = 1$, as we have choice of sign. In particular $P[4n]$ requires that $a_n^2 = 1$, *i.e.* sufficiently $a_n = 1$.

I collect for reference

$$\begin{cases} P[m+n]: & a_m = -a_n, \\ P[2m+2n]: & a_m a_n = 1, \\ P[4n]: & a_n = 1, \end{cases} \tag{16}$$

When the conics become circles, we may write

$$S \equiv R^{-2}(x^2+y^2)-1 \quad \text{and} \quad \{u\} \equiv (x-d)^2+y^2-r^2.$$

Forming the discriminant of $\{u\}+kS$, we find

$$a^2 = R^2, \quad b^2+c^2 = R^2+r^2-d^2, \quad bc = Rr, \tag{17}$$

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* $a_n = 0$, *i.e.* $a_n = 0$ is the condition that $\{nu\}$, the envelope of the median diagonals of the polygon, reduce to a vertex of the common self-polar triangle. It is not equivalent to $b_n = 0$ or $c_n = 0$. But $a_m = -a_n$ and $b_m = -b_n$ *are* equivalent. This illustrates what has been said as to the "separable" case for an even-sided polygon.

and hence

$$A = \sqrt{(Rd)}, \quad a = \sqrt{(R/d)}, \tag{18}$$

and from (15), we have

$$a_2 = \frac{R^2-d^2}{2r}, \qquad \alpha_2 = \frac{R^2-d^2}{2r\sqrt{(Rd)}}. \tag{19}$$

From these formulæ emerges an odd fact, namely, that, if we interchange R, d, then a_{2n-1} becomes $1/\alpha_{2n-1}$, and a_{2n} becomes $-\alpha_{2n}$ for every n. From (18), (19) this is evidently so for $n = 1$. I shall prove it universal by induction. For suppose it true for every n up to some a_{2r-1}, a_{2r} inclusive, and let a_{2r+1}, a_{2r+2} become α'_{2r+1}, α'_{2r+2} by the interchange. Then, by (14),

$$a_{2r+1}\alpha = \frac{\alpha_r^2\alpha_{r+1}^2-1}{\alpha_r^2-\alpha_{r+1}^2},$$

and

$$\alpha'_{2r+1}\left(-\frac{1}{\alpha}\right) = \frac{\alpha_r^2/\alpha_{r+1}^2-1}{\alpha_r^2-1/\alpha_{r+1}^2}, \text{ if } r \text{ is even,}$$

or

$$\frac{\alpha_{r+1}^2/\alpha_r^2-1}{1/\alpha_r^2-\alpha_{r+1}^2}, \text{ if } r \text{ is odd.}$$

In either case

$$\alpha'_{2r+1} = -1/a_{2r+1}.$$

Similarly

$$a_2 a_{2r+2} = \frac{\alpha_r^2\alpha_{r+2}^2-1}{\alpha_r^2-\alpha_{r+2}^2}.$$

The expression on the right is unaltered by the interchange, if r is even. If r is odd,

$$-\alpha_2\alpha'_{2r+2} = \frac{1/\alpha_r^2\alpha_{r+2}^2-1}{1/\alpha_r^2-1/\alpha_{r+2}^2} = a_2 a_{2r+2},$$

i.e.

$$\alpha'_{2r+2} = -a_{2r+2}.$$

Hence the result is established universally.

Now, from (16), the condition $P[2n+1]$ is $a_n = -\alpha_{n+1}$. If we interchange R, d, this becomes $a_n a_{n+1} = 1$, *i.e.* the condition $P[2(2n+1)]$.

(20) *Thus the conditions $P[2n+1]$ and $P[2(2n+1)]$ can be derived from each other by mere interchange of R and d.*

Evidently the interchange of R, d does not alter $\alpha_n = 1$, *i.e.* the condition $P[4n]$ is symmetrical in R and d.

We can at once establish certain conditions, for, by (16),

$$P[4] \text{ requires } \alpha = 1,$$

$$P[3] \quad ,, \quad \alpha_2 = -\alpha,$$

$$P[6] \quad ,, \quad \alpha_2\alpha = 1,$$

$$P[8] \quad ,, \quad \alpha_2 = 1.$$

Hence, by (18), (19),

$$R = d \qquad \text{for the quadrilateral,}$$

$$R^2 - d^2 = 2rR \qquad \text{for the triangle,}$$

$$R^2 - d^2 = 2rd \qquad \text{for the hexagon,}$$

$$R^2 - d^2 = 2r\sqrt{(Rd)} \qquad \text{for the octagon.}$$

These are the conditions (2), (1), (8), (9) already stated in the preamble. We may notice that, in accordance with the principle of (20), the conditions for the triangle and the hexagon are related by the interchange of R and $-d$, and that those for the quadrilateral and the octagon are symmetrical (save for permissible changes of sign) in R and d. I should add that the porisms for the three even-sided figures refer to the "separable" case.

Of the other conditions quoted in the preamble, (3), (5) give the "inseparable" cases of the quadrilateral and the hexagon. They contrast unfavourably, as regards simplicity, with the formulæ for the corresponding "separable" case. Steiner's condition (6) for the octagon was presumably (from its complexity) intended for the "inseparable" cases.

It is certainly not satisfied by the condition (9) for the "separable" configuration, and is inconsistent with the actual "inseparable" condition, which may be written

$$(21) \quad \{(R^2-d^2)^2-2r^2(R^2+d^2)\}^4 = 16r^4(R^2-d^2)^4\{(R^2+r^2-d^2)^2-4R^2r^2\}.*$$

It is sometimes a useful test of accuracy to put $d = 0$ in a formula.

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* The inconsistency is seen, for instance, by putting $R^2 + d^2 = 0$ in the two formulæ.

The circles become concentric and the polygons regular, *i.e.* $r = R\cos\pi/n$. If n is even, this applies only to the "inseparable" configuration. The "separable" case associates $R = 0$ with $d = 0$.

4. *The Pentagon and the Decagon.*

For the pentagon we have, by (16),

$$a_3 + a_2 = 0.$$

From (14) we derive $$aa_3 = \frac{a^2a_2^2 - 1}{a^2 - a_2^2},$$

and hence $$aa_2^3 - a^2a_2^2 - a^3a_2 + a^4 = a^4 - 1.$$

This gives $$a(a_2 - a)^2(a_2 + a) = a^4 - 1,$$

and $$a(a_2 + a)^3 = (a^2 + 2aa_2)^2 - 1.$$

Dividing and taking the square root,

$$\frac{a_2 + a}{a_2 - a} = \left\{1 + \frac{2aa_2}{a^2+1}\right\}^{\frac{1}{2}} \left\{1 + \frac{2aa_2}{a^2-1}\right\}^{\frac{1}{2}}.$$

By (18) and (19),

$$\left\{1 + \frac{2aa_2}{a^2+1}\right\}^{\frac{1}{2}} = \left\{\frac{R - d + r}{r}\right\}^{\frac{1}{2}} \equiv u, \text{ say,}$$

and $$\left\{1 + \frac{2aa_2}{a^2-1}\right\}^{\frac{1}{2}} = \left\{\frac{R + d + r}{r}\right\}^{\frac{1}{2}} \equiv v, \text{ say.}$$

Write also $$w \equiv \sqrt{(u^2 + v^2 - 2)} = \sqrt{(2R/r)}.$$

We have from these definitions

$$\frac{u^2v^2 - u^2 - v^2 + 1}{u^2 + v^2 - 2} = \frac{a_2}{a},$$

and the poristic condition may be written

$$\frac{uv}{uv - 1} = \frac{a_2 + a}{2a} = \frac{u^2v^2 - 1}{2w^2}.$$

This gives $$(1 - uv)^2(1 + uv) = 2uvw^2,$$

and $$(1 + uv)^3 = 2uv(u + v)^2.$$

By division and extraction of the square root,

$$(1+uv)w = (1-uv)(u+v).$$

If we use the equation $w^2 = u^2+v^2-2$, we may write this condition in the symmetrical forms

$$(22) \qquad (v^2+w^2-u^2)u+(w^2+u^2-v^2)v+(u^2+v^2-w^2)w+2uvw = 0,$$

$$(23) \qquad (R+d)\sqrt{(R-d+r)}+(R-d)\sqrt{(R+d+r)}+r\sqrt{(2R)} \\ +\sqrt{\{2R(R-d+r)(R+d+r)\}} = 0.$$

The condition (22) may also be written

$$(23a) \qquad (u+v+w)^{-1}+\sum_{u,\,v,\,w}(-u+v+w)^{-1} = 0.$$

Hence

$$(24) \qquad [\sqrt{(R-d+r)}+\sqrt{(R+d+r)}+\sqrt{(2R)}]^{-1} \\ +[-\sqrt{(R-d+r)}+\sqrt{(R+d+r)}+\sqrt{(2R)}]^{-1} \\ +[\sqrt{(R-d+r)}-\sqrt{(R+d+r)}+\sqrt{(2R)}]^{-1} \\ +[\sqrt{(R-d+r)}+\sqrt{(R+d+r)}-\sqrt{(2R)}]^{-1} = 0.$$

By interchange of R and d, we have for the decagon

$$(25) \qquad (d+R)\sqrt{(d-R+r)}+(d-R)\sqrt{(d+R+r)} \\ +r\sqrt{(2d)}+\sqrt{\{2d(d-R+r)(d+R+r)\}} = 0,$$

and an equivalent condition deducible from (24).

It is just a little odd that, if in (23) we take an alternative arrangement of signs, viz.

$$(R+d)\sqrt{(R-d+r)}+(R-d)\sqrt{(R+d+r)}+r\sqrt{(2R)} \\ -\sqrt{\{2R(R-d+r)(R+d+r)\}} = 0,$$

we have merely a disguised form of $P[3]$. For with such signs (22) has become

$$\sum_{u,\,v,\,w}(v^2+w^2-u^2)u-2uvw = 0,$$

i.e.

$$\prod_{u,\,v,\,w}(-u+v+w) = 0.$$

The three factors are essentially indistinguishable, and it is sufficient to

write $u+v=w$, that is

$$\sqrt{(R-d+r)}+\sqrt{(R+d+r)}=\sqrt{(2R)}, \tag{26}$$

i.e. $$r^2=(R+r)^2-d^2,$$

i.e. $$R^2-d^2=-2Rr,$$

which is $P[3]$. This also establishes as a form for the triangle

$$u+v+w=0, \tag{26a}$$

which I shall subsequently need.

By interchange of R, d, we have the associated form for the hexagon

$$\sqrt{(d-R+r)}+\sqrt{(d+R+r)}=\sqrt{(2d)}.$$

Steiner's condition (4) for the pentagon may be written, after change of sign of r, as

$$(R+d)\sqrt{(R-d+r)}\{\sqrt{(R+d+r)}+\sqrt{(2R)}\}+r(R-d)=0,$$

i.e. $$2u(v+w)(v^2+w^2-u^2)+(u^2+v^2-w^2)(u^2-v^2+w^2)=0.$$

The expression on the left can be split into two factors, of which one is the expression on the left of (22), and the other is $v+w-u$ representing $P[3]$.

5. *The Octagon and the* 16*-agon.*

From the condition (9), $R^2-d^2=2r\sqrt{(Rd)}$, already established for the octagon, we can derive various other forms. Thus we can verify, by substitution for r, that

$$\begin{aligned}
2(r+R-d)\sqrt{(Rd)} &= (\sqrt{R}+\sqrt{d})^3(\sqrt{R}-\sqrt{d}),\\
2(r-R+d)\sqrt{(Rd)} &= (\sqrt{R}-\sqrt{d})^3(\sqrt{R}+\sqrt{d}),\\
2(r+iR+id)\sqrt{(Rd)} &= (\sqrt{R}+i\sqrt{d})^3(\sqrt{R}-i\sqrt{d}),\\
2(r-iR-id)\sqrt{(Rd)} &= (\sqrt{R}-i\sqrt{d})^3(\sqrt{R}+i\sqrt{d}).
\end{aligned}$$

The simplest deductions are perhaps

$$\frac{\sqrt{(r-R+d)}}{\sqrt{(r+R-d)}}=\frac{\sqrt{R}-\sqrt{d}}{\sqrt{R}+\sqrt{d}}, \quad \frac{\sqrt{(r-iR-id)}}{\sqrt{(r+iR+id)}}=\frac{\sqrt{R}-i\sqrt{d}}{\sqrt{R}+i\sqrt{d}} \tag{27}$$

and

$$\begin{cases} R-d=\sqrt[4]{\{4Rd(r-R+d)(r+R-d)\}},\\ R+d=\sqrt[4]{\{4Rd(r-iR-id)(r+iR+id)\}}. \end{cases} \tag{28}$$

For $P[16]$ we have, from (16), $\alpha_4=1$.

From (14) $$a_1a_2 = \frac{a_3^2a^2-1}{a^2-a_3^2}, \quad a_3a = \frac{a_2^2a^2-1}{a^2-a_2^2}.$$

Eliminating a_3, we have

(29) $$a_1a_2(a^2+a^{-2}-2a_2^2) = a_2^4-1.$$

Hence, under the condition $P[16]$,

$$a_2(a^2+a^{-2}) = a_2^4+2a_2^3-1.$$

This gives

(30) $$a_2(a-a^{-1})^2 = (a_2-1)(a_2+1)^3$$

and $$a_2\{(a+a^{-1})^2-4a_2^2\} = (a_2+1)(a_2-1)^3,$$

and so $$\frac{a_2-1}{a_2+1} = \frac{\sqrt{\{(a+a^{-1})^2-4a_2^2\}}}{a-a^{-1}}.$$

Since $a_2 = \dfrac{R^2-d^2}{2r\sqrt{(Rd)}}$ and $1-\dfrac{4a_2^2}{(a+a^{-1})^2} = \dfrac{(r-R+d)(r+R-d)}{r^2}$,

we then have

(31) $$\frac{R^2-d^2-2r\sqrt{(Rd)}}{R^2-d^2+2r\sqrt{(Rd)}} = \frac{(R+d)\sqrt{\{(r-R+d)(r+R-d)\}}}{r(R-d)}$$

and $$\frac{R^2-d^2}{2r\sqrt{(Rd)}} = \frac{r(R-d)+(R+d)\sqrt{\{(r-R+d)(r+R-d)\}}}{r(R-d)-(R+d)\sqrt{\{(r-R+d)(r+R-d)\}}}.$$

Write now

$$u,\ u' \equiv (\sqrt{R}+\sqrt{d})\sqrt{(r+R-d)} \pm (\sqrt{R}-\sqrt{d})\sqrt{(r-R+d)},$$

and $$v,\ v' \equiv (\sqrt{R}-\sqrt{d})\sqrt{(r+R-d)} \pm (\sqrt{R}+\sqrt{d})\sqrt{(r-R+d)},$$

where in each case the first-named symbol on the left refers to the upper sign on the right. Sheer multiplication gives

$$\tfrac{1}{2}uv,\ \tfrac{1}{2}u'v' = r(R-d) \pm (R+d)\sqrt{\{(r-R+d)(r+R-d)\}}$$

and $$\tfrac{1}{2}uu',\ \tfrac{1}{2}vv' = R^2-d^2 \pm 2r\sqrt{(Rd)},$$

with the same reference of the double sign. Thus, from (31),

$$\frac{(R+d)\sqrt{\{(r-R+d)(r+R-d)\}}}{r(R-d)} = \frac{vv'}{uu'}$$

and $$\frac{R^2-d^2}{2r\sqrt{(Rd)}} = \frac{uv}{u'v'}.$$

Hence, if we multiply and take the square root,

$$(32)\quad \frac{R+d}{r}\left\{\frac{(r-R+d)(r+R-d)}{4Rd}\right\}^{\frac{1}{2}} = \frac{(\sqrt{R}-\sqrt{d})\sqrt{(r+R-d)}+(\sqrt{R}+\sqrt{d})\sqrt{(r-R+d)}}{(\sqrt{R}+\sqrt{d})\sqrt{(r+R-d)}-(\sqrt{R}-\sqrt{d})\sqrt{(r-R+d)}};$$

if we divide and take the square root,

$$(33)\quad \frac{\{4Rd(r-R+d)(r+R-d)\}^{\frac{1}{2}}}{R-d} = \frac{(\sqrt{R}-\sqrt{d})\sqrt{(r+R-d)}-(\sqrt{R}+\sqrt{d})\sqrt{(r-R+d)}}{(\sqrt{R}+\sqrt{d})\sqrt{(r+R-d)}+(\sqrt{R}-\sqrt{d})\sqrt{(r-R+d)}}.$$

We may also write equations (30) as

$$\{R^2-d^2+2r\sqrt{(Rd)}\}^3\{R^2-d^2-2r\sqrt{(Rd)}\} = (2r)^3(R-d)^3(R+d)\sqrt{(Rd)},$$

$$\{R^2-d^2+2r\sqrt{(Rd)}\}\{R^2-d^2-2r\sqrt{(Rd)}\}^3 = 8r(R-d)(R+d)^3(r-R+d)(r+R-d)\sqrt{(Rd)}.$$

Hence

$$(34)\quad R^2-d^2+2r\sqrt{(Rd)} = 2r(R-d)(Rd)^{\frac{1}{8}}\{4(r-R+d)(r+R-d)\}^{-\frac{1}{8}},$$

and $$R^2-d^2-2r\sqrt{(Rd)} = (R+d)(Rd)^{\frac{1}{8}}\{4(r-R+d)(r+R-d)\}^{\frac{3}{8}}.$$

6. *The Dodecagon and the* 24*-agon.*

For $P[12]$ we have $$a_3 = -1$$

and from (14) $$aa_3 = \frac{a_2^2a^2-1}{a^2-a_2^2}.$$

Hence $$a_2^2 = -(a+1+a^{-1})$$

and so $$a+2+a^{-1} = -\sqrt{\{(a+a^{-1}-2a_2)(a+a^{-1}+2a_2)\}}.$$

This gives

$$(35)\quad a^{\frac{1}{2}}+a^{-\frac{1}{2}} = i\{(a+a^{-1}-2a_2)(a+a^{-1}+2a_2)\}^{\frac{1}{4}}$$

and $$(a+a^{-1}-2a_2)^{\frac{1}{2}}+(a+a^{-1}+2a_2)^{\frac{1}{2}} = -2i.$$

Accordingly

$$(36)\quad (a^{\frac{1}{2}}+a^{-\frac{1}{2}})\left\{\left(1-\frac{2aa_2}{a^2+1}\right)^{\frac{1}{2}}+\left(1+\frac{2aa_2}{a^2+1}\right)^{\frac{1}{2}}\right\}$$
$$=2\left\{\left(1-\frac{2aa_2}{a^2+1}\right)\left(1+\frac{2aa_2}{a^2+1}\right)\right\}^{\frac{1}{4}}.$$

But $$a=\sqrt{(R/d)},\quad \frac{2aa_2}{a^2+1}=\frac{R-d}{r}.$$

Hence

$$(37)\quad (\sqrt{R}+\sqrt{d})\{\sqrt{(r+R-d)}+\sqrt{(r-R+d)}\}$$
$$=2\{Rd(r+R-d)(r-R+d)\}^{\frac{1}{4}}.$$

Had we substituted directly into the two equations (35) we should have obtained the alternative forms for the dodecagon

$$(38)\quad \frac{(\sqrt{R}+\sqrt{d})\sqrt{r}}{\sqrt{(R+d)}}=i\{(r-R+d)(r+R-d)\}^{\frac{1}{4}}$$

and $$\sqrt{(r-R+d)}+\sqrt{(r+R-d)}=-2ir^{\frac{1}{2}}(Rd)^{\frac{1}{4}}(R+d)^{-\frac{1}{2}}.$$

From this last equation, by squaring,

$$2r+2\sqrt{\{(r-R+d)(r+R-d)\}}=-\frac{4r\sqrt{(Rd)}}{R+d},$$

and so $$2r-2\sqrt{\{(r-R+d)(r+R-d)\}}=\frac{4r\{R+\sqrt{(Rd)}+d\}}{R+d},$$

i.e. $$\sqrt{(r-R+d)}-\sqrt{(r+R-d)}=\frac{2\sqrt{\{R+\sqrt{(Rd)}+d\}}\sqrt{r}}{\sqrt{(R+d)}}.$$

From the product of this and the second equation of (38) we have, after reduction, the form, explicit for r,

$$(39)\quad \frac{1}{r}=\frac{2i\{R+\sqrt{(Rd)}+d\}^{\frac{1}{2}}(Rd)^{\frac{1}{4}}}{R^2-d^2}.$$

For $P[24]$ it is enough to double the suffix of every a in any $P[12]$. Hence from (36) we derive

$$(40)\quad (a_2+1)\left\{\left(1-\frac{2a_2a_4}{a_2^2+1}\right)^{\frac{1}{2}}+\left(1+\frac{2a_2a_4}{a_2^2+1}\right)^{\frac{1}{2}}\right\}$$
$$=2a_2^{\frac{1}{2}}\left\{\left(1-\frac{2a_2a_4}{a_2^2+1}\right)\left(1+\frac{2a_2a_4}{a_2^2+1}\right)\right\}^{\frac{1}{4}}.$$

But from (29) we have *identically*

$$\frac{2a_2a_4}{a_2^2+1} = \frac{2a_2^2-2}{a^2+a^{-2}-2a_2^2}.$$

Hence

$$\left\{1+\frac{2a_2a_4}{a_2^2+1}\right\} \Big/ \left\{1-\frac{2a_2a_4}{a_2^2+1}\right\} = \frac{(a-a^{-1})^2}{(a+a^{-1}-2a_2)(a+a^{-1}+2a_2)}$$

$$= \frac{(R-d)^2r^2}{(R+d)^2(r-R+d)(r+R-d)}.$$

Thus for $P[24]$ we have the condition

$$(41)\quad [R^2-d^2+2r\sqrt{(Rd)}][(R-d)r+(R+d)\sqrt{\{(r-R+d)(r+R-d)\}}]$$
$$= 2r(R^2-d^2)\{4Rd(r-R+d)(r+R-d)\}^{\frac{1}{2}}.$$

7. *The Eicosagon.*

For $P[20]$ we have $\quad a_5 = 1$

and from (14) $\quad aa_5 = \dfrac{a_3^2a_2^2-1}{a_2^2-a_3^2}, \quad aa_3 = \dfrac{a_2^2a^2-1}{a^2-a_2^2}.$

If we eliminate a_3, a_5 from these equations and write $z \equiv a_2^2$, we have

$$(a^3-a^4)z^3+3(a^2-a^5)z^2-z(1-2a^3+2a^4-a^7)-(a-a^6) = 0.$$

If we divide by $a^3(1-a)$ and write $c \equiv a+a^{-1}$, we have

$$z^3+3z^2(c+1)+3z(c+1) = z(c^3+c^2+c)+(c^2+c-1).$$

This gives $\quad (z+c+1)^3 = c(c+2)^2(z+1)$

and $\quad (z+c+1)(z-c+1)^2 = c(c^2-4z)(z+1).$

Hence $\quad \dfrac{z+1-c}{z+1+c} = \dfrac{\sqrt{(c^2-4z)}}{c+2}.$

If we replace $4z$ by $c^2-\theta^2$, we have

$$\theta^3-(c+2)\theta^2-(c+2)^2\theta+(c+2)(c-2)^2 = 0.$$

This gives $\quad (\theta-c-2)^2(\theta+c+2) = 8c(c+2)$

and $\quad (\theta+c-2)^2(\theta+c+2) = 4\theta c(c+\theta).$

Hence $$\frac{\theta-c-2}{\theta+c-2} = \frac{\surd\{2(c+2)\}}{\surd\{\theta(c+\theta)\}}$$

(42) *i.e.* $$\frac{c}{\theta-2} = \frac{\surd\{\theta(\theta+c)\}-\surd\{2(2+c)\}}{\surd\{\theta(\theta+c)\}+\surd\{2(2+c)\}}.$$

Write $$u, u' \equiv \surd\{\theta(c+2)\} \pm \surd\{2(\theta+c)\}$$

and $$v, v' \equiv \surd\{(\theta+c)(c+2)\} \pm \surd(2\theta),$$

the first-named letter on the left referring to the upper sign on the right.

Then $$uv', u'v = c\surd\{\theta(\theta+c)\} \pm c\surd\{2(2+c)\}$$

and $$uu' = c(\theta-2).$$

Thus (42) is $$c^2uv' = uu'.u'v,$$

i.e. $$c^2v' = vu'^2.$$

But $$v'^2-u'^2 = c^2.$$

Hence we have

(43) $$vv'^2 = c^2(v+v') \quad \text{and} \quad v'^3 = u'^2(v+v').$$

Now $$c = a+a^{-1} = (R+d)/\surd(Rd),$$

$$\surd(c+2) = (\surd R+\surd d)/\sqrt[4]{(Rd)},$$

$$\theta = \surd(c^2-4a_2^2) = (R+d)\surd\{(r-R+d)(r+R-d)\}/r\surd(Rd)$$

and $$\theta+c = (R+d)\{\surd(r-R+d)+\surd(r+R-d)\}^2/2r\surd(Rd).$$

Hence

$$\frac{u'\surd(rRd)}{\surd(R+d)} = (\surd R+\surd d)\{(r-R+d)(r+R-d)\}^{\frac{1}{4}} - \{\surd(r-R+d)+\surd(r+R-d)\}(Rd)^{\frac{1}{4}},$$

$$\frac{v'\surd(2rRd)}{\surd(R+d)} = (\surd R+\surd d)\{\surd(r-R+d)+\surd(r+R-d)\} - 2\{Rd(r-R+d)(r+R-d)\}^{\frac{1}{4}},$$

with a similar expression for v.

For $P[20]$ we accordingly have from (43) the equivalent conditions

$$(44)\quad \{(\sqrt{R}+\sqrt{d})[\sqrt{(r-R+d)}+\sqrt{(r+R-d)}] \\ +2\sqrt[4]{[Rd(r-R+d)(r+R-d)]}\} \\ \times\{(\sqrt{R}+\sqrt{d})[\sqrt{(r-R+d)}+\sqrt{(r+R-d)}] \\ -2\sqrt[4]{[Rd(r-R+d)(r+R-d)]}\}^2 \\ =4r(R+d)(\sqrt{R}+\sqrt{d})[\sqrt{(r-R+d)}+\sqrt{(r+R-d)}]$$

and

$$(45)\quad \{(\sqrt{R}+\sqrt{d})[\sqrt{(r-R+d)}+\sqrt{(r+R-d)}] \\ +2\sqrt[4]{[Rd(r-R+d)(r+R-d)]}\}^3 \\ =4\{(\sqrt{R}+\sqrt{d})\sqrt[4]{[(r-R+d)(r+R-d)]} \\ -[\sqrt{(r-R+d)}+\sqrt{(r+R-d)}]\sqrt[4]{(Rd)}\}^2 \\ \times(\sqrt{R}+\sqrt{d})[\sqrt{(r-R+d)}+\sqrt{(r+R-d)}].$$

8. *The Enneagon, the* 15*-agon, the* 18*-agon, and the* 30*-agon.*

For $P[9]$ and $P[15]$ I recall the forms of $P[3]$ and $P[5]$ given in equations (26a) and (23a) of § 4, namely

$$(26a)\qquad u+v+w=0$$

for $P[3]$, and

$$(23a)\qquad (u+v+w)^{-1}+\sum_{u,v,w}(-u+v+w)^{-1}=0$$

for $P[5]$, where u, v, w are proportional respectively to

$$\sqrt{(R+r-d)},\quad \sqrt{(R+r+d)},\quad \sqrt{(2R)}.$$

From (17) we see that

$$(46)\qquad b+c=\sqrt{\{(R+r-d)(R+r+d)\}},$$

and so $$\sqrt{\{(a\pm b)(a\pm c)\}}=\{\sqrt{(R+r+d)}\pm\sqrt{(R+r-d)}\}\sqrt{(\tfrac{1}{2}R)},$$

upper signs all corresponding. We may therefore take $u\pm v, w$ respectively proportional to

$$\sqrt{\{(a\pm b)(a\pm c)\}},\ a.$$

If we treble every suffix in $P[3]$, $P[5]$, we obtain the conditions $P[9]$, $P[15]$. This done, we have $u_3\pm v_3$, w_3 (let us so write them now)

proportional to

$$\surd\{(a_3\pm b_3)(a_3\pm c_3)\},\ a_3.$$

Now in (11) and the analogous equations for b_{m+n}, b_{m-n} write $m, n = 2, 1$. We then have after reduction

$$\frac{a_3-b_3}{a-b} = \frac{c+c_2}{c-c_2}.$$

Since $c^2-c_2^2 = b^2-b_2^2$, we see that

$$\frac{c+c_2}{c-c_2}\cdot\frac{b+b_2}{b-b_2} = \left(\frac{c+c_2}{b-b_2}\right)^2.$$

From (15) $$b_2 = \frac{b^2(c^2+a^2)-c^2a^2}{2abc},\ \text{etc.}$$

and so, after reduction,

$$\frac{c+c_2}{b-b_2} = \frac{bc+ca+ab}{-bc+ca+ab}.$$

Accordingly

$$\surd\{(a_3-b_3)(a_3-c_3)\} = \frac{bc+ca+ab}{-bc+ca+ab}\surd\{(a-b)(a-c)\}.$$

Since $$a_3^2-b_3^2 = a^2-b^2,\ \text{etc.},$$

$$\surd\{(a_3+b_3)(a_3+c_3)\} = \frac{-bc+ca+ab}{bc+ca+ab}\surd\{(a+b)(a+c)\}.$$

From (14) $$a_3a = \frac{a_2^2a^2-1}{a^2-a_2^2},$$

i.e. $$R^{-1}a_3 = -\frac{(R^2-d^2-2rd)(R^2-d^2+2rd)}{(R^2-d^2+2rR)(R^2-d^2-2rR)}.$$

Write $\epsilon, \epsilon' \equiv r+R\pm d$; then from (17) and (46)

$$R^{-1}(ca+ab\pm bc) = \surd(\epsilon\epsilon')\pm r.$$

Hence $$R^{-1}\surd\{(a_3+b_3)(a_3+c_3)\} = \frac{\surd(\epsilon\epsilon')-r}{\surd(\epsilon\epsilon')+r}\cdot\frac{\surd\epsilon+\surd\epsilon'}{\surd(2R)}$$

$$= \frac{\{\surd(\epsilon\epsilon')-r\}^2(\surd\epsilon+\surd\epsilon')}{(\epsilon\epsilon'-r^2)\ \surd(2R)}$$

So $$R^{-1}\sqrt{\{(a_3-b_3)(a_3-c_3)\}} = \frac{\{\sqrt{(\epsilon\epsilon')}+r\}^2(\sqrt{\epsilon}-\sqrt{\epsilon'})}{(\epsilon\epsilon'-r^2)\sqrt{(2R)}}.$$

Now $$\epsilon\epsilon'-r^2 = R^2-d^2+2rR,$$

and $$\{\sqrt{(\epsilon\epsilon')}\mp r\}^2(\sqrt{\epsilon}\pm\sqrt{\epsilon'}) = (R^2-d^2+2rd)\sqrt{\epsilon}\pm(R^2-d^2-2rd)\sqrt{\epsilon'}.$$

Hence $u_3\pm v_3$, w_3 are proportional to

$$\frac{\sqrt{\epsilon}}{R^2-d^2-2rd}\pm\frac{\sqrt{\epsilon'}}{R^2-d^2+2rd}, \qquad \frac{-\sqrt{(2R)}}{R^2-d^2-2rR}.$$

We may change the sign of $\sqrt{(2R)}$ and take u_3, v_3, w_3 themselves proportional to

$$\frac{\sqrt{(R+r+d)}}{R^2-d^2-2rd}, \quad \frac{\sqrt{(R+r-d)}}{R^2-d^2+2rd}, \quad \frac{\sqrt{(2R)}}{R^2-d^2-2rR}.$$

Hence for the enneagon

$$\frac{\sqrt{(R+r+d)}}{R^2-d^2-2rd}+\frac{\sqrt{(R+r-d)}}{R^2-d^2+2rd}+\frac{\sqrt{(2R)}}{R^2-d^2-2rR} = 0. \tag{47}$$

For the 15-agon

$$\Sigma\left[\pm\frac{\sqrt{(R+r+d)}}{R^2-d^2-2rd}\pm\frac{\sqrt{(R+r-d)}}{R^2-d^2+2rd}\pm\frac{\sqrt{(2R)}}{R^2-d^2-2rR}\right]^{-1} = 0, \tag{48}$$

the summation being taken for those *four* terms in which *at least two* of the double signs are positive.

For $P[18]$ and $P[30]$ we interchange R and d by the principle of (20), and thus have for the 18-agon

$$\frac{\sqrt{(d+r+R)}}{R^2-d^2+2rR}+\frac{\sqrt{(d+r-R)}}{R^2-d^2-2rR}+\frac{\sqrt{(2d)}}{R^2-d^2+2rd} = 0, \tag{49}$$

and for the 30-agon

$$\Sigma\left[\pm\frac{\sqrt{(d+r+R)}}{R^2-d^2+2rR}\pm\frac{\sqrt{(d+r-R)}}{R^2-d^2-2rR}\pm\frac{\sqrt{(2d)}}{R^2-d^2+2rd}\right]^{-1} = 0, \tag{50}$$

where the summation is interpreted as in (48).

Added Jan., 1923. — The following results, obtained since this paper was approved for publication, may be stated without proof.

For the heptagon I find

$$\frac{1}{(R+d)\sqrt{(R-d+r)}}+\frac{1}{(R-d)\sqrt{(R+d+r)}}+\frac{1}{r\sqrt{(2R)}}$$
$$+\frac{1}{\sqrt{\{2R(R-d+r)(R+d+r)\}}}=0,$$

which should be compared with (23) for the pentagon.

We may also write $P[7]$ in the form

$$\begin{vmatrix} (u+v+w)^{-1} & (u+v-w)^{-1} & (u-v+w)^{-1} \\ (u+v-w)^{-1} & (u+v+w)^{-1} & (-u+v+w)^{-1} \\ (u-v+w)^{-1} & (-u+v+w)^{-1} & (u+v+w)^{-1} \end{vmatrix}=0$$

where u, v, w are $\sqrt{(R\pm d+r)}$, $\sqrt{(2R)}$.

The transformations of (20) and § 8 enable us to derive forms for $P[14]$, $P[21]$, $P[42]$.

Added May, 1923.—I find a paper by Weill* in which, by considering special degenerate polygons, he obtains porisms for the octagon and decagon, as well as that for the hexagon. He gives $P[8]$ in the form

$$r^2=(R^2-d^2)/4dR,$$

equivalent to (9) above, and $P[10]$ in the form

$$(R^2-d^2-2dr)(R^2-d^2+2dr)^2=8dr^3(R^2-d^2),$$

which is just the rationalized form of (25).

* *Bull. Soc. Math.* (1901), vol. 29, pp. 202, 203.

ON THE DISCONTINUITIES OF MONOTONE FUNCTIONS OF SEVERAL VARIABLES

By W. H. Young *and* Grace Chisholm Young.

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1. The following communication deals with the distribution and discusses the character of the discontinuities of a monotone function of any number of variables. The theorems obtained are generalisations of the theorem that the discontinuities of a monotone function of a single variable are countable, and that such a function has a unique limit of approach on the right and on the left, that is

$$\operatorname*{Lt}_{0<h\to 0} f(x+h) = \phi(x), \qquad \operatorname*{Lt}_{0>h\to 0} f(x+h) = \psi(x),$$

where $\phi(x)$ and $\psi(x)$ are uniquely determinate functions and coincide with $f(x)$ except at a countable set of points.

Monotone functions in higher space fall naturally into a number of types, according as one or more of the possible increments (simple, double, ... n-ple) are hypothecated to be always $\geqslant 0$ or always $\leqslant 0$; and each type is subdivided into classes according to the sign chosen for each of the selected increments. Thus in the plane we have for a function $f(x, y)$ two single increments

$$f(x+h, y)-f(x, y), \qquad f(x, y+k)-f(x, y),$$

and one double increment

$$f(x+h, y+k)-f(x, y+k)-f(x+h, y)+f(x, y).$$

In the case which originally occupied us in the present investigation all these increments are of the same sign. Taking them all to be $\geqslant 0$, we have what we have been accustomed to call *a monotonely monotone ascending function*, since the condition laid on the double increment ensures that *the monotone increase with respect to either variable has an*

*increasing rate as the other variable increases.** In view, however, of the extension to higher space, one might be inclined to call them *entirely*† *monotone ascending functions of the ensemble* $(x_1, x_2, \ldots, x_n)$, omitting the word "*entirely*" when there is no danger of misconception. These functions lie at one end of the gamut of types, and at the other extremity we find those functions which are merely monotone with respect to all the variables singly and separately. These we call *monotone functions of each variable separately.*‡ Intermediate between these there will be functions which are monotone with respect to all the ensembles of the n variables of dimensions $\leqslant k$, where k is any integer greater than unity and less than n. Beside all these there will still be types where, for only some and not all of these ensembles, the function is monotone.

Let us take for simplicity two variables only. Since the functions considered are monotone ascending with respect to each variable separately, the limit, as we approach the point along the axial cross through the point from the right, or the left, from above or below, is unique. Denoting§ these limits by v_1, v_3, v_4, v_2 respectively, and the sets of limits in the completely open quadrants bounded by them, beginning with the $(+, +)$-quadrant and proceeding clockwise, by Π_1, Π_2, Π_3, Π_4, we have by the monotony

$$\Pi_3 \leqslant v_2 \leqslant \Pi_2 \leqslant v_1 \leqslant \Pi_1, \qquad \Pi_3 \leqslant v_3 \leqslant \Pi_4 \leqslant v_4 \leqslant \Pi_1,$$

$$v_3 \leqslant f \leqslant v_1, \qquad v_2 \leqslant f \leqslant v_4.$$

Hence it is immediately evident that *wherever the function is continuous with respect to both variables separately it is continuous in the* $(+, -)$ *and* $(-, +)$ *quadrants.*

This result can then evidently be extended by induction for any number of variables. *Wherever the function is continuous with respect to each of the ensembles of* $(n-1)$ *variables it is continuous in all the n-dimensional hyperquadrants except the first and its opposite one in which all the signs are all the same.*

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* W. H. Young, "On Multiple Integrals," 1917, *Proc. R. S.* (A), Vol. 93, p. 31.

† Hobson, in the new edition of his "Theory of Functions of a Real Variable" (Camb. Univ. Press, 1921), § 255, p. 325, calls these functions *Quasi-monotone.* They may, however, be said to be *more* and not *less* monotone than the functions which he, following Arzelà, terms *monotone* (see next foot-note).

‡ Called by Hobson *monotone functions.* The definition given in § 253, p. 322 of his new edition, is formally different from that given above, but is easily seen to be equivalent.

§ As in our paper on the general theory of the crystalline structure of a function, *Proc. London Math. Soc.*, Ser. 2, Vol. 17, pp. 1–16.

The dominant feature is now that *there is a unique limit* in each of the two excepted quadrants or hyperquadrants, viz. in the first, or* $(+, +, \ldots, +)$*-hyperquadrant, and its opposite, the* $(-, -, \ldots, -)$ *one.*

No such unicity of limit occurs of necessity in the non-excepted quadrants,† as is obvious by considering the function y/x, or, if we wish to have a function which is monotone ascending and finite everywhere, the function which is zero for all but negative values of y, and, when y is negative, is equal to -1, unless x is positive, while

$$f(x, y) = \frac{2}{\pi} \tan^{-1} \frac{y}{x} \quad (0 < x \leqslant 1;\ -1 \leqslant y < 0).$$

This function has all possible values from -1 to 0 as limits in the $(+, -)$ neighbourhood of the origin.

If we choose now to apply the general theory of the crystalline structure of a function, already quoted, we can easily convince ourselves that *if the function belongs to the narrower class, that is, if it is entirely monotone with respect to the ensemble of all the variables, it is continuous with respect to the ensemble of all the variables except on a countable set of parallels to the axes, if there are two variables, or of planes, or hyperplanes parallel to the axial planes or hyperplanes in the general case.*‡

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* This result, proved below, Theorem 1, was quite correctly given in a "Note on Monotone Functions", 1909, *Quarterly Journal*, Vol. 41, p. 82, but was incorrectly given in a casual statement in the introductory remarks to a "Note on Multiple Integrals" by W. H. Young, 1917, *Proc. R. S.* (A), Vol. 93, p. 31, to the effect that in the special case when the function belongs to the narrower class of entirely monotone functions of the ensemble of all the variables, there is a unique limit in *each* of the completely open quadrants, instead of *in one pair of opposite quadrants;* u_2 and u_4 must be regarded as sets of limits in the passage quoted; they are the same as Π_2 and Π_4 in the present text. One has only to consider the product of a monotone ascending function of x and a monotone ascending function of y to see immediately that, at a point of discontinuity, there may be a plurality of limits in the second and fourth quadrants. It is quite unaccountable how this misstatement could have passed into print. For the rest, it does not affect anything in the Note in question.

† There is an obvious error on this point in § 307 of Hobson's new edition, it being there stated that these functions have unique limits in all four open quadrants. The *soi-disant* proof that follows contains a somewhat subtle fallacy; it occurs on p. 391, lines 7–10. In fact Hobson's k_1 depends not only on x, y, and z, but also on h_1, and though the relation he uses then holds for all smaller values of h, this is only true if k be correspondingly modified. Thus when Hobson writes "By choosing the numbers h_1, k_1, h_1', k_1' so that $k_1 > k_1'$, $h_1 < h_1'$," &c., he involves himself in a *non possumus*.

‡ This result was given in the *R. S.* note, already quoted, and indications of proof given. Hobson has filled in the proof in § 308 of his book, but it seems to be still too condensed to be quite clear. It was in consequence of seeing the confusion in this section of Hobson's book, and the consequent detection of the error in the *R. S.* note, that we took up again our unpublished work on the subject and prepared it for presentation to the Society.

It is better, however, to discuss our problem from first principles, using this intuitive result as a guide; we obtain in this way more information and a more complete view of the situation.

Summing up the results obtained below, we may state that *the discontinuities of a function $f(x, y)$ which is monotone ascending with respect to each variable separately form, in the case of two variables, a plane set of content zero which consists of four separate subsets:*

(i) *The set of points where $f(x, y)$ is discontinuous both with respect to x and with respect to y; this set is countable on every straight line parallel to the axes;*

(ii) *The set of points where $f(x, y)$ is continuous with respect to x, but discontinuous with respect to y; this set lies on a countable set of parallels to the axis of x;* *

(iii) *The set of points where $f(x, y)$ is discontinuous with respect to x, but continuous with respect to y; this set lies on a countable set of parallels to the axis of y;* *

(iv) *The set of points where $f(x, y)$ is continuous with respect to x and with respect to y, but discontinuous with respect to (x, y); this set is countable.*†

The set (i) *need not be countable on a line which is not parallel to either of the axes, and may even fill up whole stretches on such a line; but the stretches which consist entirely of points of one of these sets,* (i), (ii), *and* (iii), *are* in toto *only countable in number.*‡

It is no longer necessary, as it is when the function belongs to the narrower class, that these discontinuities should lie on a countable set of lines, and an example is given below where this is, in fact, not the case.

In the case of more variables than two a precisely analogous classification of the singularities can at once be written down. The following relates to a function of three variables, for instance:—

The discontinuities of a function $f(x, y, z)$ which is monotone ascending with respect to each variable separately, form a set of content zero, which consists of eight separate subsets:

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* By Theorem 3, Cor. 2.

† By Theorem 3, Cor. 3.

‡ In the statement of this theorem given in the *Quarterly Journal* (*loc. cit.*) these results are only in embryo.

(i) *The set of points where* $f(x, y, z)$ *is discontinuous with respect to* (x, y), (y, z), *and* (z, x); *this set is countable on every straight line parallel to the axes;*

(ii) *The set of points where* $f(x, y, z)$ *is discontinuous with respect to* (z, x) *and* (y, z), *but continuous with respect to* (x, y); *this set lies on a countable set of planes parallel to the axial plane* (x, y);

(iii) *and* (iv) *are got from* (ii) *by cyclic change of the coordinates;*

(v) *The set of points where* $f(x, y, z)$ *is discontinuous with respect to* (x, y), *but continuous with respect to* (y, z) *and* (z, x); *this set lies on a countable set of straight lines parallel to the axes;*

(vi) *and* (vii) *are again got by cyclical change from the preceding;*

(viii) *The set of points where* $f(x, y, z)$ *is continuous with respect to* (x, y), (y, z), *and* (z, x) *but discontinuous with respect to* (x, y, z); *this set is countable.*

The set (i) *need not be countable on a line which is not parallel to the axes, and may even fill up whole rectangles, but the rectangles which consist entirely of points of one of the sets* (i)–(vii) *are* in toto *at most countably infinite in number.*

The existence of these exceptional stretches, rectangles or hyper-rectangles, throughout which the function is discontinuous in the perpendicular direction, is at once evident in the case of the narrower class of functions, and the singularities are confined to these entities and their boundaries. It is interesting to find these same entities occurring in the case of the functions which are monotone with respect to each variable separately, although the points and boundary points of these entities do not exhaust the singularities, and to find that they are countable. The proof given here depends on a Lemma from the Theory of Sets of Points, which seems to be new. This asserts that, *if we have a more than countable set of stretches, none of which are parallel to a given direction, say vertical, there is at least one vertical line which meets a more than countable set of the stretches, each in a different point, and that there is at least one complete block of such vertical lines.*

This Lemma may be extended, not only, as is done in the text, for rectangles in space of three dimensions, or hyper-rectangles in S_n, but also for monotone arcs in place of stretches.

2. We now state and prove the following theorem :—

THEOREM 1.—*If $f(x, y)$ is monotone increasing with respect to x alone, and with respect to y alone, then $f(x, y)$ has a unique limit in the $(+, +)$-neighbourhood and in the $(-, -)$-neighbourhood of any point. and the value at the point lies between these two limits; moreover each such limit may be obtained by moving up to the point along any chosen half ray in the quadrant considered.**

Let (x, y) be the point P; then any point in its $(+, +)$-neighbourhood is, by definition, one of the form $(x+h, y+k)$, where $0 < h$, $0 < k$, and the limits to be discussed are obtained by letting h and k tend to zero, in any and every manner, consistent with their remaining positive (not zero).

Now if we take any half ray

$$y+k = m(x+h) \quad (0 < m),$$

in the neighbourhood considered, $f(x+h, y+k)$ will be monotone increasing on it, as we move away from the point (x, y), and will therefore have a unique limit, if we move contrariwise up to the point. On the other hand, if we take a point $(x+h', y+k')$ off our ray, and project it by means of parallels to the axes on to the ray, the value of f at this point evidently lies between the values at its projections.† We therefore again obtain the same unique limit, when $h' \to 0$, $k' \to 0$, since the same limit results from each of the two projections, by what has been already pointed out.

This proves the theorem as far as the $(+, +)$-neighbourhood is concerned. Similarly the part relating to the $(-, -)$-neighbourhood is proved.

That the value at the point lies between the two limits is evident from the monotony, and that the remaining limits lie between was indicated in the introduction.

3. In the next theorem we come across the countable set of stretches which serve throughout our subsequent investigations as the basis of orientation.

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* A somewhat different proof was given in the *Quarterly Journal*, *loc. cit.*, p. 81.

† This is intuitive geometrically; analytically it results from considering the coordinates relative to P of the point (h', k') and its projections (h', mh') and $(k'/m, k')$; it being evident that $k' - mh'$ and $h' - k'/m$ have opposite signs, since $0 < m$.

Theorem 2.—*The points at which a function $f(x, y)$, monotone increasing with respect to each variable separately, is discontinuous with respect to y, form a set of content zero, whose section on any vertical line is countable. This set may fill up certain non-vertical stretches* whose end-points may, or may not, be included, but these then form a countable set of stretches.*

On any vertical line our function $f(x, y)$ is monotone, and therefore continuous with respect to y except possibly at a countable set of points. The set is therefore certainly of content zero.

Suppose now, if possible, that there is a more than countable set of non-vertical stretches, every point of which† is a point of discontinuity of $f(x, y)$ with respect to y; then by a Lemma from the Theory of Sets of Points, proved immediately below, there would be at least one vertical line which met a more than countable set of these stretches each in a different point; such a vertical line would thus contain a superfluity of points of discontinuity of $f(x, y)$ with respect to y, so that this cannot occur. Thus such exceptional non-vertical stretches can be at most countable.

This proves the theorem.

4. In the above we use the following lemma:—

Lemma from the Theory of Sets of Points.—*Given a more than countably infinite set of stretches, none of which is parallel to a fixed axis, say vertical, then there is at least one vertical line which meets a more than countable set of the stretches.*

Moreover, there must be at least one vertical strip, such that every vertical line in it meets a more than countable set of the stretches each in a different point.

For, suppose the first statement were not true, then every vertical line meets only a countable set of the stretches.

Take the rational points on a horizontal axis and index them in the usual way, and transfer the same index to the vertical line through each

* For definiteness it must be understood that each *stretch*, as meant in the present paper, is the greatest possible; any greater segment inside which it lies contains points at which $f(x, y)$ is continuous with respect to y.

† This expression must be understood strictly to include only the internal points, if the stretch is open at both ends, but to include any end-point at which the segment is closed. This is, of course, only calling attention to the precise mode in which the expression is used. It is impossible in the present connexion to use terms like the French *au sens large*, *au sens étroit*, since some of the stretches may and others may not be closed.

such point. We thus have a numbered set of verticals, dense everywhere, say

$$v_1, v_2, v_3, \ldots.$$

Each of these, say v_n, meets only a countable set of the stretches, which we denote by

$$s_{n,1}, s_{n,2}, s_{n,3}, \ldots.$$

Here one and the same segment will, in general, appear repeatedly for different indices n, but, on the whole, there will be only a countable set of actually different stretches thus numbered. There must therefore remain over unnumbered stretches; let s be one of these.

Since s is not vertical, it projects into a segment s' of the horizontal axis, which accordingly contains rational points; let the first of these be the rth. Then the vertical v_r intersects s', and therefore also s. But this is in contradiction with the definition of s, as a stretch not intersected by any of the numbered verticals. Thus, by a *reductio ad absurdum*, the first part of the theorem is proved.

Now let v denote a vertical which meets a more than countable set of the stretches, and let us denote by v_{r_1} the first of our numbered verticals on the left of v, by v_{r_2} the first between v and v_{r_1}, and so on. Then these verticals form a sequence with v as limit. If only a countable set of the stretches which meet v meet each v_{r_n}, it follows by the same argument as before that we can index all of these stretches which do not lie entirely to the right of the vertical v, by means of a double system of indices, so that these are only countable in number. The same holds on the right of v; therefore on one side or the other we must arrive at a numbered vertical, say v_m, which meets a more than countable set of the stretches which meet v. Each of these stretches will therefore project on to the horizontal axis into a segment containing the feet of v and v_m, and therefore the feet of every vertical in the strip bounded by v and v_m. Thus *every vertical in this strip meets each of those stretches, more than countable in number.*

Now either this strip is one such as is described in the enunciation, or else there is one of the verticals in our strip—and we shall suppose for definiteness it is v—which contains only a countable set of points of the stretches, although it meets a more than countable set of them. It is then evident that there must be at least one of these points through which there passes a more than countable set of the stretches, and that a more than countable number of these stretches will protrude on one side of v, say on the right. These stretches, radiating from a point on v, form a set

having the property with which we started. Forming then our vertical strip, as before, on the right of v, every vertical in which meets a more than countable set of these stretches, each of these verticals will meet in a different point each of these stretches, since they have no common point except on v. This proves the second part of the Lemma.

5. It is unnecessary to state in full the corresponding theorem for the discontinuities with respect to x; we only have, in the enunciation of Theorem 2, to change y into x and "vertical" into "horizontal." We proceed to deduce the following:—

Theorem 3.—*If $f(x, y)$ is monotone ascending with respect to each variable separately, the unique limits in the $(+, +)$- and $(-, -)$-neighbourhoods respectively may be obtained by approaching the point along the adjacent part of the vertical boundary of that neighbourhood, and the same limit may be obtained by moving suitably up to the point inside the adjacent quadrant with the same vertical boundary, provided the point is neither an internal nor end-point (included or not) of any of the exceptional* vertical stretches of Theorem* 2.

We remark that, since even two of the exceptional vertical stretches which *abut* must have between them a non-included end-point, for otherwise they would have already fused into one, we may be sure that between any two of the stretches on the same vertical there is a point at which $f(x, y)$ is continuous with respect to x.

Hence it at once follows that, if (x, y) is a point not belonging to one of the exceptional vertical stretches, nor an end-point of such a stretch, there will be a sequence of points $(x, y+k)$ on the same vertical, and having the point (x, y) as limit, at each of which the function is continuous with respect to x. Therefore we can find a corresponding sequence of points in either of the two adjacent quadrants, having that same vertical boundary, such that, as we approach (x, y) along either of these latter sequences, $f(x+h, y+k)$ has the same unique limit as when we move along the feet of the perpendiculars from these points on the vertical boundary of these adjacent quadrants, namely

$$\operatorname*{Lt}_{h\to 0,\, k\to 0} f(x+h, y+k) = \operatorname*{Lt}_{k\to 0} f(x, y+k).$$

The values of h and k in this relation are conveniently chosen values, but, since by one-dimensional theory, the limit along the vertical is unique,

* *i.e.* where there is discontinuity with respect to x.

and by our first theorem the same is true for approach in the first or the third quadrant, it follows from our relation that the unique limits alluded to in the enunciation are equal, and that the same limit occurs among the set of possible limits in the adjacent quadrant, the fourth adjacent to the first, or the second adjacent to the third. This proves the required result.

COR. 1.—*The corresponding result with "vertical" replaced by "horizontal" is true.*

COR. 2.—*If at a point P, which is neither an internal point nor an end-point (included or not) of one of the exceptional horizontal stretches, $f(x, y)$ is continuous with respect to x, then $f(x, y)$ is continuous with respect to (x, y) at P.*

In this result we may change "horizontal" to "vertical" and x to y.

For, taking the "vertical" case for example, the vertical limit above the point, as we move down towards it, or the vertical limit below, as we move up to the point, is equal to the value at the point and they are the same as the unique limit in the adjacent extreme quadrant, the $(+, +)$ in the former and the $(-, -)$ in the latter case. By Theorem 1 the remaining limits must also coincide with this value, which proves the result stated.

COR. 3.—*If at a point P, which is not a common non-included end-point of a pair of exceptional stretches, one vertical and one horizontal,* our function $f(x, y)$ is continuous with respect to x and also with respect to y, it is continuous with respect to (x, y).*

For P cannot be a point belonging to an exceptional stretch, since $f(x, y)$ is continuous with respect to x and to y at P. The result is therefore an immediate consequence of Cor. 2.

6. The corollary last stated was, in an embryonic form, in part anticipated in an earlier paper.† Using the reasoning there employed, we can render the preceding result more precise. We have, in fact, the following theorem :—

* Not excluding the possibility of there being another exceptional stretch abutting with either of those mentioned at the point, and having it for non-included end-point.

† W. H. Young, "On Uniform Oscillation", 1909, *Trans. Camb. Phil. Soc.*, Vol. 21, p. 255, Theorem 6 and foot-note.

Theorem 4.—*If $f(x, y)$ is monotone increasing with respect to each variable separately, then at any point where it is continuous with respect to x and is continuous with respect to y, it is continuous in the $(+, -)$- and $(-, +)$-quadrants; moreover there is continuity with respect to (x, y) all round, except possibly at a countable set of points, namely those mentioned in Cor. 3 to the preceding theorem.*

Indeed, by the monotony, the limits in these two quadrants lie between the limits along the bounding lines, which in our case coincide.

7. For two or more dimensions we have, of course, a Lemma exactly similar in enunciation of proof to that of § 4. Instead of a stretch, we now have in the plane a rectangle with sides parallel to the axes. The lemma is then as follows:—

Lemma.—*Given a more than countably infinite set of rectangles in space S_3, none of which are parallel to a fixed line, say vertical, then there is at least one vertical line which meets a more than countable set of the rectangles. Moreover, there is at least one cylindrical block of such verticals, each of which meets a more than countable set of the rectangles, each in a different point.*

In higher space the lemma is the same, with "hyper-rectangle" in place of "rectangle."

Hence the theorems given for two dimensions can be at once extended to three or more dimensions, the proof given being perfectly general in character. Thus, for instance, Theorem 2 holds for n-dimensional space, if we remember that "vertical" means "parallel to the axis of y," and that "stretch" must be replaced by "hyper-rectangle," and Theorem 3 then tells us that the unique limits, which we know exist in the first quadrant or its opposite quadrant, can be obtained by moving up to the point in one of the bounding planes (hyperplanes) provided the point is not an internal or boundary point of the exceptional rectangles (hyper-rectangles) lying in that plane (hyperplane). The corollaries then follow at once:—

Cor. 2 (Theorem 2).—*If at a point P, which is neither an internal point nor a boundary point (included or not), of one of the exceptional rectangles* parallel to the plane of (x, y), $f(x, y, z)$ is continuous with respect to (x, y), then it is continuous at P with respect to (x, y, z).*

* Rectangles at every point of which $f(x, y, z)$ is discontinuous with respect to z. These, as we saw, are at most countable in number.

COR. 3 (THEOREM 2).—*If at a point P, which is not a common non-included boundary point of a pair of exceptional rectangles, parallel to the planes of (x, z) and (y, z), our function $f(x, y, z)$ is continuous with respect to (x, z) and also with respect to (y, z), then it is continuous at P with respect to (x, y, z).*

Hence we have the generalisation for higher space of the theorem as to the distribution and character of the singularities of a function which is monotone ascending with respect to each variable separately as stated in the introduction.

And a further corollary :—

COR. 4 (THEOREM 2).—*If at a point P, which is not a common non-included end-point of a triad of exceptional rectangles, parallel to the planes of (x, y), (y, z), and (z, x), our function $f(x, y, z)$ is continuous with respect to (x, y), (y, z), and (z, x), then it is continuous with respect to (x, y, z).*

8. We begin now the discussion of the singularities of the functions of the more restricted type, (entirely) monotone functions of the ensemble $(x_1, x_2, \ldots, x_n)$. In the case of two variables, we then have the two simple and the one double increment, so that, taking the signs all positive, the function $f(x, y)$ which we are considering is such that

$$f(x+h, y)-f(x, y) \geqslant 0,$$

$$f(x, y+k)-f(x, y) \geqslant 0,$$

$$f(x+h, y+k)-f(x+h, y)-f(x, y+k)+f(x, y) \geqslant 0.$$

We shall throughout for brevity denote the direction of the axis of x as "vertical" and that of the axis of x as "horizontal." We then have the following theorem :—

THEOREM 1.—*If $f(x, y)$ is an entirely monotone function of the ensemble (x, y) defined in a certain closed fundamental rectangle with sides horizontal and vertical, then its singularities lie on a countable set of horizontal and vertical lines, and are distributed as follows :—*

(a) The points (black) at which $f(x, y)$ is discontinuous with respect to x form a countable set of vertical stretches, closed at the top, which lies*

* For convenience of imagining, we think of the points of discontinuity with respect to x, or y, as marked black, and the remaining points of discontinuity with respect to (x, y) as marked red, while the points of continuity are left white.

on the top-side of the fundamental rectangle, and which may be closed or open at the bottom. Such a stretch may degenerate into a single point on the top-side of the fundamental rectangle.

A similar statement holds for the points of discontinuity with respect to y, vertical and horizontal being interchanged, and the right-hand side replacing the top.

(*b*) *The points of discontinuity with respect to* (x, y), *not included in* (*a*), *are countable; each of them* (*red*) *is a non-included end-point of a pair of exceptional stretches, one horizontal on its right and the other vertically above it.*

At each such (*red*) *singularity there is continuity of approach, except in the completely open* $(+, +)$*-quadrant, in which there is a unique limit greater than the value* $f(x, y)$ *at the singularity itself.*

(*c*) *If a pair of exceptional stretches have a common end-point, which is included in the horizontal* (*vertical*) *one, and not included in the vertical* (*horizontal*) *one, there is continuity of approach on and below the horizontal line through the point* (*on and to the left of the vertical through the point*).

Consider the upper side of the fundamental rectangle. On this $f(x, y)$ is a monotone function of the single variable x, and therefore continuous with respect to x except at a countable set of points; the verticals through these points will be proved to be the exceptional lines.

In fact, taking any point P on the side in question, such that P is one of the points of continuity with respect to x, and a neighbouring point Q on the same side of the rectangle, we have

$$|f(Q)-f(P)|\to 0,$$

as Q moves up to P as limit.

Also, if P' and Q' are points in the rectangle such that $PQQ'P'$ is a rectangle, we have, since the double increment over this rectangle is not negative,

$$|f(Q)-f(P)|\geqslant|f(Q')-f(P')|.$$

Thus both sides of this inequality tend to zero as Q' moves up to P' as limit; in other words, $f(x, y)$ is continuous with respect to x at P' as well as at P.

Thus the only vertical lines on which there lie discontinuities with respect to x are those through the countable set of points on the upper side of the fundamental rectangle at which $f(x, y)$ is discontinuous with respect to x.

Moreover, if on any such vertical there is a second point of discontinuity with respect to x, it is clear from the preceding argument that every point above this in the rectangle is also such a discontinuity. Thus the whole set of such points on the vertical in question forms a (black) stretch or segment, which may be open or closed at the bottom.

This proves the statements of the enunciation about discontinuity with respect to x. The same argument applies to discontinuity with respect to y, *mutatis mutandis*.

To prove the statements as to discontinuity with respect to (x, y), we use the fact, already proved for any function monotone with respect to each variable separately, that, *at a point where $f(x, y)$ is continuous with respect to x and with respect to y separately, there is one unique limit for approach to the point (x, y) in any one of the four axial neighbourhoods, and the unique limit, in the cases of the $(+, -)$- and $(-, +)$-neighbourhood is the value $f(x, y)$ at the point.* We have only to complete this by discussing the value of the limit in each of the remaining quadrants, the $(+, +)$ and the $(-, -)$. The point P at which these limits are considered, being supposed not to be a common end-point of two of the "black" segments already discussed (and, of course, not itself a black point), is, by what has been proved, such that both the horizontal and the vertical bounding lines of the $(-, -)$-neighbourhood of the point P are "white," that is to say, they consist of points at which $f(x, y)$ is continuous in the perpendicular direction.

We also know that at least one of the bounding lines of the $(+, +)$-neighbourhood is "white," provided that neighbourhood be small enough, whereas in the case of the $(-, -)$-neighbourhood no restriction was necessary.

If M denote then any point on such a "white" boundary, we can, inside the neighbourhood under consideration, whether $(+, +)$ or $(-, -)$, find a point Q in the direction from M perpendicular to that boundary, such that

$$f(Q)-f(M)$$

is numerically less than the length PM.

Let now M move up to P, then Q will also move up to P, and $f(Q)$ will have the same limit as $f(M)$, namely $f(P)$, since our function is continuous with respect both to x and y at P. Having found that we can move up to P in both the $(+, +)$- and the $(-, -)$-neighbourhoods so as to get the limit $f(P)$, it follows, by what was already remarked, that this is the unique limit obtainable by approach in any manner in these or the other quadrants. Thus $f(x, y)$ is continuous at P for all possible modes

of approach; in other words, P is a point of continuity of the function $f(x, y)$.

If, however, P is a common non-included end-point of two black stretches, these will bound the $(+, +)$-neighbourhood of P so that we cannot apply to it the result we had found. But the $(-, -)$-neighbourhood will still be bounded by white stretches, so that, as before, there is continuity of approach in the $(-, -)$-neighbourhood.

This proves all the statements of the enunciation.

9. This argument applies for space of any number of dimensions. We have only to remember that *the n-ple increment of a function of n variables* $(x_1, x_2, \ldots, x_n)$ *is the difference of its* $(n-1)$*-ple increments with respect to* $(n-1)$ *of the variables for values of the remaining variable differing by the length of the side of the hyper-rectangle considered corresponding to that variable.*

Thus, using the notation $f_{h,k,l}(x, y, z)$ for the triple increment of $f(x, y, z)$ over the hyper-rectangle (here a rectangular parallelopiped) whose opposite vertices are (x, y, z) and $(x+h, y+k, z+l)$, where

$$0 < h,\ 0 < k,\ 0 < l$$

and using $f_{h,k}(x, y, z)$ for the double increment over the rectangle $(x, y\ ;\ x+h, y+k)$ in the plane $z = \text{const.}$, we have

$$f_{h,k}(x, y, z+l) = f(x+h, y+k, z+l) - f(x+h, y, z+l) - f(x, y+k, z+l) + f(x, y, z+l),$$

$$f_{h,k}(x, y, z) = f(x+h, y+k, z) - f(x+h, y, z) - f(x, y+k, z) + f(x, y, z),$$

and therefore

$$f_{h,k,l}(x, y, z) = f_{h,k}(x, y, z+l) - f_{h,k}(x, y, z).$$

Moreover, we see that we may choose out whichever of the coordinates we please in place of z, *e.g.*

$$f_{h,k,l}(x, y, z) = f_{k,l}(x+h, y, z) - f_{k,l}(x, y, z),$$

where

$$f_{k,l}(x, y, z) = f(x, y+k, z+l) - f(x, y+k, z) - f(x, y, z+l) + f(x, y, z).$$

10. *By a monotonely monotone or entirely monotone increasing function of the ensemble of n variables*, we mean, as above explained,

a function $f(x_1, x_2, \ldots, x_n)$, which we will suppose defined in a closed hyper-rectangle

$$\left.\begin{aligned} a_1 &\leqslant x_1 \leqslant b_1, \\ a_2 &\leqslant x_2 \leqslant b_2, \\ \cdots \quad &\cdots \quad \cdots, \\ a_n &\leqslant x_n \leqslant b_n, \end{aligned}\right\}$$

this function being such that all its increments are $\geqslant 0$.

We denote by *the extreme corner of the super-rectangle* the point $(b_1, b_2, \ldots, b_n)$, and by *the extreme edge parallel to the axis of* x_1 the edge in this direction through the extreme corner; a similar mode of expression will be used for the faces and hyperfaces of various dimensions belonging to the hyper-rectangle.

We may then evidently assert that *the increment of* $f(x_1, x_2, \ldots, x_n)$ *over an extreme edge, face or hyperface, is not less than that over any parallel one nearer the origin.* It was this simple property that we used in the discussion of $f(x, y)$, where we had only simple increments. In three dimensions we have simple and double increments over the edges and faces of our hyper-rectangle, which is nothing more than a rectangular parallelopiped.

The theorem we obtain for space of three dimensions is as follows:—

THEOREM.—*If* $f(x, y, z)$ *is a monotonely monotone increasing function of* (x, y, z), *defined in a closed super-rectangle (rectangular parallelopiped) with edges parallel to the axes of coordinates, then the points at which it is discontinuous with respect to* x *alone form a countable set of rectangles (slabs) parallel to the plane of* (y, z), *each closed at its extreme corner, which lies on the extreme edge of the super-rectangle; such a rectangle (black) may be closed or open at other points of its periphery, only with this restriction, that if it is closed at any point of the periphery it is also closed at every point of the periphery between that point and the extreme corner in the progressive direction.* In special cases such a rectangle may degenerate into its extreme corner, or into a single stretch from that corner parallel to the axis of* y *or of* z.

A similar statement mutatis mutandis holds for the points of discontinuity with respect to either of the other variables alone.

Apart from the points above specified, at which there is discontinuity

* That is the direction of increase of the variable coordinate.

with respect to x or y alone or both together, the points of discontinuity with respect to (x, y) form a countable set of stretches (red), parallel to the axis of z. Each of these, however, must lie on a common vertical side of two black slabs, one parallel to the plane of (y, z) and the other to that (z, x); and the upper end of the stretch lies on the extreme face of the super-rectangle, parallel to the plane of (x, y).

A similar statement may be made for the discontinuities with respect to (y, z) and (z, x).

Apart from the discontinuities already mentioned, the points of discontinuity with respect to (x, y, z) are countable in number, each being a common non-included end-point (violet) of three red stretches, one parallel to each axis, proceeding from the point in the progressive direction.

Moreover, at each red point the discontinuity with respect to (x, y) only occurs for approach in the $(+, +, +)$- and $(+, +, -)$-neighbourhoods, and at each violet point the discontinuity with respect to (x, y, z) occurs only for approach in the $(+, +, +)$-neighbourhood.

In dealing with the points of continuity with respect to x, we compare, as in the case of a function of two variables, the single increment

$$f(x+h, y, z)-f(x, y, z) \quad (0 < h)$$

with the corresponding increment

$$f(x+h, b, c)-f(x, b, c)$$

on the extreme edge parallel to the axis of x.

The application of our principle to the edges of the super-rectangle

$$(x, y, z;\ x+h, b, c)$$

proves that, if the second of these increments tends to zero as $h \to 0$, so does the first, whence we get the result given in the enunciation.

Similarly, in dealing with the points of continuity with respect to (x, y), we compare the double incrementary ratios over the faces parallel to the plane of (x, y) of the super-rectangle

$$(x, y, z;\ x+h, y+k, c).$$

Finally, in examining the points where f is continuous with respect to (x, y), (y, z), and (z, x), but discontinuous with respect to x, y, z, we use the theorem that there is a unique limit to $f(x, y, z)$ as we approach the point in any one of the axial neighbourhoods, which are now eight octants;

and that $f(x, y, z)$ is itself the limit except in the $(+, +, +)$ and $(-, -, -)$ octants. We then find, precisely on the same lines as in the earlier discussion, a sequence in each of these excepted octants leading to the same limit $f(x, y, z)$ as for approach along the planes bounding the octant in question, provided (x, y, z) is not one of a countable set of points, which are common corners of three of the exceptional slabs, one parallel to each of the axial planes.

In this way the theorem is proved, and similarly it is proved for any number of dimensions.

11. The following example of a function of a single variable which is monotone ascending with discontinuities which are dense everywhere must, we think, have occurred to anyone who has interested himself in monotone functions of a single variable, since Cantor's invention of his typical ternary set. The value of the function at each of the rational binary points,

$$x = .\, e_1 e_2 e_3 \ldots e_n,$$

where the notation is that of the fractions to the base 2, the e's being each a zero or a 1, and the last e_n is 1, is got by replacing each of these digits e by its double, and reading off in the ternary notation,

$$f(x) = .\, (2e_1)(2e_2)(2e_3) \ldots (2e_n).$$

Geometrically, the points $[x, f(x)]$ project on to the axis of y into the upper end-points of the black intervals of Cantor's typical ternary set. The function $f(x)$ is that which habitually yields the (1, 1)-correspondence between the points of the continuum and those of a perfect set dense everywhere, maintaining the order.

This function enables us now to give an example of a function of two variables which is monotone with respect to each variable separately, but whose points of discontinuity with respect to y are in a certain region dense everywhere, of potency c, incapable of being arranged on a countable set of parallels to the axis of x, and indeed are incapable of being arranged on any countable set of lines whatever, only a countable number of them lying on any particular straight line. In the example given below, these points of discontinuity with respect to y are also points of discontinuity with respect to x, and with respect to (x, y) of the most pronounced type, and fill up a countable set of concentric circles; but on the same principle we could construct functions with all sorts of varieties of peculiarities.

We take, for all values of x and y from zero to 1 inclusive,

$$F(x, y) = f(x^2+y^2),$$

where $f(x)$ is the monotone function just alluded to. The discontinuities fill up the concentric circular arcs

$$x^2+y^2 = c,$$

where c is any terminating binary fraction.

Since there are no exceptional stretches, we get all possible limits by approaching along the horizontal and vertical lines through the point (Theorem 3). Thus we have v_1 on the right and v on the left of the point (*horizontal limits*), and Π_1 above and Π_3 below the point (*vertical limits*), while Π_2 and Π_4 consist of the limits Π_1 and Π_3 together with the value of the function itself.

Indeed, (x, y) can only move up to (X, Y) in three essentially different ways: x^2+y^2 remaining constant, or increasing up to X^2+Y^2, or decreasing down to that limit. Any other sequence can be split up into these types.

THE RATIONAL LINEAR ALGEBRAS OF MAXIMUM AND MINIMUM RANKS

By L. E. DICKSON.

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1. In the older theories of hypercomplex numbers $\Sigma x_i e_i$, each coordinate x_i ranges over either all complex numbers or all real numbers. In recent theories,* each coordinate ranges over all numbers of any field F, which need not be the field of all complex numbers, nor the field of all real numbers, but may be any algebraic field, or the field of all rational numbers. The latter field is the important one for the theory† of the arithmetics of hypercomplex integers. In particular, the theory of algebraic numbers is a special case of our subject.

The present paper provides an elementary introduction to rational algebras in n units. We now define the few terms employed. Let S be a linear associative algebra, the coordinates of whose numbers range over any given field F. Let S have a principal unit ϵ (usually denoted by 1) such that $\epsilon z = z\epsilon = z$ for every number z of S. Let the units of S be $e_1, \ldots, e_n, E_1, \ldots, E_N$, where the e_i are the units of a sub-algebra c (whence every $e_i e_k$ is a linear homogeneous function of $e_1, \ldots, e_n$ with coefficients in F), and the E_j are units of a sub-algebra C, such that every $e_i E_j = 0$, $E_j e_i = 0$. Then S is said to be rationally reducible (*i.e.* reducible with respect to F), and to be the direct sum $S = c + C$ of the component algebras c and C. Conversely, given any two associative algebras $c = (e_1, \ldots, e_n)$ and $C = (E_1, \ldots, E_N)$, the coordinates of whose numbers range over F, and having principal units e and E, we obtain an algebra A

* Wedderburn, *Proc. London Math. Soc.*, Ser. 2, Vol. 6 (1907); Dickson, "Linear Algebras", *Cambridge Tracts in Mathematics*, No. 16 (1914), pp. 66–73.

† See the writer's memoir to be published in *Jour. de Math.*; also his "Algebras and their Arithmetics," University of Chicago Press, 1923, Chap. 10.

with the units $e_1, \ldots, e_n, E_1, \ldots, E_N$ by equating the products

$$e_1E_1, \ldots, e_nE_N, \quad E_1e_1, \ldots, E_Ne_n$$

to linear functions of $e_1, \ldots, E_N$ with any coefficients in F chosen so that the associative law shall hold. An important case is that in which every $e_iE_j = 0$, $E_je_i = 0$; then A is the direct sum of c and C, and its principal unit is $e+E$. Hence we can write down at once all rationally reducible algebras in m units if we know all rationally irreducible algebras in fewer than m units. Thus we need find only the irreducible algebras.

The numbers of $S = c+C$ are $z = x+X$, where x belongs to c and X to C. Then $ze = xe = x = ex = ez$, $e^2 = e$, $e \neq \epsilon$, $e \neq 0$. Conversely, any associative algebra S over F, having the principal unit ϵ, is reducible with respect to F if it contains a number e such that $e \neq \epsilon$, $e \neq 0$, $e^2 = e$, and $ez = ze$ for every number z of S. For, if we write $E = \epsilon - e$ and denote the totality of products ze by c and the totality zE by C, where z ranges over all numbers of S, we readily verify* that $S = c+C$.

Let $e_0 = 1, e_1, \ldots, e_{n-1}$ be the units of any linear associative algebra A over the field F. The powers of $x = \Sigma x_ie_i$ are expressible as linear functions of the e_i. Hence x^n is a linear function of $1, x, \ldots, x^{n-1}$ with coefficients in F. The equation $x^r - c_1x^{r-1} - \ldots - c_r = 0$ of lowest degree with coefficients in F, which is satisfied by the number x with arbitrary coordinates in F, is called the rank equation of the algebra A, and its degree r is the rank of A.

In § 2 we give a simple determination of all rationally irreducible algebras in n units of maximum rank n. Since the numbers are polynomials in a single unit e_1, the associate law shows that multiplication is the same as for polynomials in elementary algebra, and the problem is merely that of distinguishing the reducible from the irreducible algebras. It is shown that the algebra is rationally irreducible if, and only if, in the rank equation $f(x) = 0$, $f(x)$ is a power of a polynomial irreducible in F.

In §§ 3–5 we determine explicitly all rational algebras of the minimum rank 2 except for the special case in which the square of every number not involving the unit 1 is zero. That case itself is a special case of the outstanding problem of nilpotent algebras (some power of every number of which is zero). For this case we find in § 6 all algebras with fewer than eight units.

* Dickson, "Linear Algebras", p. 27.

Algebras of rank 2 in more than 2 units are evidently irreducible. Hence the problem is that of utilizing the conditions implied by the associative law and the reduction of the resulting algebras to normal forms by the introduction of new units.

The simplicity of the problem of finding the algebras of the maximum and minimum ranks lies in the fact that either the associative law or irreducibility is satisfied automatically. But for intermediate ranks we would have to consider both conditions simultaneously.

In § 7 the general theory is applied to the complete determination of all non-equivalent rational algebras in 2, 3, and 4 units.

I. *Algebras of Maximum Rank.*

2. When the rank is the number n of basal units, the algebra A_n contains a number x for which $1, x, x^2, \ldots, x^{n-1}$ are linearly independent with respect to the field F. Hence they may be taken as new basal units $1, e_1, e_2, \ldots, e_{n-1}$. Since every number of the algebra is a polynomial in e_1, multiplication is commutative. We have

$$e_2 = e_1^2, \quad e_3 = e_1 e_2 = e_1^3, \quad e_4 = e_1 e_3 = e_2^2 = e_1^4, \ldots, \quad e_{n-1} = e_1 e_{n-2} = \ldots = e_1^{n-1},$$

$$e_1 e_{n-1} = c_0 + c_1 e_1 + \ldots + c_{n-1} e_{n-1}.$$

Thus e_1 is a root of

$$f(x) \equiv x^n - c_0 - c_1 x - \ldots - c_{n-1} x^{n-1} = 0.$$

If $f(x)$ is irreducible in F, the algebra is the field $F(e_1)$ whose properties are treated in the theory of algebraic numbers.

LEMMA 1.—*If* $f(x) \equiv f_r(x)\phi_r(x)$, *where* f_r *is irreducible in* F, *and*

$$f_r(x) = x^r - a_1 x^{r-1} - \ldots - a_r, \qquad \phi_r(x) = x^r - b_1 x^{r-1} - \ldots - b_r$$

are distinct polynomials with coefficients in F, *the algebra is reducible with respect to* F.

We employ the new basal units $1, e_1, \ldots, e_{r-1}, E_r, \ldots, E_{n-1}$, where

$$E_r = e_r - b_1 e_{r-1} - \ldots - b_r, \quad E_{r+1} = e_1 E_r, \quad E_{r+2} = e_1 E_{r+1}, \ldots,$$

$$E_{n-1} = e_1 E_{n-2}.$$

Denote $\phi_r(x) - f_r(x)$ by $h(x)$. Since $f(e_1) = 0$, $\phi_r(e_1) = E_r$, we have

$$E_r f_r(e_1) = 0, \quad E_r - f_r(e_1) = h(e_1), \quad E_r^2 = E_r h(e_1).$$

Let ρ be a root of $f_r(x) = 0$. In the resulting algebraic field $F(\rho)$, $h(\rho)$ is not zero and hence has an inverse $l(\rho)$. Thus

$$h(x)\,l(x) \equiv 1 + f_r(x)\,q(x),$$

identically in x, where $q(x)$ is a polynomial with coefficients in F. Replace x by e_1 and write Q for $q(e_1)$. Then

$$h(e_1)\,l(e_1) = 1 + [E_r - h(e_1)]\,Q,$$

$$[l(e_1)E_r]^2 = h(e_1)\,l(e_1)\,.\,l(e_1)E_r = l(e_1)E_r,$$

since $[E_r - h(e_1)]E_r = 0$. Hence $z^2 = z$ for $z = l(e_1)E_r$, which is equal to a polynomial in E_r, $e_1E_r = E_{r+1}, \ldots, e_{r-1}E_r = E_{n-1}$. Thus $z \neq 1$, $z \neq 0$. Hence, by § 1, the algebra is reducible with respect to F.

LEMMA 2.—*If $f(x) \equiv f_r(x)f_s(x)$, where $r < s$, f_r is irreducible in the field F, and f_r is not a factor of f_s, the algebra is reducible with respect to F.*

Let $f_r(x) = x^r - a_1x^{r-1} - \ldots - a_r$, $f_s(x) = x^s - b_1x^{s-1} - \ldots - b_s$.

We employ the new basal units $1, e_1, \ldots, e_{r-1}, E_r, \ldots, E_{n-1}$, where

$$(1) \qquad E_r = e_r - a_1e_{r-1} - \ldots - a_r, \quad E_s = e_s - b_1e_{s-1} - \ldots - b_s,$$

$$(2) \qquad E_{r+1} = e_1E_r, \quad E_{r+2} = e_1E_{r+1}, \quad \ldots, \quad E_{s-1} = e_1E_{s-2},$$

$$(3) \qquad E_{s+1} = e_1E_s, \quad E_{s+2} = e_1E_{s+1}, \quad \ldots, \quad E_{n-1} = e_1E_{n-2}.$$

Each E_k is the sum of e_k and a linear function of $e_{k-1}, \ldots, e_1, 1$. For example, if $r < t < s$,

$$(4) \qquad E_t = e_1E_{t-1} = e_1^2E_{t-2} = e_1^{t-r}E_r = e_t - a_1e_{t-1} - \ldots - a_re_{t-r}.$$

We have $E_r = f_r(e_1)$, $E_s = f_s(e_1)$, $f(e_1) = 0$. Hence $E_rE_s = 0$. By repeated multiplications of the first factor by e_1, we get

$$(5) \qquad E_rE_s = 0, \quad E_{r+1}E_s = 0, \quad \ldots, \quad E_{s-1}E_s = 0.$$

Next $e_1E_{s-1} = e_1^2E_{s-2} = \ldots = e_1^{s-r}E_r = e_s - a_1e_{s-1} - \ldots - a_re_{s-r}$.

Eliminate e_s by means of (1), then e_{s-1} by means of (4) for $t = s-1$, then e_{s-2} by means of (4) for $t = s-2$, etc., and finally e_r by means of (1). We get a relation of the form

$$(6) \quad e_1E_{s-1} = E_s - k_{s-1}E_{s-1} - \ldots - k_rE_r - h, \quad h \equiv k_{r-1}e_{r-1} + \ldots + k_1e_1 + k_0,$$

in which the values of the constants k_i will not be needed below.

If $h = 0$, we replace E_{r+1} by $e_1 E_r$, E_{r+2} by $e_1^2 E_r$, ..., E_{s-1} by $e_1^{s-r-1} E_r$ in (6) and obtain

$$E_s = KE_r, \quad K \equiv e_1^{s-r} + k_{s-1} e_1^{s-r-1} + \ldots + k_{r+1} e_1 + k_r.$$

But $f_s(x)$ would then be the product of $f_r(x)$ by $x^{s-r} + \ldots + k_r$, contrary to hypothesis. Hence $h \neq 0$.

By (5) and (6),

$$0 = e_1(E_{s-1}E_s) = (e_1 E_{s-1})E_s = E_s^2 - hE_s. \tag{7}$$

We employ a root ρ of the irreducible equation $f_r(x) = 0$, and write

$$h(x) \equiv k_{r-1}x^{r-1} + \ldots + k_1 x + k_0, \quad h(e_1) = h.$$

As just proved, not all of these k's are zero, whence $h(\rho) \neq 0$. Hence $h(\rho)$ has an inverse $l(\rho) \neq 0$ in the algebraic field $F(\rho)$. Thus

$$h(x)\, l(x) \equiv 1 + f_r(x)\, q(x),$$

identically in x, where $q(x)$ is a polynomial with coefficients in F. Take $x = e_1$ and write Q for $q(e_1)$, and l for $l(e_1)$. Thus

$$hl = 1 + E_r Q.$$

By (7) and $E_r E_s = 0$,

$$(lE_s)^2 = l^2 hE_s = (1 + E_r Q) lE_s = lE_s.$$

Hence $z^2 = z$ for $z = lE_s$, which is equal to a polynomial in

$$E_s, \quad e_1 E_s = E_{s+1}, \quad \ldots, \quad e_{r-1} E_s = E_{n-1}.$$

Thus $z \neq 1$, $z \neq 0$, and the algebra is reducible.

THEOREM 1.—*The algebra A_n is reducible with respect to F, if $f(x)$ is not a power of a polynomial irreducible in F.*

Let $f_r(x)$ be an irreducible polynomial of minimum degree r which divides $f(x)$. Let $f(x) \equiv f_r^t f_c(x)$, where $f_c(x)$ is not divisible by $f_r(x)$. Either f_c is irreducible or it is a product of irreducible factors, the degree of each of which is $\geqslant r$, whence $c \geqslant r$. We shall prove by induction on t that basal units $1, e_1, \ldots, e_{r-1}, E_r, \ldots, E_{n-1}$ may be chosen for algebra A_n such that it contains a number z for which $z^2 = z$, $z \neq 0$, $z \neq 1$, z being a linear function of the E_i (*i.e.* free of $1, e_1, \ldots, e_{r-1}$), this result

being assumed when t is replaced by $t-1$. For $t=1$, this result was proved in Lemma 1 when $c=r$, and in Lemma 2 when $c>r$.

For $t>1$, write $f_s=f_r^{t-1}f_c$. Then $f(x)=f_rf_s$, $s>r$. We may proceed as in the proof of Lemma 2. Since f_s is now divisible by f_r, we have $h=0$ in (6). Then $E_s^2=0$ by (7). By repeated multiplications by e_1, we get

$$E_iE_j=0 \quad (i,j=s,\ s+1,\ \ldots,\ n-1). \tag{8}$$

Since $r+s=n$ and $E_rE_s=0$,

$$e_1^rE_s=e_1^{r-1}(e_1E_s)=e_1^{r-1}E_{s+1}=e_1^{r-2}E_{s+2}=\ldots=e_1E_{n-1},$$

$$e_1E_{n-1}=e_rE_s=(E_r+a_1e_1^{r-1}+\ldots+a_r)E_s=a_1E_{n-1}+\ldots+a_rE_s,$$

by (1). This and (3) show that the product of any one of

$$E_s,\ E_{s+1},\ \ldots,\ E_{n-1} \tag{9}$$

by e_1 is a linear homogeneous function of the units (9). The same is therefore true of the products of (9) by any power of e_1 and hence by every number of the algebra A_n.

Consider any two numbers $a'+a$ and $b'+b$ of algebra A_n, where a and b are linear functions of the units (9), while a' and b' are linear functions of the remaining units

$$1,\ e_1,\ \ldots,\ e_{r-1},\ E_r,\ E_{r+1},\ \ldots,\ E_{s-1}. \tag{10}$$

By the preceding result, ab' and ba' are linear functions of the units (9). Also, $ab=0$ by (8). Hence, if we let $a'+a$ correspond to a', and, similarly, $b'+b$ to b', then $(a'+a)(b'+b)$ and $a'b'$ both correspond to the same linear function of the units (10). In other words, if we suppress the terms involving (9) in all the numbers of A_n, we obtain an isomorphic algebra A'. Since E_s is suppressed, we see from (1) that e_1 is a root of $f_s(x)=0$ in algebra A'. Thus A' is the algebra A_s whose $f(x)$ is $f_s(x)=f_r^{t-1}f_c$. Hence, by the hypothesis for the induction on t, algebra A' has the units $1, e_1, \ldots, e_{r-1}, E_r, \ldots, E_{s-1}$, defined as in (1)–(3) with s replaced by $s-r$, and n by s, and A' contains a linear combination z' of $E_r, \ldots, E_{s-1}$ such that $z'^2=z'$, $z'\neq 0$, $z'\neq 1$.

In algebra A_n, let $z'^2-z'=v$, so that v is a linear function of the units (9). Thus $v^2=0$ by (8), and $z'v=0$ by (5) and the relations

$$E_iE_{s+1}=0,\ \ldots,\quad E_iE_{n-1}=0 \quad (i=r,\ \ldots,\ s-1),$$

derived from (5) by repeated multiplications by e_1. Write $z = z' + v$. Hence $z^2 = z'^2 = z' + v = z$. If $z = 0$, its component v in $E_s, \ldots, E_{n-1}$ and component z' in $E_r, \ldots, E_{s-1}$ would both be zero, contrary to the above. Similarly, $z \neq 1$. Hence the induction is complete.

THEOREM 2.—*Algebra A_n is irreducible with respect to the field F if $f(x)$ is a power f_r^t of a polynomial irreducible in F.*

For, if A_n is reducible with respect to F and has the component algebras c and C, then $f(x) \equiv rR$, where $r(x) = 0$ is the rank equation of e_1 for c, and $R(x) = 0$ is that of e_1 for C. Thus r and R are powers of $f_r(x)$ with positive integral exponents $< t$. The general number z of A_n may be written in the form $x + X$, where x is in c, and X is in C. Since $xX = Xx = 0$,

$$z^k = x^k + X^k, \quad \lambda(z) = \lambda(x) + \lambda(X),$$

where λ is any polynomial with coefficients in F. Take λ to be f_r^{t-1}, and take $z = e_1$. Then $\lambda(x)$ has the factor $r(x)$ and is zero. Likewise, $\lambda(X) = 0$. Hence $\lambda(z) = 0$, whereas $f(z) = 0$ is the equation of lowest degree satisfied by e_1.

For example, let $f_r(x) = x^2 - \nu$, $t = 3$, where ν is not a square in F. As in (1)–(3), introduce the units

$$1, \quad e_1, \quad E_2 = e_1^2 - \nu, \quad E_3 = e_1 E_2, \quad E_4 = E_2^2, \quad E_5 = e_1 E_4.$$

We obtain the rationally irreducible algebra

$$e_1^2 = E_2 + \nu, \quad e_1 E_2 = E_3, \quad e_1 E_3 = \nu E_2 + E_4, \quad e_1 E_4 = E_5, \quad e_1 E_5 = \nu E_4,$$

$$E_2^2 = E_4, \quad E_2 E_3 = E_5, \quad E_2 E_4 = E_2 E_5 = 0, \quad E_3^2 = \nu E_4,$$

$$E_3 E_4 = E_3 E_5 = E_4^2 = E_4 E_5 = E_5^2 = 0,$$

having $(E_2, \ldots, E_5)$ as a nilpotent invariant sub-algebra. For $t = 2$, see (37).

II. *Algebras of Minimum Rank.*

3. Let the coordinates of the numbers of a linear associative algebra A range over a field F; let A have the principle unit 1, and let every number of A satisfy a quadratic equation with coefficients in F, the leading coefficient being unity. After completing the square, we may assume

that the units are $1, e_1, \ldots, e_m$, where e_i^2 is equal to a number s_{ii} of F. For $i \neq j$, $(e_i \pm e_j)^2$ is a linear function of $e_i \pm e_j$. Hence $e_ie_j + e_je_i$ is a linear function of $e_i + e_j$ and also a linear function of $e_i - e_j$, and hence is a number of F:

$$e_ie_j + e_je_i = 2s_{ij} = 2s_{ji} \quad (i, j = 1, \ldots, m).$$

Let $u_1, \ldots, u_m$ be arbitrary numbers of F and write $U = \Sigma u_k e_k$. Then

$$U^2 = Q, \quad Q = \sum_{k,l=1}^{m} s_{kl} u_k u_l.$$

This quadratic form can be reduced to $\Sigma c_i v_i^2$ by a linear transformation, with coefficients in F of determinant $\neq 0$,

$$u_k = \sum_{l=1}^{m} a_{kl} v_l \quad (k = 1, \ldots, m).$$

We may introduce the linearly independent new units

$$E_l = \sum_{k=1}^{m} a_{kl} e_k \quad (l = 1, \ldots, m).$$

Then $U = \sum_{k,l} a_{kl} v\, e_k = \sum_l v_l E_l, \quad U^2 = \sum_{k,l} v_k v_l E_k E_l = \Sigma c_i v_i^2.$

Hence

$$E_k^2 = c_k, \quad E_kE_l + E_lE_k = 0 \quad (l, k = 1, \ldots, m;\ l \neq k). \tag{11}$$

By the definition of a linear algebra, we have

$$E_iE_j = \gamma_{ij} + \sum_{k=1}^{m} \gamma_{ijk} E_k \quad (i, j = 1, \ldots, m;\ i \neq j), \tag{12}$$

where the constants γ are numbers of F. By the associative law and (11)

$$E_i \,.\, E_iE_j = c_iE_j, \quad -E_iE_j \,.\, E_i = E_jE_i \,.\, E_i = c_iE_j.$$

Subtracting, applying (12) and then (11), we get

$$2\gamma_{ij}E_i + \sum_{k=1}^{m} \gamma_{ijk}(E_iE_k + E_kE_i) = 2\gamma_{ij}E_i + 2\gamma_{iji}c_i = 0,$$

$$\gamma_{ij} = 0, \quad \gamma_{iji}c_i = 0 \quad (i, j = 1, \ldots, m;\ i \neq j). \tag{13}$$

By (11) and (12),

$$\gamma_{ijk} = -\gamma_{jik} \quad (i, j, k = 1, \ldots, m;\ i \neq j). \tag{14}$$

4. *The case* $c_1c_2 \neq 0$. By (13), $\gamma_{1j1} = 0 \ (j \neq 1)$, $\gamma_{2k2} = 0 \ (k \neq 2)$. Then $\gamma_{122} = 0$ by (14). Thus

$$E_1E_2 = \sum_{k=3}^{m} \gamma_{12k}E_k, \qquad -c_1c_2 = (E_1E_2)^2 = \sum_{k=3}^{m} \gamma_{12k}^2 c_k \neq 0.$$

Denote the final sum by c_3'. Thus $m \geqslant 3$. Permuting $E_3, \ldots, E_m$, we may take $\gamma_{123} \neq 0$. Thus we may introduce the new units

$$e_1 = E_1, \quad e_2 = E_2, \quad e_3 = \sum_{k=3}^{m} \gamma_{12k}E_k, \quad e_i = E_i + \rho_i e_3 \quad (i \geqslant 4).$$

Taking $\rho_i = -c_i\gamma_{12i}/c_3'$, we find that

$$e_3e_i + e_ie_3 = 0, \quad e_je_3 + e_3e_j = 0, \quad e_je_i + e_ie_j = 0$$
$$(i \geqslant 4, \ j = 1, 2).$$

The latter relations obviously hold true when any $e_i (i \geqslant 4)$ is replaced by a linear homogeneous function e_i' of $e_4, \ldots, e_m$. By the argument leading to (11), we can choose linearly independent functions $e_4', \ldots, e_m'$ with coefficients in F such that

$$e_k'^2 = c_k', \quad e_k'e_l' + e_l'e_k' = 0 \quad (k, l = 4, \ldots, m; \ l \neq k).$$

Dropping accents, we have units for which

$$e_i^2 = c_i, \quad e_ie_j + e_je_i = 0 \quad (i, j = 1, \ldots, m; \ i \neq j), \tag{15}$$

and $e_1e_2 = e_3$, $c_3 = -c_1c_2 \neq 0$. If $m > 3$,

$$e_3e_4 = e_1e_2e_4 = -e_1 \cdot e_4e_2 = e_4e_1 \cdot e_2 = e_4e_3, \quad e_3e_4 + e_4e_3 = 0,$$

whence $e_3e_4 = 0$, $0 = e_3^2e_4 = c_3e_4$, whereas $c_3 \neq 0$. Hence $m = 3$. Then

$$e_1e_3 = e_1 \cdot e_1e_2 = c_1e_2, \quad e_3e_2 = e_1e_2 \cdot e_2 = c_2e_1,$$

and we obtain the algebra*

$$\begin{cases} e_1^2 = c_1, \quad e_2^2 = c_2, \quad e_3^2 = -c_1c_2, \quad e_1e_2 = e_3, \quad e_2e_1 = -e_3, \\ e_1e_3 = c_1e_2, \quad e_3e_1 = -c_1e_2, \quad e_3e_2 = c_2e_1, \quad e_2e_3 = -c_2e_1. \end{cases} \tag{16}$$

By a transformation of units usually irrational with respect to the field F, we may divide e_1, e_2, e_3 by the square roots of $-c_1$, $-c_2$, c_1c_2,

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* Here and henceforth we omit the relations $1^2 = 1$, $1e_i = e_i1 = e_i$.

respectively, and obtain the case $c_1 = c_2 = -1$, in which (16) is the algebra of quaternions. Since the associative law holds for the latter, it holds also for (16).

THEOREM 3.—*Every associative algebra* (15) *in which at least two* c_i *are not zero is rationally equivalent to the generalization* (16) *of quaternions.*

We readily deduce all linear associative algebras with a principal unit 1 such that each coordinate of every number of the algebra ranges over all real numbers and such that a product is zero only when one factor is zero. The rank equation $f(\rho) = 0$ has real coefficients. Since $f(\rho)$ is a product of real linear and quadratic factors, every number of the algebra satisfies a real quadratic equation. Hence the assumptions in § 3 hold when F is the field of all real numbers. If there are only one or two units, the algebra is evidently that of ordinary real or complex numbers. In the contrary case, $m \geqslant 2$ and $c_1 c_2 \neq 0$, since $E_i E_i = 0$ contradicts our hypothesis, and the algebra is equivalent to that of quaternions with real coordinates. This result was first proved by Frobenius.*

5. *The case* $c_1 = 0, \ldots, c_{m-1} = 0, c_m \neq 0$. By (12) and (13),

$$e_m e_j = \sum_{k=1}^{m-1} \gamma_{mjk} e_k \quad (j = 1, \ldots, m-1). \tag{17}$$

If $m = 2$, $e_1^2 = 0$, $e_2 e_1 = \gamma e_1$, $c_2 e_1 = e_2^2 e_1 = \gamma^2 e_1$, $c_2 = \gamma^2$. Introduce e_2/γ as a new unit e_2. We obtain the algebra

$$(1, e_1, e_2): \quad e_1^2 = 0, \quad e_2^2 = 1, \quad e_1 e_2 = -e_1, \quad e_2 e_1 = e_1. \tag{18}$$

Henceforth, let $m \geqslant 3$. First, let

$$e_m e_j = b_j e_j \quad (j = 1, \ldots, m-1). \tag{19}$$

Then $$c_m e_j = e_m^2 e_j = b_j^2 e_j, \quad c_m = b_j^2.$$

Introducing e_m/b_1 as a new unit in place of e_m, we have $c_m = b_1 = 1$.

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* *Jour. für Math.*, Vol. 84 (1878), p. 59. *Cf.* C. S. Peirce, *Amer. Jour. Math.*, Vol. 4 (1881), p. 225.; Dickson, "Linear Algebras," *Cambridge Tracts*, p. 10.

Permuting $e_2, \ldots, e_{m-1}$, we may assume that the positive b_j precede the negative b_j. Thus

$$e_m^2 = 1, \quad e_m e_i = +e_i \quad (i = 1, \ldots, k), \tag{20}$$

$$e_m e_j = -e_j \quad (j = k+1, \ldots, m-1).$$

For $1 \leqslant i \leqslant k$, $k+1 \leqslant j \leqslant m-1$,

$$e_i e_j = e_m e_i \,.\, e_j = e_m \,.\, e_i e_j = -e_m e_j e_i = +e_j e_i = -e_i e_j, \quad e_i e_j = 0.$$

If $k > 1$, let i and i_1 be integers selected from $1, \ldots, k$. Then

$$e_i e_{i_1} = e_i e_m e_{i_1} = -e_m e_i \,.\, e_{i_1} = -e_i e_{i_1}, \quad e_i e_{i_1} = 0.$$

Similarly, $e_j e_{j_1} = 0$. Hence

$$e_r e_s = 0 \quad (r, s = 1, \ldots, m-1). \tag{21}$$

In case the number $m-k-1$ of minus signs in (20) exceeds the number k of plus signs, replace e_m by $-e_m$ and interchange e_i with e_{m-i} for $i = 1, \ldots, k$; we obtain (20) with k and $m-k+1$ interchanged. Hence we may assume that $k \geqslant m-k-1$.

No two of the resulting algebras defined by* (20) and (21) are equivalent. For, if $x = x_0 + \Sigma x_i e_i$ does not reduce to x_0 and if x^2 is a multiple of the unit 1, then $x_0 x_i = 0$ $(i = 1, \ldots, m)$ and hence $x_0 = 0$. Thus any transformation on the units which replaces an algebra (15) by another such algebra must replace every number $x = \Sigma x_i e_i$ by another such number. Consider a number $y = \Sigma y_i e_i$ for which $xy = y$ is solvable for such an x. By (20) and (21), xy has no term involving e_m. Hence $y_m = 0$. Then

$$xy = x_m e_m y = x_m \sum_{i=1}^{k} y_i e_i - x_m \sum_{j=k+1}^{m-1} y_j e_j.$$

This is equal to y if and only if

$$(x_m - 1) y_i = 0 \quad (i = 1, \ldots, k), \qquad (x_m + 1) y_j = 0 \quad (j = k+1, \ldots, m-1).$$

If $x_m^2 \neq 1$, then $y = 0$. If $x_m = +1$, y is a linear combination of the units e_i $(i = 1, \ldots, k)$. If $x_m = -1$, y is a linear combination of the units e_j $(j = k+1, \ldots, m-1)$. But $k \geqslant m-k-1$. Hence k is the maximum number of linearly independent numbers $y = \Sigma y_i e_i$ for which $xy = y$ is

* And $e_i e_j = -e_j e_i$ $(i, j = 1, \ldots, m;\ i \neq j)$. These relations are always understood when we speak below of an algebra.

solvable for $x = \Sigma x_i e_i$. Thus k is an invariant under every transformation on the units.

Second, let one of the relations (19) fail. Permuting the e_j, we may take $e_m e_1 \neq b_1 e_1$. Introducing a new unit e_2, we may write $e_m e_1 = e_2$. Then $e_m e_2 = c_m e_1$. Either we have

$$e_m e_j = a_{j1} e_1 + \ldots + a_{j,2r} e_{2r} + f_j e_j \quad (j = 2r+1, \ldots, m-1), \tag{22}$$

for $r = 1$, or else, for a certain value of j, $3 \leqslant j \leqslant m-1$, $e_m e_j$ is not a linear function of e_1, e_2, e_j. In the latter case, we interchange e_3 with e_j, introduce the value of $e_m e_3$ as a new unit e_4, and have

$$e_m e_3 = e_4, \quad e_m e_4 = c_m e_3.$$

Proceeding in this manner we ultimately obtain

$$e_m e_{2i-1} = e_{2i}, \quad e_m e_{2i} = c_m e_{2i-1} \quad (i = 1, \ldots, r), \tag{23}$$

and also (22) if $m-1 > 2r$.

Consider the case $m-1 > 2r$. Then each $f_j \neq 0$ in (22). For, if $f_j = 0$ for a certain j, $c_m e_j = e_m \Sigma a_{jk} e_k$ would be equal, by (23), to a linear function of $e_1, \ldots, e_{2r}$, whereas $c_m \neq 0$. Introduce new units

$$E_j = e_j - a_{j2} e_1 - a_{j4} e_3 - a_{j6} e_5 - \ldots - a_{j,2r} e_{2r-1} \quad (j = 2r+1, \ldots, m-1).$$

Then, by (22) and (23),

$$e_m E_j = b_1 e_1 + b_3 e_3 + b_5 e_5 + \ldots + b_{2r-1} e_{2r-1} + f_j E_j,$$

in which the values of the b_i will not be needed. Then

$$c_m E_j = e_m^2 E_j = b_1 e_2 + b_3 e_4 + \ldots + b_{2r-1} e_{2r} + f_j [b_1 e_1 + b_3 e_3 + \ldots + b_{2r-1} e_{2r-1} + f_j E_j].$$

Hence every $b_i = 0$ and $c_m = f_j^2$. Replacing e_m/f_j by a new e_m, we may assume that $c_m = 1$ from the start. We have $e_m E_j = f_j E_j$ $(j \geqslant 2r+1)$, $f_j^2 = 1$. Introduce the further new units

$$E_{2i-1} = e_{2i-1} + e_{2i}, \quad E_{2i} = e_{2i-1} - e_{2i} \quad (i = 1, \ldots, r). \tag{24}$$

Then, by (23),

$$e_m E_{2i-1} = E_{2i-1}, \quad e_m E_{2i} = -E_{2i}. \tag{25}$$

Hence we obtain an earlier algebra (20).

Next, let $m-1=2r$ and c_m be the square of a number of the field F. We may divide e_m by a number of F and take $c_m=1$. Introducing the new units (24), we obtain (25) and hence an algebra (20),

Finally, let $m-1=2r$ and c_m be not a square in F. We have

$$e_m^2=c_m, \quad e_j^2=0 \quad (j=1, \ldots, m-1), \tag{26}$$

and (23). We shall prove relations (21) and hence obtain a complete multiplication table of the units $e_1, \ldots, e_m$ of our algebra A. For this proof we make use of an irrational transformation on the units; we employ

$$E_m=\rho e_m, \quad E_{2i-1}=e_{2i-1}+\rho e_{2i}, \quad E_{2i}=e_{2i-1}-\rho e_{2i} \tag{27}$$

$$(i=1, \ldots, r),$$

where $\rho=1/\sqrt{c_m}$. Then, by (23) and (26),

$$E_m^2=1, \quad E_m E_{2i-1}=+E_{2i-1}, \quad E_m E_{2i}=-E_{2i}$$

$$(i=1, \ldots, r).$$

which is algebra (20) with $k=r$. By the proof below (20),

$$E_j E_k=0 \quad (j, k=1, \ldots, m-1).$$

By solving (27), we see that e_{2i-1} and e_{2i} are linear homogeneous functions of E_{2i-1} and E_{2i}. Hence (21) follows at once.

However, our algebra A is not rationally equivalent to any algebra (20). In fact, A does not contain numbers x and y corresponding to e_m and e_1 in (20) and hence such that $xy=y$, $y\neq 0$. For, by the remark below (21), neither x nor y involves the unit 1. Then, by (23) and (26), xy does not involve e_m. Write $x=\Sigma x_j e_j$, $y=\Sigma y_j e_j$. Then, by (21) and (23),

$$xy=x_m e_m y=x_m \sum_{i=1}^{r} (y_{2i-1} e_{2i}+y_{2i} c_m e_{2i-1}).$$

This is equal to y if and only if

$$x_m y_{2i-1}=y_{2i}, \quad x_m y_{2i} c_m=y_{2i-1} \quad (i=1, \ldots, r).$$

Elimination of y_{2i-1} gives $(x_m^2 c_m-1)y_{2i}=0$. Since c_m is not a square in the field F, $y_{2i}=0$, and hence $y=0$.

THEOREM 4.—*Every associative algebra* (15) *with*

$$c_1 = 0, \ldots, c_{m-1} = 0, c_m \neq 0,$$

and having the principal unit 1, *is rationally equivalent either to one of the algebras defined by* (20) *and* (21) *for* $k \geqslant \frac{1}{2}(m-1)$ *or to the algebra defined by* (21), (23), *and* (26), *with* $m = 2r+1$, c_m *being* not a square in the field* F. *In each case, it is understood that every* $e_ie_j = -e_je_i$ $(i \neq j)$. *No two of these algebras are rationally equivalent.*

6. *The case* $c_1 = 0, \ldots, c_m = 0$. By (12) and (13), e_ie_j is a linear homogeneous function of $e_1, \ldots, e_m$ with coefficients in F. We shall lay aside the unit 1 and discuss the algebras A in $e_1, \ldots, e_m$ in a field F for which

$$e_i^2 = 0, \quad e_je_i = -e_ie_j \quad (i, j = 1, \ldots, m;\ i \neq j). \tag{28}$$

For any numbers x and y of A, $xy+yx = 0$. In particular, $x^2 = 0$. Hence, if we introduce as new units any linear homogeneous functions of $e_1, \ldots, e_m$ with coefficients in the field F, we again have relations (28).

LEMMA 1.—*Algebra* A *does not contain two linearly independent numbers* x *and* y *such that* $xy = ax+by$, *where* a *and* b *are not both zero.*

For we may assume that $a \neq 0$ after interchanging x and y if necessary. Write $X = ax+by$, $Y = y/a$. Then

$$XY = X, \quad 0 = XY^2 = XY.Y = XY = X.$$

The algebra in which every $e_ie_j = 0$ needs no attention. Hence let $e_1e_2 \neq 0$. Then $e_1e_2 \neq ae_1+be_2$ by Lemma 1. Thus we may choose a new unit e_3 such that

$$e_1e_2 = e_3, \quad 0 = e_1^2e_2 = e_1 . e_1e_2 = e_1e_3, \quad 0 = e_1e_2^2 = e_3e_2. \tag{29}$$

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* We retain only one of a set of values of c_m which differ by square factors. For, the algebra A′ having $c'_m = s^2c_m$ is obtained from A by introducing the new units

$$e'_{2i-1} = e_{2i-1}, \quad e'_{2i} = se_{2i} \ (i = 1, \ldots, r), \quad e'_m = se_m.$$

Finally, an algebra A' whose c'_m is not the product of c_m by the square of a number of F is not rationally equivalent to A. For, $e'_m = \Sigma x_ie_i$ does not involve the unit 1 by the remark below (21), whence $c'_m = e'^2_m = x_m^2c_m$.

First, let every e_ie_j be a multiple of e_3. Write $e_1e_j = c_je_3$. For $j > 3$, take $e_j - c_je_2$ as a new unit in place of e_j; thus $e_1e_j = 0$. Similarly, if $e_2e_j = d_je_3$ for $j > 3$, we take $e_j + d_je_1$ as a new unit in place of e_j and have

$$e_1e_j = 0, \quad e_2e_j = 0, \quad e_3e_j = 0 \quad (j = 4, \ldots, m). \tag{30}$$

If not every $e_ie_j = 0$ $(i \geqslant 4,\ j \geqslant 4)$, we may write $e_4e_5 = e_3$. Then $e_3e_4 = e_3e_5 = 0$. As above we add a multiple of e_5 to e_j and a multiple of e_4 to e_j and get

$$e_4e_j = 0, \quad e_5e_j = 0 \quad (j = 6, \ldots, m).$$

THEOREM 5.—*Let* 1 2 3 *denote the pair of equations*

$$e_1e_2 = e_3, \quad e_2e_1 = -e_3.$$

Let $[\alpha, \beta, \ldots]$ *denote the algebra for which relations* $\alpha, \beta, \ldots$ *hold, while all further products* e_ie_j *are zero. Then the associative algebras* (28), *for which every* e_ie_j *is a multiple of* e_3, *are equivalent to one of*

$$[1\,2\,3], \quad [1\,2\,3,\ 4\,5\,3], \quad [1\,2\,3,\ 4\,5\,3,\ 6\,7\,3], \quad \ldots,$$

or the algebra in which every $e_ie_j = 0$.

No two of these algebras are equivalent. For, let $x = \Sigma x_ie_i$, etc. Then for the first and second algebras,

$$XY = Be_3, \quad xy = be_3, \quad B = X_1Y_2 - X_2Y_1, \quad b = x_1y_2 - x_2y_1 + x_4y_5 - x_5y_4.$$

But the bilinear forms B and b are of ranks 2 and 4 and hence are not equivalent under a linear transformation on the x's and one on the y's.

Second, let e_1e_j and e_2e_j $(j > 3)$, but not all e_ie_j $(i > 3,\ j > 3)$, be multiples of e_3. We have (30) and, say, $e_4e_5 \neq ce_3$. Then $e_1e_4e_5$ and $e_2e_4e_5$ vanish, so that e_4e_5 lacks e_2 and e_1. If $e_4e_5 = ce_3 + ae_4 + be_5$, the proof of Lemma 1 (with $x = e_4$, $y = e_5$) shows that $a = b = 0$, contrary to hypothesis. Hence we may choose a new unit e_6 such that $e_4e_5 = e_6$. Hence the algebra has the sub-algebra $[1\,2\,3,\ 4\,5\,6]$.

Third, let not all e_1e_j, e_2e_j $(j > 3)$ be multiples of e_3. Hence we may take $e_1e_4 \neq ce_3$. If $e_1e_4 = ae_1 + be_2 + ce_3 + de_4$, then $0 = e_1^2e_4 = be_3 + de_1e_4$ has the term d^2e_4, whence $d = 0$, $b = 0$, and

$$e_3e_4 = e_1e_2e_4 = -e_2e_1e_4 = ae_3, \quad 0 = e_1e_4^2 = a^2e_1 + 2ace_3, \quad a = 0,$$

contrary to $e_1e_4 \neq ce_3$. Hence we may choose a new unit e_5 such that $e_1e_4 = e_5$. But $ab = c$ implies

$$xc = xab = -axb = abx = cx = -xc.$$

LEMMA 2.—*If* $ab = c$, *then* $xc = 0$ *for every* x.

Hence every $e_i e_3 = 0$, $e_i e_5 = 0$. If $e_2 e_4$ is independent of $e_1, \ldots, e_5$, we may take $e_2 e_4 = e_6$. Then every $e_i e_6 = 0$, and the algebra has the sub-algebra [1 2 3, 1 4 5, 2 4 6].

There remains the case

$$e_2 e_4 = re_1 + se_2 + te_3 + pe_4 + qe_5.$$

Multiplication by e_1 and e_4 gives $0 = se_3 + pe_5$, $0 = re_5$. Thus

$$e_2 e_4 = te_3 + qe_5.$$

For the new units $E_2 = e_2 - qe_1$, $E_4 = e_4 + te_1$, we get $e_1 E_2 = e_3$, $e_1 E_4 = e_5$, $E_2 E_4 = 0$. Hence the algebra has the sub-algebra [1 2 3, 1 4 5].

THEOREM 6.—*Every algebra* (28) *not given by Theorem* 5 *has one of the sub-algebras*

$$[1\,2\,3,\ 1\,4\,5],\quad [1\,2\,3,\ 1\,4\,5,\ 2\,4\,6],\quad [1\,2\,3,\ 4\,5\,6].$$

To complete the investigation for $m = 6$, it remains to treat the algebra having [1 2 3, 1 4 5] as a sub-algebra. By Lemma 2, every $e_i e_3 = 0$, $e_i e_5 = 0$. Write

$$e_1 e_6 = ae_1 + be_2 + ce_3 + de_4 + fe_5 + ge_6.$$

Then $0 = e_1^2 e_6 = be_3 + de_5 + ge_1 e_6$ has the term $g^2 e_6$, whence $g = 0$, $b = d = 0$. Then $0 = e_1 e_6^2 = ae_1 e_6$, $a = 0$. Thus $e_1 e_6 = ce_3 + fe_5$. Taking $e_6 - ce_2 - fe_4$ as a new unit e_6, we have $e_1 e_6 = 0$. Since $e_2^2 e_6 = 0$, we see that

$$e_2 e_6 = \beta e_2 + \gamma e_3 + \delta e_4 + \rho e_5.$$

Then $0 = e_3 e_6 = e_1 e_2 e_6 = \beta e_3 + \delta e_5$, $\beta = \delta = 0$. Taking $e_6 + \gamma e_1$ as a new unit e_6, we have

$$(31)\qquad e_1 e_2 = e_3,\quad e_1 e_4 = e_5,\quad e_2 e_6 = \rho e_5,\quad e_1 e_6 = e_3 e_6 = e_5 e_6 = 0 \qquad (\rho = 0 \text{ or } 1).$$

From $e_1 e_4 e_6 = e_5 e_6 = 0$, $e_4^2 e_6 = 0$, we see that $e_4 e_6 = ce_3 + pe_5$. If $\rho = 0$, $c \neq 0$, we interchange e_2 with e_4, and e_3 with e_5, and obtain

$$e_1 e_4 = e_5,\quad e_1 e_2 = e_3,\quad e_2 e_6 = pe_3 + ce_5,\quad e_i e_6 = 0 \quad (i = 1, 3, 4, 5).$$

Taking $E_6 = (e_6 + pe_1)/c$ as a new unit e_6, we obtain (31) with $\rho = 1$.

Next, if $\rho = c = 0$, either $p = 0$ and the algebra is $[1\,2\,3,\ 1\,4\,5]$, or we may set $p = -1$ and, after taking $e_1 - e_6$ as a new unit e_1, obtain the algebra $[1\,2\,3,\ 6\,4\,5]$ of earlier type.

Finally, consider (31) for $\rho = 1$. Taking

$$E_4 = e_4 - \tfrac{1}{2}pe_2, \quad E_5 = e_5 - \tfrac{1}{2}pe_3, \quad E_6 = e_6 + \tfrac{1}{2}pe_1,$$

we have $e_1E_4 = E_5$, $e_2E_6 = E_5$, $E_4E_6 = ke_3$, and hence

$$A_k = [1\,2\,3,\ 1\,4\,5,\ 2\,6\,5,\ e_4e_6 = ke_3]. \tag{32}$$

It remains to discuss the equivalence of the algebras

$$A_k, \quad B = [1\,2\,3,\ 4\,6\,5], \quad C = [1\,2\,3,\ 1\,4\,5],$$

in each of which every product is a linear function of two units e_3 and e_5. For C, the only numbers y such that $xy = 0$ for every x are the linear combinations of e_3, e_5, e_6, while for A_k and B the y's are the linear combinations of e_3, e_5 only. Hence C is equivalent to neither A_k nor B.

In B, $xe_2 = e_3$ for any $x = \Sigma x_i e_i$ having $x_1 = 1$, and $xe_6 = e_5$ for any x having $x_4 = 1$. But in A_k the two conditions for $xy = ce_3 + de_5$, where c and d are given numbers not both zero, are

$$\begin{array}{cccc|l} x_1 & x_2 & x_4 & x_6 & \\ \hline y_2 & -y_1 & -ky_6 & -ky_4 & = c \\ y_4 & y_6 & -y_1 & -y_2 & = d. \end{array}$$

If any two-rowed determinant of this matrix is not zero, the corresponding two x_i are uniquely determined in terms of the remaining two x_i, while x_3 and x_5 are arbitrary, and y does not have the property noted for both e_2 and e_6 in B. Hence let every two-rowed determinant vanish. Then $y_1^2 - ky_6^2 = 0$, $y_2^2 - ky_4^2 = 0$. If k is not a square in the field F,

$$y_6 = y_1 = y_4 = y_2 = 0 \quad \text{and} \quad xy \equiv 0,$$

contrary to hypothesis. If $k = 0$, then $y_1 = y_2 = 0$, $c = 0$, and $xe_6 = e_5$ for any x having $x_2 = 1$; but for $E_3 = ce_3 + de_5$, $c \neq 0$, there exists no y for which $xy = E_3$ for every x having a single condition on the x_i. Hence, if $k = 0$ or if k is not the square of a number of F, algebra A_k is not rationally equivalent to B. Finally, let $k = \alpha^{-2}$, where α is a number $\neq 0$ of F. In B introduce the new (linearly independent) units

$$E_1 = \alpha(e_1 - e_4), \quad E_2 = \alpha(e_6 - e_2), \quad E_3 = -\alpha^2(e_3 + e_5),$$
$$E_4 = e_2 + e_6, \quad E_5 = \alpha(e_3 - e_5), \quad E_6 = e_1 + e_4.$$

We obtain A_k. Taking te_2, te_3, $t^{-1}e_6$ as new units in place of e_2, e_3, e_6, we see that A_k becomes $A_{k'}$ where $k' = k/t^2$, whence we may remove any square factor from k.

Theorem 7.—*The non-equivalent associative algebras* (28) *are those in Theorem* 5 *and, if* $m \leqslant 6$, *the algebras* [1 2 3, 1 4 5], [1 2 3, 1 4 5, 2 4 6], *and* A_k *given by* (32), *in which we retain only one of a set of* k*'s differing by square factors. Instead of* A_1 *we may take* [1 2 3, 4 6 5].

III. *All Rational Algebras in* 2, 3, 4 *Units.*

7. Let the coordinates of the numbers of a linear associative algebra A range over a field F and let A have the principal unit 1 and the further units $e_1, \ldots, e_{n-1}$.

If $n = 2$, we may evidently take $e_1^2 = 0$, 1, or a number ν which is not a square in F. If $e_1^2 = 1$, we introduce the new units $e = \frac{1}{2}(1+e_1)$, $E = \frac{1}{2}(1-e_1)$, and obtain the reducible algebra

$$(e)+(E): \quad e^2 = e, \quad E^2 = E, \quad eE = Ee = 0.$$

If $n = 3$, the rationally reducible algebras are evidently $A+(e_2)$, where $e_2^2 = e_2$ and A is one of the preceding algebras in two units, and hence are $(e_0)+(e_1)+(e_2)$ and

$$(33) \quad (e_0, e_1)+(e_2): \quad e_0^2 = e_0, \quad e_0e_1 = e_1e_0 = e_1, \quad e_1^2 = 0 \text{ or } \nu e_0, \quad e_2^2 = e_2.$$

These algebras are of rank $2+1 = 3$. By § 2, the rationally irreducible algebras of rank 3 in three units are the cubic fields $F(e_1)$, where e_1 is a root of a cubic equation irreducible in F, and

$$(34) \qquad (1, e_1, e_2): \quad e_1^2 = e_2, \quad e_1e_2 = e_2e_1 = 0, \quad e_2^2 = 0,$$

for which the rank equation of e_1 is $x^3 = 0$. The (irreducible) algebras of rank 2 are (18) and

$$(35) \qquad (1, e_1, e_2): \quad e_1^2 = e_2^2 = e_1e_2 = e_2e_1 = 0.$$

If $n = 4$, the rationally reducible algebras are $B+(e_3)$, where $e_3^2 = e_3$ and B is one of the preceding algebras in three units with the principal unit 1 replaced by e_0. By § 2, the rationally irreducible algebras of rank 4 are the fields $F(e_1)$, where e_1 is a root of a quartic equation irreducible

in F, and the commutative algebras

$$(36) \qquad e_1^2 = e_2, \quad e_1e_2 = e_3, \quad e_1e_3 = e_2^2 = e_2e_3 = e_3^2 = 0;$$

$$(37) \quad e_1^2 = E_2 + \nu, \quad e_1E_2 = E_3, \quad e_1E_3 = \nu E_2, \quad E_2^2 = E_2E_3 = E_3^2 = 0,$$

for which the rank equations of e_1 are $x^4 = 0$, $(x^2 - \nu)^2 = 0$, respectively, ν being not a square in F. By §§ 4–6, the (irreducible) algebras of rank 2 are the generalized quaternion algebras (16); the two algebras defined by (20) and (21) for $m = 3$, $k = 1$ or 2; (23) for $r = 1$, viz.,

$$(38) \quad e_1^2 = e_2^2 = 0, \quad e_3^2 = \nu, \quad e_1e_2 = 0, \quad e_3e_1 = e_2, \quad e_3e_2 = \nu e_1, \quad e_je_i = -e_ie_j \quad (i \neq j);$$

the algebra in which every $e_ie_j = 0$; and the algebra in which every square and product is zero except $e_1e_2 = e_3$, $e_2e_1 = -e_3$.

Finally, a sufficiently general number α of an algebra α of rank 3 in 4 units together with α^2 and 1 are the units of an algebra β of rank 3 in 3 units. Similarly, if α' is a suitably general number, not in β, of α, then 1, α', α'^2 are the units of an algebra β' of the same type as β. Since α has only four units, β and β' have a common sub-algebra with 2 units, one of which is 1. The types for β are $D = (e_0) + (e_1) + (e_2)$, a cubic field $F(e_1)$, (33) and (34).

The case $\beta = F(\alpha_1)$ is excluded since the degree of any sub-field is a divisor of the degree of the field.

If $\beta = D$, a sub-algebra S containing $1 = e_0 + e_1 + e_2$ and $x = \Sigma x_ie_i$ contains $x - x_2 = ae_0 + be_1$ and its square $a^2e_0 + b^2e_1$. Hence $a^2 : b^2 = a : b$, or $ab(a - b) = 0$. Thus S has the units 1 and e_0 or e_1 or $1 - (e_0 + e_1) = e_2$. In view of the symmetry, we may take $S = (1, e_1)$. Then $\beta = (S, e_2)$, $\beta' = (S, e_3)$, where $e_1^2 = e_1$, $e_1e_2 = e_2e_1 = 0$, $e_2^2 = e_2$, and e_3 has the same relation to e_1 that e_2 has to e_1, whence $e_1e_3 = e_3e_1 = 0$, $e_3^2 = e_3$. Thus the algebra α is $(1 - e_1, e_2, e_3) + (e_1)$ and is reducible.

Let β be of type (33) with $e_1^2 = \nu e_0$ and principal unit $e_0 + e_2$. The only sub-algebra S with 2 units of β is $(e_0) + (e_2)$. For, by subtracting a multiple of $1 = e_0 + e_2$ from any number of S, we may delete the term in e_0 and obtain $t = re_1 + se_2$. Then $t^2 - s^2 = ke_0$, where $k = \nu r^2 - s^2$ is not zero if r and s are not both zero. Hence β' contains $S = (e_0) + (e_2)$ and a new unit ϵ_1 having the same relation to S that e_1 has to S. Eliminating e_2 by means of $e_2 = 1 - e_0$, we obtain α:

$$e_0^2 = e_0, \quad e_0e_1 = e_1e_0 = e_1, \quad e_1^2 = \nu e_0, \quad e_0\epsilon_1 = \epsilon_1e_0 = \epsilon_1, \quad \epsilon_1^2 = \nu e_0.$$

Let $e_1\epsilon_1 = a+be_0+ce_1+d\epsilon_1$. Multiplying on the left by e_1, we get

$$\nu\epsilon_1 = ae_1+be_1+cve_0+d(a+be_0+ce_1+d\epsilon_1),$$

whence $\nu = d^2$, contrary to the assumption that ν is not a square in the field F.

If β is (33) with $e_1^2 = 0$ or (34), viz. algebras II or III of Study*, we note that his discussion is rational, except in his case (*b*), page 303, where he divided a unit by $\sqrt{c}$ to make $c = 1$, and leads to the following rationally irreducible algebras of rank 3 (in which we include under VII its reciprocal algebra):

(VII) $e_1^2 = 1, \quad e_1e_2 = \pm e_2, \quad e_2e_1 = \mp e_2, \quad e_1e_3 = e_3e_1 = e_3, \quad e_ie_j = 0$ $(i, j = 2, 3)$;

(IX) $e_1^2 = e_3, \quad e_1e_2 = e_3, \quad e_2e_1 = -e_3, \quad e_2^2 \mp ce_3, \quad e_ie_3 = e_3e_i = 0$ $(i = 1, 2, 3)$;

(X) $e_1^2 = e_3, \quad e_2^2 = -ke_3, \quad e_3^2 = 0, \quad e_ie_j = 0 \quad (i \neq j), \quad k = 1$ or ν;

(XI) $e_1^2 = e_2, \quad e_2^2 = e_3^2 = 0, \quad e_ie_j = 0 \quad (i \neq j)$,

where c is an arbitrary number of the field and ν is not a square.

* *Monatshefte Math. Phys.*, Vol. 1 (1890), pp. 293–303.

THE THEOREMS OF CLAUSEN AND CAYLEY ON PRODUCTS OF HYPERGEOMETRIC FUNCTIONS

By G. N. WATSON.

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WHEN, two years ago, I discovered the formulae* which express the product $F(a, \beta\,;\, \gamma\,;\, z)\times F(a, \beta\,;\, \gamma\,;\, Z)$ in terms of the fourth type of Appell's hypergeometric functions of the two variables zZ and $(1-z)(1-Z)$, I was so obsessed by potential applications of the results to theorems concerning Bessel functions† that I failed to observe that I had taken the essential step towards solving two problems which had baffled me for ten years. The problems in question consist in proving *by direct transformations* (I) Clausen's theorem that

$$\{F(a, \beta\,;\ a+\beta+\tfrac{1}{2}\,;\ z)\}^2 = {}_3F_2(2a,\ a+\beta,\ 2\beta\,;\ a+\beta+\tfrac{1}{2},\ 2a+2\beta\,;\ z),$$

and (II) Cayley's theorem that, if

$$(1-z)^{a+\beta-\gamma}F(2a,\ 2\beta\,;\ 2\gamma\,;\ z) = \sum_{n=0}^{\infty} a_n z^n,$$

then

$$F(a, \beta\,;\ \gamma+\tfrac{1}{2}\,;\ z)\times F(\gamma-a,\ \gamma-\beta\,;\ \gamma+\tfrac{1}{2}\,;\ z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\gamma+\frac{1}{2})_n} a_n z^n,$$

where $(\gamma)_n$ stands for $\gamma(\gamma+1)(\gamma+2)\ldots(\gamma+n-1)$, and $(\gamma)_0 = 1$. In the special case in which $\gamma = a+\beta$, Cayley's theorem obviously reduces to Clausen's theorem.

Clausen's proof of his theorem appeared in the *Journal für Math.*.

* *Proc. London Math. Soc.*, Ser. 2, Vol. 20 (1921), pp. 189–195.

† See my *Theory of Bessel Functions* (1922), § 11.6.

Vol. 3 (1828), pp. 89–95. Cayley's theorem was stated without proof in the *Phil. Mag.*, Ser. 4, Vol. 16 (1858), pp. 356–357 [*Collected Papers*, Vol. 3, pp. 268–269]; it was remarked by Cayley that he was led to it by attempting to verify a formula of planetary theory.

A full discussion of both theorems, with a proof of Cayley's theorem, has been given by Orr* whose investigations consisted in the construction of the linear differential equations satisfied by the expressions whose equality was in question.

So far as I know, the Clausen-Cayley theorems are the only theorems concerning hypergeometric functions which have never hitherto been proved by direct methods; and consequently this paper, notwithstanding its apparent triviality, does fill a definite gap.

In spite of the fact that Clausen's theorem is a particular case of Cayley's theorem, I propose to give independent proofs of the theorems, because in the particular case the analysis is considerably shortened by making use of a quadratic transformation† of hypergeometric functions. In the general case the quadratic transformation does not exist, and the absence of it necessitates the use of somewhat devious analysis.

2. To prove Clausen's theorem, take the equation established on p. 195 of my previous paper :—

$$\frac{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}{\{\Gamma(\gamma)\}^2}\,F(\alpha,\,\beta\,;\;\gamma\,;\;z)\,F(\alpha,\,\beta\,;\;\gamma\,;\;Z)$$

$$=\sum_{m=0}^{\infty}\frac{\Gamma(\alpha+m)\,\Gamma(\beta+m)\,\Gamma(\gamma-\alpha+m)\,\Gamma(\gamma-\beta+m)}{m!\;\Gamma(\gamma+m)\,\Gamma(\gamma+2m)}\,(zZ)^m$$

$$\times F(\alpha+m,\,\beta+m\,;\;\gamma+2m\,;\;z+Z-zZ),$$

which may be written in the form

$$F(\alpha,\,\beta\,;\;\gamma\,;\;z)\,F(\alpha,\,\beta\,;\;\gamma\,;\;Z)$$

$$=\sum_{m=0}^{\infty}\frac{(\alpha)_m\,(\beta)_m\,(\gamma-\alpha)_m\,(\gamma-\beta)_m}{m!\;(\gamma)_m\,(\gamma)_{2m}}\,(zZ)^m\,F(\alpha+m,\,\beta+m\,;\;\gamma+2m\,;\;z+Z-zZ)\,;$$

after replacing Z by z, observe that, *when* γ *has the value* $\alpha+\beta+\frac{1}{2}$, the

* *Trans. Camb. Phil. Soc.*, Vol. 17 (1899), pp. 1–15.

† A direct proof of the quadratic transformation has been constructed by Barnes, *Quarterly Journal*, Vol. 39 (1908), pp. 115–118.

hypergeometric functions on the right admit of the quadratic transformation

$$F(\alpha+m,\ \beta+m;\ \alpha+\beta+2m+\tfrac{1}{2};\ 2z-z^2)$$
$$= F(2\alpha+2m,\ 2\beta+2m;\ \alpha+\beta+2m+\tfrac{1}{2};\ \tfrac{1}{2}z).$$

It is thus found that

$$\{F(\alpha,\ \beta;\ \alpha+\beta+\tfrac{1}{2};\ z)\}^2$$
$$= \sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta)_m(\alpha+\tfrac{1}{2})_m(\beta+\tfrac{1}{2})_m z^{2m}}{m!\ (\alpha+\beta+\tfrac{1}{2})_m(\alpha+\beta+\tfrac{1}{2})_{2m}}$$
$$\times F(2\alpha+2m,\ 2\beta+2m;\ \alpha+\beta+2m+\tfrac{1}{2};\ \tfrac{1}{2}z)$$
$$= \sum_{m=0}^{\infty} \frac{(2\alpha)_{2m}(2\beta)_{2m}(\tfrac{1}{4}z)^{2m}}{m!\ (\alpha+\beta+\tfrac{1}{2})_m(\alpha+\beta+\tfrac{1}{2})_{2m}}$$
$$\times F(2\alpha+2m,\ 2\beta+2m;\ \alpha+\beta+2m+\tfrac{1}{2};\ \tfrac{1}{2}z)$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(2\alpha)_{2m+n}(2\beta)_{2m+n}(\tfrac{1}{2}z)^{2m+n}}{2^{2m}.\,m!\ n!\ (\alpha+\beta+\tfrac{1}{2})_m(\alpha+\beta+\tfrac{1}{2})_{2m+n}}.$$

Rearrange this double series by writing $r-2m$ in place of n, and it becomes

$$\sum_{r=0}^{\infty}\sum_{m=0}^{\frac{1}{2}r} \frac{(2\alpha)_r(2\beta)_r(\tfrac{1}{2}z)^r}{2^{2m}.\,m!\ (r-2m)!\ (\alpha+\beta+\tfrac{1}{2})_m(\alpha+\beta+\tfrac{1}{2})_r}$$
$$= \sum_{r=0}^{\infty} \frac{(2\alpha)_r(2\beta)_r(\tfrac{1}{2}z)^r}{r!\ (\alpha+\beta+\tfrac{1}{2})_r} F(-\tfrac{1}{2}r,\ \tfrac{1}{2}-\tfrac{1}{2}r;\ \alpha+\beta+\tfrac{1}{2};\ 1)$$
$$= \sum_{r=0}^{\infty} \frac{(2\alpha)_r(2\beta)_r(\tfrac{1}{2}z)^r}{r!\ (\alpha+\beta+\tfrac{1}{2})_r} \cdot \frac{\Gamma(\alpha+\beta+\tfrac{1}{2})\,\Gamma(\alpha+\beta+r)}{\Gamma(\alpha+\beta+\tfrac{1}{2}+\tfrac{1}{2}r)\,\Gamma(\alpha+\beta+\tfrac{1}{2}r)}$$
$$= \sum_{r=0}^{\infty} \frac{(2\alpha)_r(2\beta)_r(\tfrac{1}{2}z)^r}{r!\ (\alpha+\beta+\tfrac{1}{2})_r} \cdot \frac{2^r.\,(\alpha+\beta)_r}{(2\alpha+2\beta)_r}$$
$$= {}_3F_2(2\alpha,\ \alpha+\beta,\ 2\beta;\ \alpha+\beta+\tfrac{1}{2},\ 2\alpha+2\beta;\ z),$$

and this establishes Clausen's theorem. The reader will appreciate the significance of the condition that the third element of the hypergeometric functions on the left should be $\alpha+\beta+\frac{1}{2}$; the reason for the necessity of the condition had, to me at any rate, previously been obscure.

[*Added February* 8, 1923.—It has been pointed out to me by Professor Hardy that Clausen's theorem is deducible from a theorem discovered by Dougall, *Proc. Edinburgh Math. Soc.*, Vol. 25 (1907), pp. 114–132. In

the notation of this paper, Dougall's theorem is that

$$\sum_{n=0} \frac{c+2n}{c} \cdot \frac{(c)_n}{n!} \frac{(x+y+z+w+2c+1)_n}{(-x-y-z-w-c)_n} \prod_{x,y,z,w} \frac{(-x)_n}{(x+c+1)_n}$$

$$= \frac{1}{\Gamma(c+1)\,\Gamma(x+y+z+w+c+1)} \prod_{x,y,z,w} \frac{\Gamma(x+c+1)\,\Gamma(y+z+w+c+1)}{\Gamma(x+y+c+1)},$$

wherein one of the numbers x, y, z, w is a positive integer, and the summation extends over integral values of n (zero included which do not exceed that integer. If, in Dougall's theorem, we make $z = \frac{1}{2}m$, $w = \frac{1}{2}(m-1)$, where m is an integer, and then make

$$x = -\alpha, \quad y = -\beta, \quad c = -m,$$

Dougall's sum becomes a fairly simple multiple of the coefficient of ξ^m in the product

$$F(\alpha, \beta\,;\ \alpha+\beta+\tfrac{1}{2}\,;\ \xi) \times (F(\alpha, \beta\,;\ \alpha+\beta+\tfrac{1}{2}\,;\ \xi).$$

But my contention that no direct proof of Clausen's theorem has been given previously is nevertheless valid, because Dougall's proof of his theorem is of an indirect character, consisting of the deduction of the identity of two polynomials from the consideration that the polynomials are equal for a number of values of the variable greater than the degree of the polynomials in that variable.]

3. In the course of proving Cayley's theorem, we shall make use of the two following theorems:—

(I) *Saalschütz' theorem**

$${}_3F_2(-n, \beta, \delta+\epsilon+n-\beta-1\,;\ \delta\,;\ \epsilon\,;\ 1) = \frac{\Gamma(\delta)\,\Gamma(\epsilon)\,\Gamma(\delta-\beta+n)\,\Gamma(\epsilon-\beta+n)}{\Gamma(\delta+n)\,\Gamma(\epsilon+n)\,\Gamma(\delta-\beta)\,\Gamma(\epsilon-\beta)},$$

where n is a positive integer. This is most easily proved by equating coefficients of z^n in the expansions of both sides of Euler's identity

$$(1-z)^{\epsilon-n-1}F(\delta+\epsilon+n-\beta-1, \beta\,;\ \delta\,;\ z) \equiv F(\beta-n+1-\epsilon, \delta-\beta\,;\ \delta\,;\ z).$$

(II) *The transformation of Thomae and Dixon*

$${}_3F_2(\alpha, \beta, \gamma\,;\ \delta, \epsilon\,;\ 1)$$

$$= \frac{\Gamma(\delta)\,\Gamma(\delta+\epsilon-\alpha-\beta-\gamma)}{\Gamma(\delta-\alpha)\,\Gamma(\delta+\epsilon-\beta-\gamma)} \cdot {}_3F_2(\alpha, \epsilon-\beta, \epsilon-\gamma\,;\ \delta+\epsilon-\beta-\gamma, \epsilon\,;\ 1).$$

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* *Zeitschrift für Math.*, Vol. 35 (1890), pp. 186–188.

Proofs of this theorem* have been given by J. Thomae, *Journal für Math.*, Vol. 87 (1879), p. 33, A. C. Dixon, *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 2 (1905), pp. 8–15, and later by Barnes, *Quarterly Journal*, Vol. 41 (1910), pp. 136–140.

We now investigate Cayley's theorem, by *defining* the coefficients a_n by means of the expansion

$$F(\alpha, \beta;\ \gamma+\tfrac{1}{2};\ z)\, F(\gamma-\alpha, \gamma-\beta;\ \gamma+\tfrac{1}{2};\ z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\gamma+\frac{1}{2})_n} a_n z^n,$$

and then it has to be proved that

$$\sum_{n=0}^{\infty} a_n z^n = (1-z)^{\alpha+\beta-\gamma} F(2\alpha, 2\beta;\ 2\gamma;\ z).$$

It is evident that

$$\sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\gamma+\frac{1}{2})_n} a_n z^n = \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \frac{(\alpha)_m (\beta)_m (\gamma-\alpha)_r (\gamma-\beta)_r}{m!\ r!\ (\gamma+\frac{1}{2})_m (\gamma+\frac{1}{2})_r} z^{m+r},$$

and so $$\sum_{n=0}^{\infty} a_n z^n = \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \frac{(\alpha)_m (\beta)_m (\gamma-\alpha)_r (\gamma-\beta)_r (\gamma+\frac{1}{2})_{m+r}}{m!\ r!\ (\gamma+\frac{1}{2})_m (\gamma+\frac{1}{2})_r (\gamma)_{m+r}} z^{m+r}.$$

We now assume that $|\arg z| < \pi$ and we replace the expression on the right by a double contour integral, transform it into a quadruple integral and then back again into a double integral by repeatedly using Barnes' lemma that

$$\frac{1}{2\pi i} \int_{-\infty i}^{\infty i} \Gamma(\alpha_1+w)\, \Gamma(\alpha_2+w)\, \Gamma(\beta_1-w)\, \Gamma(\beta_2-w)\, dw$$

$$= \frac{\Gamma(\alpha_1+\beta_1)\, \Gamma(\alpha_1+\beta_2)\, \Gamma(\alpha_2+\beta_1)\, \Gamma(\alpha_2+\beta_2)}{\Gamma(\alpha_1+\alpha_2+\beta_1+\beta_2)}.$$

It is supposed in this, as in all the following contour integrals, that the contours have loops, when necessary, to ensure they separate the positive sequences of poles of the integrand from the negative sequences of poles; it is also assumed that $R(\gamma)$ is so large that $R(\gamma+\frac{1}{2}+\phi+\psi)$ is positive for all positions of ϕ and ψ on their respective contours; this restriction may subsequently be removed by analytic continuation; the ambiguous sign which occurs is to be so chosen that $|\pm\pi+\arg(-z)| < \pi$.

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* The proof due to Thomae is the simplest; the real significance of the theorem that the hypergeometric function in question exhibits a symmetry in five elements was first pointed out by Dixon.

It is thus found that

$$\frac{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}{\Gamma(\gamma)\,\Gamma(\gamma+\frac{1}{2})}\sum_{n=0}^{\infty}a_n z^n$$

$$=\frac{1}{2\pi i}\sum_{m=0}^{\infty}\frac{\Gamma(\alpha+m)\,\Gamma(\beta+m)\,z^m}{m!\,\Gamma(\gamma+\frac{1}{2}+m)}$$

$$\times\int_{-\infty i}^{\infty i}\frac{\Gamma(\gamma-\alpha+\psi)\,\Gamma(\gamma-\beta+\psi)(\gamma+\frac{1}{2}+m+\psi)}{\Gamma(\gamma+\frac{1}{2}+\psi)\,\Gamma(\gamma+m+\psi)}\,\Gamma(-\psi)(-z)^{\psi}\,d\psi$$

$$=\left(\frac{1}{2\pi i}\right)^2\int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i}\frac{\Gamma(\alpha+\phi)\,\Gamma(\beta+\phi)\,\Gamma(\gamma+\frac{1}{2}+\phi+\psi)}{\Gamma(\gamma+\frac{1}{2}+\phi)\,\Gamma(\gamma+\frac{1}{2}+\psi)\,\Gamma(\gamma+\phi+\psi)}$$

$$\times\Gamma(\gamma-\alpha+\psi)\,\Gamma(\gamma-\beta+\psi)\,\Gamma(-\phi)\,\Gamma(-\psi)(-z)^{\phi+\psi}\,d\phi\,d\psi$$

$$=\left(\frac{1}{2\pi i}\right)^3\int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i}\frac{\Gamma(\alpha+\phi)\,\Gamma(\beta+\phi)\,\Gamma(-s)\,e^{\pm s\pi i}}{\Gamma(\gamma+\frac{1}{2}+s)\,\Gamma(\gamma+\phi+\psi)}$$

$$\times\Gamma(\gamma-\alpha+\psi)\,\Gamma(\gamma-\beta+\psi)\,\Gamma(s-\phi)\,\Gamma(s-\psi)(-z)^{\phi+\psi}\,ds\,d\phi\,d\psi$$

$$=\left(\frac{1}{2\pi i}\right)^4\int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i}\frac{\Gamma(\alpha+s+t)+\Gamma(\beta+s+t)\,\Gamma(-s)\,e^{\pm s\pi i}}{\Gamma(\gamma+\frac{1}{2}+s)}$$

$$\times\Gamma(\phi-s-t)\,\Gamma(\psi+\gamma-\alpha-\beta-s-t)\,\Gamma(s-\phi)$$

$$\times\Gamma(s-\psi)(-z)^{\phi+\psi}\,ds\,dt\,d\phi\,d\psi.$$

After changing the order of the integrations, we write $\phi+s$ and $\psi+s$ in place of ϕ and ψ; the quadruple integral then becomes

$$\left(\frac{1}{2\pi i}\right)^4\int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i}\frac{\Gamma(\alpha+s+t)\,\Gamma(\beta+s+t)\,\Gamma(-s)\,e^{\pm s\pi i}}{\Gamma(\gamma+\frac{1}{2}+s)}$$

$$\times\Gamma(\phi-t)\,\Gamma(\psi+\gamma-\alpha-\beta-t)\,\Gamma(-\phi)\,\Gamma(-\psi)(-z)^{\phi+\psi+2s}\,d\phi\,d\psi\,ds\,dt$$

$$=\left(\frac{1}{2\pi i}\right)^2\int_{-\infty i}^{\infty i}\int_{-\infty i}^{\infty i}\frac{\Gamma(\alpha+s+t)\,\Gamma(\beta+s+t)\,\Gamma(-s)\,e^{\pm s\pi i}}{\Gamma(\gamma+\frac{1}{2}+s)}$$

$$\times\Gamma(-t)\,\Gamma(\gamma-\alpha-\beta-t)(1-z)^{2t+\alpha+\beta-\gamma}(-z)^{2s}\,ds\,dt$$

$$=\frac{1}{2\pi i}\sum_{m=0}^{\infty}\int_{-\infty i}^{\infty i}\frac{\Gamma(\alpha+m+t)\,\Gamma(\beta+m+t)\,z^{2m}}{m!\,\Gamma(\gamma+\frac{1}{2}+m)}$$

$$\times\Gamma(-t)\,\Gamma(\gamma-\alpha-\beta-t)(1-z)^{2t+\alpha+\beta-\gamma}\,dt$$

$$=(1-z)^{\alpha+\beta-\gamma}\sum_{m=0}^{\infty}\frac{\Gamma(\alpha+m)\,\Gamma(\beta+m)\,\Gamma(\gamma-\alpha+m)\,\Gamma(\gamma-\beta+m)\,z^{2m}}{m!\,\Gamma(\gamma+\frac{1}{2}+m)\,\Gamma(\gamma+2m)}$$

$$\times F(\alpha+m,\ \beta+m;\ \gamma+2m;\ 2z-z^2).$$

It therefore remains to be proved that

$$\sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta)_m(\gamma-\alpha)_m(\gamma-\beta)_m z^{2m}}{m!\,(\gamma+\tfrac{1}{2})_m(\gamma)_{2m}} F(\alpha+m,\ \beta+m;\ \gamma+2m;\ 2z-z^2)$$

$$= F(2\alpha,\ 2\beta;\ 2\gamma;\ z).$$

Now it is evident by rearrangement of double series that

$$\sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta)_m(\gamma-\alpha)_m(\gamma-\beta)_m z^{2m}}{m!\,(\gamma+\tfrac{1}{2})_m(\gamma)_{2m}} F(\alpha+m,\ \beta+m;\ \gamma+2m;\ 2z-z^2)$$

$$= \sum_{m=0}^{\infty}\sum_{p=0}^{\infty} \frac{(\alpha)_{m+p}(\beta)_{m+p}(\gamma-\alpha)_m(\gamma-\beta)_m}{m!\,p!\,(\gamma+\tfrac{1}{2})_m(\gamma)_{2m+p}} z^{2m}(2z-z^2)^p$$

$$= \sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{p} \frac{(\alpha)_{m+p}(\beta)_{m+p}(\gamma-\alpha)_m(\gamma-\beta)_m(-)^r 2^{p-r} z^{2m+p+r}}{m!\,r!\,(p-r)!\,(\gamma+\tfrac{1}{2})_m(\gamma)_{2m+p}}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\frac{1}{2}n}\sum_{p=0}^{n-2m} \frac{(\alpha)_{m+p}(\beta)_{m+p}(\gamma-\alpha)_m(\gamma-\beta)_m(-)^{n+p}2^{2m+2p-n} z^n}{m!\,(n-2m-p)!\,(2p+2m-n)!\,(\gamma+\tfrac{1}{2})_m(\gamma)_{2m+p}}$$

$$= \sum_{n=0}^{\infty}\sum_{k=\frac{1}{2}n}^{n}\sum_{m=0}^{n-k} \frac{(\alpha)_k(\beta)_k(\gamma-\alpha)_m(\gamma-\beta)_m(-)^{n+k+m}2^{2k-n} z^n}{m!\,(n-m-k)!\,(2k-n)!\,(\gamma+\tfrac{1}{2})_m(\gamma)_{m+k}}$$

$$= \sum_{n=0}^{\infty}\sum_{k=\frac{1}{2}n}^{n} \frac{(\alpha)_k(\beta)_k(-)^{n+k}2^{2k-n} z^n}{(n-k)!\,(2k-n)!\,(\gamma)_k}$$

$$\times {}_3F_2(k-n,\ \gamma-\alpha,\ \gamma-\beta;\ \gamma+k,\ \gamma+\tfrac{1}{2};\ 1)$$

$$= \sum_{n=0}^{\infty}\sum_{k=\frac{1}{2}n}^{n} \frac{(\alpha)_k(\beta)_k(-)^{n+k}2^{2k-n} z^n}{(n-k)!\,2k-n)!\,(\gamma)_k} \cdot \frac{\Gamma(\gamma+k)\,\Gamma(\alpha+\beta+n+\tfrac{1}{2})}{\Gamma(\gamma+n)\,\Gamma(\alpha+\beta+k+\tfrac{1}{2})}$$

$$\times {}_3F_2(k-n,\ \alpha+\tfrac{1}{2},\ \beta+\tfrac{1}{2};\ \alpha+\beta+k+\tfrac{1}{2},\ \gamma+\tfrac{1}{2};\ 1)$$

(Dixon's transformation)

$$= \sum_{n=0}^{\infty}\sum_{k=\frac{1}{2}n}^{n}\sum_{m=0}^{n-k} \frac{(\alpha)_k(\beta)_k(\alpha+\tfrac{1}{2})_m(\beta+\tfrac{1}{2})_m(\alpha+\beta+\tfrac{1}{2})_n(-)^{m+n+k}2^{2k-n} z^n}{m!\,(n-k-m)!\,(2k-n)!\,(\gamma)_n(\gamma+\tfrac{1}{2})_m(\alpha+\beta+\tfrac{1}{2})_{m+k}}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\frac{1}{2}n}\sum_{r=0}^{\frac{1}{2}n-m} \frac{(\alpha)_{n-m-r}(\beta)_{n-m-r}(\alpha+\tfrac{1}{2})_m(\beta+\tfrac{1}{2})_m(\alpha+\beta+\tfrac{1}{2})_n(-)^r 2^{n-2m-2r} z^n}{m!\,r!\,(n-2m-2r)!\,(\gamma)_n(\gamma+\tfrac{1}{2})_m(\alpha+\beta+\tfrac{1}{2})_{n-r}}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\frac{1}{2}n} \frac{(\alpha)_{n-m}(\beta)_{n-m}(\alpha+\tfrac{1}{2})_m(\beta+\tfrac{1}{2})_m 2^{n-2m} z^n}{m!\,(n-2m)!\,(\gamma)_n(\gamma+\tfrac{1}{2})_m}$$

$$\times {}_3F_2(m-\tfrac{1}{2}n,\ m-\tfrac{1}{2}n+\tfrac{1}{2},\ \tfrac{1}{2}-\alpha-\beta-n;\ m+1-n-\alpha,\ m+1-n-\beta;\ 1)$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\frac{1}{2}n} \frac{(\alpha)_{n-m}(\beta)_{n-m}(\alpha+\frac{1}{2})_m(\beta+\frac{1}{2})_m 2^{n-2m} z^n}{m!\,(n-2m)!\,(\gamma)_n(\gamma+\frac{1}{2})_m}$$

$$\times \frac{\Gamma(m+1-n-\alpha)\,\Gamma(m+1-n-\beta)\,\Gamma(\frac{1}{2}-\alpha-m)\,\Gamma(\frac{1}{2}-\beta-m)}{\Gamma(\frac{1}{2}-\frac{1}{2}n-\alpha)\,\Gamma(1-\frac{1}{2}n-\alpha)\,\Gamma(\frac{1}{2}-\frac{1}{2}n-\beta)\,\Gamma(1-\frac{1}{2}n-\beta)}$$

(Saalschütz' theorem)

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\frac{1}{2}n} \frac{2^{n-2m} z^n}{m!\,(n-2m)!\,(\gamma)_n(\gamma+\frac{1}{2})_m}$$

$$\times \frac{\Gamma(\frac{1}{2}-\alpha)\,\Gamma(1-\alpha)\,\Gamma(\frac{1}{2}-\beta)\,\Gamma(1-\beta)}{\Gamma(\frac{1}{2}-\frac{1}{2}n-\alpha)\,\Gamma(1-\frac{1}{2}n-\alpha)\,\Gamma(\frac{1}{2}-\frac{1}{2}n-\beta)\,\Gamma(1-\frac{1}{2}n-\beta)}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\frac{1}{2}n} \frac{(2\alpha)_n(2\beta)_n z^n}{2^{n+2m}.\,m!\,(n-2m)!\,(\gamma)_n(\gamma+\frac{1}{2})_m}$$

$$= \sum_{n=0}^{\infty} \frac{(2\alpha)_n(2\beta)_n z^n}{2^n n!\,(\gamma)_n} F(-\tfrac{1}{2}n,\ \tfrac{1}{2}-\tfrac{1}{2}n;\ \gamma+\tfrac{1}{2};\ 1)$$

$$= \sum_{n=0}^{\infty} \frac{(2\alpha)_n(2\beta)_n z^n}{2^n.\,n!\,(\gamma)_n} \cdot \frac{\Gamma(\gamma+\frac{1}{2})\,\Gamma(\gamma+n)}{\Gamma(\gamma+\frac{1}{2}n+\frac{1}{2})\,\Gamma(\gamma+\frac{1}{2}n)}$$

$$= F(2\alpha,\ 2\beta;\ 2\gamma;\ z),$$

and this establishes Cayley's theorem.

[*Added October* 1, 1923.—Since this paper was set up in type, a proof of Cayley's theorem has been published by D. Edwardes, *Messenger of Maths.*, Vol. 52 (1923), pp. 129–136; the proof consists of an inductive proof of a special case of Dougall's theorem followed by transformations of hypergeometric functions and of integrals of Eulerian type.]

TERNARY PERPETUANTS

By ALFRED YOUNG.

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1. In what follows, functions of the coefficients of the quantics and of the variables are alone considered, and line coordinates are not introduced at all. So that pure covariants alone are treated of and not contravariants or mixed concomitants. Further, the covariants are supposed to be linear in the coefficients of each quantic, that is they are covariant types.

In § 2 a set of symbolical products is obtained in terms of which all perpetuant types of given degree can be linearly expressed.

In § 3 it is proved that the members of this set are linearly independent.

In § 4 a generating function is found which gives the number of members of this set of given weight and degree.

In § 5 various useful expressions for this generating function are given.

In § 6 a reduced generating function is obtained, giving the number of perpetuant types of given weight and degree, which are linearly independent of those which have one of the quantics as a factor.

In § 7 the generating functions as far as degree 8 are given with the numerical coefficients calculated.

In § 8 perpetuants of degree 4 are discussed.

In §§ 9, 10, 11, 12 forms of degrees 5, 6, 7, 8, and of the lowest weights are considered. In the cases of degrees 7 and 8 it is shown that the generating functions indicate the existence of syzygies which are obtained.

In § 13 a lower limit is obtained for the weight of an irreducible perpetuant of degree r. It is shown that if r is even, no perpetuant of

weight less than $r-3$ can be irreducible, and if r is odd, the lower limit is $r-2$. It is not by any means established that there are always irreducible perpetuants of these weights.

In § 14 it is proved that there is no family of syzygies for ternary forms analogous to the Stroh syzygies for binary forms.

2. We use the symbolical notation. As a rule the factors a_x are not written, the covariants being completely defined by the determinant factors of the symbolical product.

The quantics dealt with will be written

$$a_{1x}^n,\ a_{2x}^n,\ \ldots,\ a_{rx}^n$$

where n is indefinitely large—or any way at least as large as the weight of the covariants discussed.

We arrange the quantics in a fixed sequence, and for convenience take the sequence to be that of the numerical order of the suffixes.

Then the letter a_1 may be introduced into every determinant factor by means of the fundamental identity; this is supposed done.

Next a product $(a_1a_pa_q)(a_1a_ra_s)$ satisfies the identity

$$(a_1a_pa_q)(a_1a_ra_s)+(a_1a_qa_r)(a_1a_pa_s)+(a_1a_ra_p)(a_1a_qa_s)=0.$$

Let $$s>r>q>p.$$

Then we see that a product of two factors, such that the two letters other than a_1 in one lie between the two corresponding letters of the other factor, can be expressed in terms of pairs of factors for which this is not the case. Hence we can express all our forms in terms of symbolical products, each determinant factor of which contains a_1, and which are such that no pair of factors have the second and third letters of one lying between the second and third letters of the other. It is evident that the fundamental identity is not capable of giving us anything further. But a formal proof of the linear independence of the members of the set is given in the next paragraph.

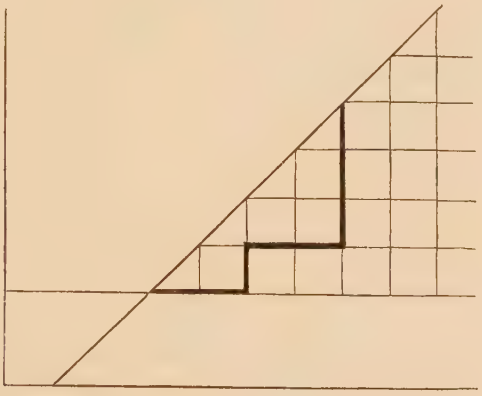
In general this set of symbolical products does not consist of a single row of determinant factors raised to different powers—as is the case with binary perpetuants; but it consists of a certain number of such rows.

Thus for degree 5 the perpetuants are expressed in terms of the forms

$$(a_1a_2a_3)^{\lambda_1}(a_1a_2a_4)^{\lambda_2}(a_1a_2a_5)^{\lambda_3}(a_1a_3a_5)^{\lambda_4}(a_1a_4a_5)^{\lambda_5}$$

and $$(a_1a_2a_3)^{\lambda_1}(a_1a_2a_4)^{\lambda_2}(a_1a_3a_4)^{\lambda_3}(a_1a_3a_5)^{\lambda_4}(a_1a_4a_5)^{\lambda_5};$$

there being two rows of factors.



We may represent the different factors of a product by points on a diagram. Thus, arrange the letters of a factor in ascending order of magnitude of the suffixes, and let the point (q, p) stand for the factor $(a_1a_pa_q)$. Then all the points will lie between the straight lines $y = 2$ and $y = x-1$, or on the boundary of the enclosed space. Then in any product we arrange the factors in the sequence of the second letters: when the second letters are equal we arrange according to the sequence of the last letters. Then, if straight lines be drawn joining points representing consecutive factors starting from the first [which is usually (3, 2)] as we pass from point to point the values of both x and y are never on the decrease, owing to the "between-ness" rule.

We may pass from a factor $(a_1a_pa_q)$ to the next $(a_1a_ra_s)$ by a series of steps, increasing the second or third suffix by unity at each step and leaving the other unchanged. In other words we can pass from one point on the diagram of the product to the next by a series of steps, each parallel to one axis and increasing one coordinate by unity.

Thus, if r is the degree of the covariant, we can represent it as a product of powers of $(2r-5)$ factors beginning with $(a_1a_2a_3)^{\lambda}$ and ending with

$(a_1 a_{r-1} a_r)^\mu$; the passage from one factor to the next, corresponding to a unit step parallel to one axis—(some of the factors, of course, may be omitted, the corresponding power being zero).

In general, the journey from (2, 3) to $(r-1, r)$ may be performed in this manner in

$$\binom{2r-8}{r-4} - \binom{2r-8}{r-6}$$

ways. And there is this number of different rows of factors to be considered for degree r. As this particular fact is not required for our purpose, it is thought unnecessary to prove it.

The figure gives the diagram for

$$(a_1 a_2 a_3)^{\lambda_1}(a_1 a_2 a_4)^{\lambda_2}(a_1 a_2 a_5)^{\lambda_3}(a_1 a_3 a_5)^{\lambda_4}(a_1 a_3 a_6)^{\lambda_5}$$

$$\times (a_1 a_3 a_7)^{\lambda_6}(a_1 a_4 a_7)^{\lambda_7}(a_1 a_5 a_7)^{\lambda_8}(a_1 a_6 a_7)^{\lambda_9}.$$

3. To prove the linear independence of the set of forms just obtained, let us suppose that there is a linear relation between our set of forms for degree r, but none of a lower degree. If every term in the relation has the factor $(a_1 a_{r-1} a_r)$, we can divide by this factor and still have an identical relation; hence we may assume that there is at least one term without this factor. The linear relation will remain true if we replace every a_r by a_{r-1}, and the "between-ness" rule will still hold for the resulting terms. But as there is no such linear relation of degree $r-1$, it must be now a formal identity, such that the coefficient of each *different* product is zero. The relation of degree r may be written

$$\Sigma A P (a_1 a_{p_1} a_{r-1})^{\lambda_1}(a_1 a_{p_2} a_{r-1})^{\lambda_2} \ldots (a_1 a_{q_1} a_r)^{\mu_1}(a_1 a_{q_2} a_r)^{\mu_2} = 0$$

where P represents factors containing neither a_r nor a_{r-1}, and A is numerical.

By the "between-ness" rule no q is less than any p. In order that this may become a formal identity when a_r is replaced by a_{r-1}, this will have to be the case with each set of factors which have the part P in common, separately. Confining our attention to one such set, we see that in the original relation this requires that corresponding to one such term written above, there must be another *different* one in which one or more pairs of factors $(a_1 a_p a_{r-1})(a_1 a_q a_r)$ is replaced by the pair $(a_1 a_q a_{r-1})(a_1 a_p a_r)$ if $p = q$, this makes no change and if $p \neq q$ this is not possible owing to

the "between-ness" rule. Hence the supposed identity for degree r cannot exist, unless there is one for degree $r-1$. But there is none for degree 3 or degree 4 and therefore our set of forms is linearly independent.

4. Let us call the generating function for linearly independent perpetuants of degree r, G_r. The number of different forms belonging to a single row of factors is $(1-x)^{-2r+5}$; when, however, one or more factors of a row are absent, the rows will be found to overlap.

Let $G_{r,s}$ be the generating function for those products of degree r which contain the factor $(a_1a_2a_s)$, where $s \geqslant 5$, but which contain no factor $(a_1a_2a_t)$ where $t > s$. And let $G_{r,4}$ be the generating function for those products of degree r which contain no factor $(a_1a_2a_t)$, $t > 4$: in $G_{r,4}$ there are included products which contain no factor $(a_1a_2a_4)$.

Looking at the diagram of § 2, $G_{r,s}$ is the generating function of those products for which the line on the diagram contains the point $(s, 2)$ but leaves the straight line $y = 2$ at this point, when $s > 4$; but, if $s = 4$, it contains no point on $y = 2$ beyond $(s, 2)$, but need not contain this point.

We proceed to obtain formulæ expressing the generating functions for perpetuants of the r forms represented by $a_1, a_2, \ldots, a_r$ in terms of the generating functions for perpetuants of the $r-1$ forms left when that represented by a_2 is removed.

Since every perpetuant belongs to one of the classes $G_{r,s}$ $(s = 4, 5, \ldots, r)$, and these classes do not overlap,

$$G_r = G_{r,4} + G_{r,5} + \ldots + G_{r,r}.$$

Now the "between-ness" rule does not exclude any factor $(a_1a_sa_t)$, $t > s > 2$, on account of the presence of the two factors $(a_1a_2a_3)(a_1a_2a_4)$, so every product giving a perpetuant of the forms represented by

$$a_1, \ a_3, \ \ldots, \ a_r$$

multiplied by $(a_1a_2a_3)^\lambda(a_1a_2a_4)^\mu$ will give a perpetuant of the class $G_{r,4}$.

Hence $$G_{r,4} = (1-x)^{-2}G_{r-1}.$$

The only factor excluded by the "between-ness" rule when $(a_1a_2a_5)$ is present is $(a_1a_3a_4)$, thus any perpetuant of class $G_{r,5}$ may be obtained by taking a perpetuant of $a_1, a_3, \ldots, a_r$, replacing the factor $(a_1a_3a_4)^\lambda$ by

$(a_1a_2a_4)^\lambda$, and multiplying by the factor $(a_1a_2a_3)^\mu(a_1a_2a_5)^{\nu+1}$, thus

$$G_{r,5} = x(1-x)^{-2}G_{r-1} = x(1-x)^{-2}(G_{r-1,4}+G_{r-1,5}+\ldots+G_{r-1,r-1}).$$

Consider any perpetuant of class $G_{r,6}$. First drop the factor

$$(a_1a_2a_3)^\lambda(a_1a_2a_4)^\mu,$$

then replace $(a_1a_2a_5)^\nu$ by $(a_1a_3a_4)^\nu$; now there is a factor $(a_1a_2a_6)^{\rho+1}$, we will replace this by $(a_1a_3a_5)^\rho(a_1a_3a_6)$—originally there were no factors $(a_1a_3a_4)$, $(a_1a_3a_5)$ in the perpetuant owing to the "between-ness" rule. We are left with a perpetuant of the forms a_1, a_3, a_4, ..., a_r which contains one factor $(a_1a_3a_6)$; that is it belongs to one of the classes

$$G_{r-1,5},\quad G_{r-1,6},\quad \ldots,\quad G_{r-1,r-1}.$$

If it contains no factor $(a_1a_3a_t)$, $t>6$, it belongs to $G_{r-1,5}$. Moreover, the steps by which we obtained it from the original perpetuant are strictly and uniquely reversible, the number of such perpetuants is then given by $G_{r-1,5}$. If it contains a factor $(a_1a_3a_h)$, but no factor $(a_1a_3a_t)$, $t>h$, it belongs to $G_{r-1,h-1}$, but here we must remember there is a factor $(a_1a_3a_6)$ which is not contained in every perpetuant of $G_{r-1,h-1}$, the number of perpetuants leading to this class is then $xG_{r-1,h-1}$. Hence

$$G_{r,6} = (1-x)^{-2}(G_{r-1,5}+xG_{r-1,6}+xG_{r-1,7}+\ldots+xG_{r-1,r-1})$$

and in the same way we find in general

$$G_{r,s} = (1-x)^{-2}(G_{r-1,s-1}+xG_{r-1,s}+xG_{r-1,s+1}+\ldots+xG_{r-1,r-1})$$

$$G_{r,r} = x(1-x)^{-2r+5}.$$

Now $$G_3 = \frac{1}{1-x},\qquad G_4 = \frac{1}{(1-x)^3},$$

$$G_5 = G_{5,4}+G_{5,5} = (1+x)(1-x)^{-5}.$$

We proceed to prove that*

$$G_r = \left((z^{r-4})\right)(1+z)^{r-4}(x+z)^{r-4}(1-z^{-2}x)(1-x)^{-2r+5}$$

and that

$$G_{r,s} = \left((z^{r-5})\right)x(1+z)^{r-s}(x+z)^{r-5}(1-z^{-2}x)(1-x)^{-2r+5}.$$

* The notation $\left((z^n)\right)F(z)$ is used to denote the coefficient of z^n in the expansion of $F(z)$.

Assume the truth of these formulæ for a given value of r. Then

$$G_{r+1,\,5} = x(1-x)^{-2}G_r$$

$$= \left((z^{r-4})\right)x(1+z)^{r-4}(x+z)^{r-4}(1-z^{-2}x)(1-x)^{-2r+3}.$$

$$G_{r+1,\,s} = (1-x)^{-2}\{G_{r,\,s-1}+xG_{r,\,s}+\ldots+xG_{r,\,r}\}$$

$$= \left((z^{r-5})\right)x(x+z)^{r-5}(1-z^{-2}x)$$

$$\times\left\{(1+z)^{r-s+1}+x\frac{(1+z)^{r-s+1}-1}{z}\right\}(1-x)^{-2r+3}$$

$$= \left((z^{r-4})\right)x(x+z)^{r-4}(1+z)^{r-s+1}(1-z^{-2}x)(1-x)^{-2r+3}.$$

For the next step we require

$$\left((z^{r-1})\right)(1+z)^r(x+z)^r(1-z^{-2}x)$$

$$= \binom{r}{1}x+\binom{r}{1}\cdot\binom{r}{2}x^2+\binom{r}{2}\cdot\binom{r}{3}x^3+\ldots$$

$$-x\left\{\binom{r}{1}+\binom{r}{2}\cdot\binom{r}{1}x+\binom{r}{3}\cdot\binom{r}{2}x^2+\ldots\right\} = 0,$$

and hence

$$\left((z^{r-3})\right)(x+z)^{r-4}(1-z^{-2}x)(1-x)^{-2r+3}(1+z)^{r-4}z^2$$

$$= \left((z^{r-5})\right)(x+z)^{r-4}(1+z)^{r-4}(1-z^{-2}x)(1-x)^{-2r+3} = 0.$$

Then

$$G_{r+1} = G_{r+1,\,4}+G_{r+1,\,5}+\ldots+G_{r+1,\,r+1}$$

$$= \left((z^{r-4})\right)(x+z)^{r-4}(1-z^{-2}x)(1-x)^{-2r+3}$$

$$\times\{(1+z)^{r-4}+x(1+z)^{r-4}+\ldots+x\}$$

$$= \left((z^{r-3})\right)(x+z)^{r-4}(1-z^{-2}x)(1-x)^{-2r+3}\{z(1+z)^{r-4}+x(1+z)^{r-3}-x\}$$

$$= \left((z^{r-3})\right)(x+z)^{r-4}(1-z^{-2}x)(1-x)^{-2r+3}(1+z)^{r-4}\{z+x+xz\};$$

by the result just proved above we may add the term z^2 in the last bracket

without affecting the result. Hence

$$G_{r+1} = \left((z^{r-3})\right)(x+z)^{r-4}(1-z^{-2}x)(1-x)^{-2r+3}(1+z)^{r-4}\{z+x+xz+z^2\}$$

$$= \left((z^{r-3})\right)(1+z)^{r-3}(x+z)^{r-3}(1-z^{-2}x)(1-x)^{-2r+3}.$$

Thus, if our formulæ are true for degree r, they are also true for degree $r+1$. But they are true for degree 5, and therefore they are true in general.

5. We may put the generating function in slightly different forms which will be useful later. Thus

$$G_r(1-x)^{2r-5} = \left((z^{r-4})\right)(1+z)^{r-4}(x+z)^{r-4}(1-z^{-2}x)$$

$$= \sum_{\alpha=0}^{r-4}\left\{\binom{r-4}{\alpha}\binom{r-4}{\alpha} - \binom{r-4}{\alpha-1}\binom{r-4}{\alpha+1}\right\}x^\alpha$$

$$= \Sigma\,\frac{1}{\alpha}\binom{r-3}{\alpha+1}\binom{r-4}{\alpha-1}x^\alpha = \frac{1}{r-3}\Sigma\binom{r-3}{\alpha+1}\binom{r-3}{\alpha}x^\alpha$$

$$= \Sigma\,\frac{1}{\alpha}x^\alpha\sum_{s=0}^{\alpha-1}\binom{\alpha}{s}\binom{2\alpha-s}{\alpha+1}\binom{r-3}{2\alpha-s}^*$$

$$= \sum_\alpha\frac{1}{\alpha}x^\alpha\sum_{s=0}^{\alpha-1}\binom{\alpha}{s}\binom{2\alpha-s}{\alpha+1}\sum_{\sigma=0}^{2\alpha-s}(-)^\sigma\binom{\sigma+2}{2}\binom{r}{2\alpha-\sigma-s}^\dagger$$

$$= \sum_\alpha\sum_\lambda B_{\alpha,\lambda}\binom{r}{\lambda}x^\alpha,$$

where $$B_{\alpha,\lambda} = \frac{1}{\alpha}\sum_s(-)^{s+\lambda}\binom{\alpha}{s}\binom{2\alpha-s}{\alpha+1}\binom{2\alpha-s-\lambda+2}{2}.$$

Thus $B_{\alpha,\lambda}$ is entirely independent of r and a function of α and λ alone. The lower limit of s is zero, the upper limit is either $\alpha-1$ or $2\alpha-\lambda+2$, whichever is the smaller.

When $2\alpha-\lambda+2 \geqslant \alpha-1$, a simple form can be found for $B_{\alpha,\lambda}$,

---

* For $\binom{r-4}{\alpha-1} = \left((x^{\alpha-1})\right)(1+x)^{r-4} = \left((x^{\alpha-1})\right)(1+x)^{r-4-\alpha}(1+x)^\alpha = \Sigma\binom{\alpha}{s}\binom{r-4-\alpha}{\alpha-1-s}$.

† For $$\binom{r-3}{2\alpha-s} = \left((x^{2\alpha-s})\right)(1+x)^{-3}(1+x)^r.$$

i.e. when $\lambda \not> a+3$. For then

$$B_{a,\lambda} = \frac{1}{a}\sum_{s=0}^{a-1}(-)^{s+\lambda}\binom{a}{s}\binom{2a-s}{a+1}\left\{\binom{2a-s+2}{2}-\lambda\binom{2a-s+1}{1}+\binom{\lambda}{2}\right\}$$

$$= (-)^{\lambda}\frac{1}{a}\left\{\binom{a+3}{2}\binom{a+2}{3}-\lambda\binom{a+2}{1}\binom{a+1}{2}+\binom{\lambda}{2}a\right\}$$

$$= (-)^{\lambda}\left\{\binom{a+4}{4}+\binom{a+3}{4}-\lambda\binom{a+2}{2}+\binom{\lambda}{2}\right\}$$

provided $a > 0$: this is also true if $a = 0$ and $\lambda \not> 2$.

The case $a = 0$ should be treated separately. The constant term in $G_r(1-x)^{-2r+5}$ is 1. Hence $B_{0,\lambda} = 0$ unless $\lambda = 0$ and then it is unity. Again it is to be observed that $B_{a,\lambda} = 0$ when $\lambda > 2a$, and

$$B_{a,2a} = \frac{1}{a}\binom{2a}{a+1}.$$

6. Let us now exclude all forms which have one of the quantics as a factor, and seek for the generating function of the perpetuants that are left. We shall call this the reduced generating function G'_r.

Its value is

$$G'_r = G_r - rG_{r-1} + \binom{r}{2}G_{r-2} - \ldots + (-)^t\binom{r}{t}G_{r-t} + \ldots$$

$$+ (-)^{r-1}G_1 + (-)^r G_0 + (-)^r x.$$

We obtain it by considering a definite product of λ quantics and a form of degree $r-\lambda$ having no separate quantic as a factor. The first term in the above sum includes this product once, the second excludes it λ times, the third includes it $\binom{\lambda}{2}$ times, and so on; finally it is found to be included

$$1 - \lambda + \binom{\lambda}{2} - \ldots = (1-1)^{\lambda} = 0$$

times.

The last term of all has to be added owing to the Jacobian syzygy of weight unity,

$$(abc)d_x - (bcd)a_x + (cda)b_x - (dab)c_x = 0.$$

Were it not for this syzygy G_r would include $\binom{r}{3}$ forms of weight one, whereas it only includes $\binom{r-1}{2}$ such forms. In G'_r we have to calculate the term to be added so that there is no term of weight 1. Taking the expression written down, we find the coefficient of x to be

$$\binom{r-1}{2}-r\binom{r-2}{2}+\binom{r}{2}\binom{r-3}{2}-\ldots+(-)^{r-3}\binom{r}{r-3}\binom{2}{2}+(-1)^r$$

$$=\left((x^2)\right)\left\{(1+x)^{r-1}-r(1+x)^{r-2}+\binom{r}{2}(1+x)^{r-3}-\ldots\right.$$

$$\left.+(-)^{r-3}\binom{r}{r-3}(1+x)^2+(-)^r x^2\right\}$$

$$=\left((x^2)\right)\left[\frac{\{(1+x)-1\}^r}{1+x}-\frac{(-1)^r}{1+x}+(-)^r x^2\right]=0.$$

It should be observed that the exactness of the value given for the reduced generating function is upset by a syzygy, but that the syzygy, having the effect of reducing the number of independent products, will make it appear that there are fewer irreducible forms than there really are. To find the value of G'_r, let us consider the sum

$$\sum_{t=0}^{r-3}(-)^t\binom{r}{t}G_{r-t}(1-x)^{2r-5}$$

$$=\Sigma(-)^t\binom{r}{t}\left[G_{r-t}(1-x)^{2r-2t-5}\right](1-x)^{2t}$$

$$=\sum_{t,\,w,\,s,\,\lambda}(-)^{s+t}\binom{2t}{s}\binom{r}{t}\binom{r-t}{\lambda}B_{w-s,\,\lambda}x^w$$

$$=\sum_{t,\,w,\,s,\,\lambda,\,\sigma}(-)^{s+t}\binom{s-\sigma}{\sigma}2^{s-2\sigma}\binom{t}{s-\sigma}\binom{r}{\lambda}\binom{r-\lambda}{t}B_{w-s,\,\lambda}x^{w*}$$

$$=\Sigma(-)^{s+t}2^{s-2\sigma}\binom{s-\sigma}{\sigma}\binom{r}{\lambda}\binom{r-\lambda}{s-\sigma}\binom{r-\lambda-s+\sigma}{t-s+\sigma}B_{w-s,\,\lambda}x^w. \qquad (A)$$

---

* For $\binom{2t}{s}=\left((x^s)\right)(1+2x+x^2)^t$.

Now when $r-\lambda-s+\sigma$ is positive and $\lambda \geqslant 3$ (noticing that $s \geqslant \sigma$)

$$\sum_{t=0}^{r-3} \binom{r-\lambda-s+\sigma}{t-s+\sigma} (-)^t = (-)^{s-\sigma}(1-1)^{r-\lambda-s+\sigma} = 0 ;$$

when $\lambda = 2$, this sum is $(-)^{r+1}$;

when $\lambda = 1$, the sum is $(-)^{r+1}(r-2-s+\sigma)$;

when $\lambda = 0$, the sum is

$$(-)^{r+1}\binom{r-1-s+\sigma}{2}.$$

Also $r-\lambda-s+\sigma$ can never be negative; it can, however, be zero. Let Γ represent the sum of those terms of the series we are discussing, for which $r-\lambda-s+\sigma = 0$.

Then

$$\sum_{t=0}^{r-3} (-)^t \binom{r}{t} G_{r-t}(1-x)^{2r-5} - \Gamma$$

$$= \sum_{w,\, s,\, \sigma} (-)^{r+s+1} x^w 2^{s-2\sigma} \binom{s-\sigma}{\sigma} \left[\binom{r}{2} \binom{r-2}{s-\sigma} B_{w-s,\,2} \right.$$

$$\left. + r\binom{r-1}{s-\sigma}(r-2-s+\sigma) B_{w-s,\,1} + \binom{r}{s-\sigma}\binom{r-1-s+\sigma}{2} B_{w-s,\,0} \right]$$

$$= \Sigma (-)^{r+s+1} x^w 2^{s-2\sigma} \binom{s-\sigma}{\sigma} \left[\binom{r}{2}\binom{r-2}{s-\sigma} \{B_{w-s,\,2} + 2B_{w-s,\,1} + B_{w-s,\,0}\} \right.$$

$$\left. - r\binom{r-1}{s-\sigma}\{B_{w-s,\,1} + B_{w-s,\,0}\} + \binom{r}{s-\sigma} B_{w-s,\,0} \right]$$

$$= \Sigma (-)^{r+s+1} x^w 2^{s-2\sigma} \binom{s-\sigma}{\sigma} \left[\binom{r}{2}\binom{r-2}{s-\sigma} - \binom{r}{1}\binom{r-1}{s-\sigma}\binom{w-s+2}{2} \right.$$

$$\left. + \binom{r}{s-\sigma}\binom{w-s+3}{3} \frac{w-s+2}{2} \right]$$

$$= \Sigma (-)^{r+s+1} x^w$$

$$\left[\binom{r}{2}\binom{2r-4}{s} - r\binom{2r-2}{s}\binom{w-s+2}{2} + \binom{2r}{s}\binom{w-s+3}{3} \frac{w-s+2}{2} \right]^*$$

---

* For $\Sigma \binom{s-\sigma}{\sigma}\binom{r-2}{s-\sigma} 2^{s-2\sigma} = \left((x^s)\right)(1+2x+x^2)^{r-2}$.

Now the limits of summation for s are 0 and w. Then we have

$$\sum_{s=0}^{w} (-)^s \binom{2r-4}{s} = (-)^w \binom{2r-5}{w},$$

$$\Sigma (-)^s \binom{2r-2}{s} \binom{w-s+2}{2} = \left((x^w)\right)(1-x)^{2r-2}(1-x)^{-3} = (-)^w \binom{2r-5}{w},$$

$$\Sigma (-)^s \binom{2r}{s} \binom{w-s+3}{3} \frac{w-s+2}{2}$$

$$= \left((x^w)\right)(1-x)^{2r}(1-x)^{-5} + \left((x^{w-1})\right)(1-x)^{2r}(1-x)^{-5}$$

$$= (-)^w \left\{ \binom{2r-5}{w} - \binom{2r-5}{w-1} \right\}.$$

Hence

$$\sum_{t=0}^{r-3} (-)^t \binom{r}{t} G_{r-t}(1-x)^{2r-5} - \Gamma$$

$$= \Sigma (-)^{r+w+1} x^w \left[\binom{2r-5}{w} \left\{ \binom{r}{2} - \binom{r}{1} + 1 \right\} - \binom{2r-5}{w-1} \right]$$

$$= -(-)^r (1-x)^{2r-5} \left[\binom{r-1}{2} + x \right].$$

Hence we find $$G'_r(1-x)^{2r-5} = \Gamma$$

where Γ is the sum of terms from (A) for which $r-\lambda-s+\sigma = 0$. Since

$$r-\lambda-s+\sigma = (t-s+\sigma)+(r-\lambda-t)$$

and neither of these latter quantities, indicated by the brackets, can be negative, they must separately vanish. Thus

$$\Gamma = \Sigma (-)^\sigma 2^{r-\lambda-\sigma} \binom{r-\lambda}{\sigma} \binom{r}{\lambda} B_{w-r+\lambda-\sigma,\,\lambda}\, x^w.$$

Now if $$\lambda > 2a, \qquad B_{a,\lambda} = 0,$$

thus we may limit the series to

$$\lambda \not> 2(w-r+\lambda-\sigma)$$

or $$\lambda \not< 2(r+\sigma-w).$$

Also from the way the series was formed

$$\lambda \not> r-\sigma,$$

therefore $$2(r+\sigma-w) \not> r-\sigma,$$

and $$w \not< \frac{r+3\sigma}{2}.$$

Hence the series Γ cannot begin before the term $x^{r/2}$.

Suppose that r is even and equal to $2w$, then to find the coefficient of x^w, we see that $\sigma = 0$, $\lambda = 2w$, the first term of Γ is

$$B_{w,\,2w}\,x^w = \frac{1}{w}\binom{2w}{w+1}x^w.$$

Next let r be odd and equal to $2w-1$, for the first term we have $\sigma = 0$, $\lambda = r$ or $r-1$, and hence this term is

$$\{2rB_{w-1,\,r-1}+B_{w,\,r}\}\,x^w = \binom{2w}{w+1}\frac{3w-5}{2w}x^w.$$

I have not succeeded in calculating the general term of Γ in a concise form. The last six terms are

$$(-)^r\Big[\Big\{\binom{2r-6}{3}\binom{r}{3}-\binom{2r-6}{1}\binom{r}{4}$$

$$-\binom{2r-5}{4}\binom{r-1}{2}-\binom{2r-5}{5}+\binom{r}{5}\Big\}x^{2r-9}$$

$$+\Big\{\binom{r-1}{4}(8r-35)-\binom{2r-5}{4}\Big\}x^{2r-8}$$

$$+\Big\{\binom{2r-5}{3}-\binom{r-1}{3}(4r-15)\Big\}x^{2r-7}$$

$$+\Big\{5\binom{r-1}{3}-\binom{2r-5}{2}\Big\}x^{2r-6}-\binom{r-3}{2}x^{2r-5}-x^{2r-4}\Big].$$

Also, if r is even and equal to 2ρ, the coefficient of $x^{\rho+1}$ is

$$3\binom{2\rho-1}{\rho+2}\frac{(\rho+1)(3\rho-4)}{\rho-1};$$

if r is odd and equal to $2\rho-1$, the coefficient of $x^{\rho+1}$ is

$$-\binom{2\rho-1}{\rho+2}\frac{9\rho^3-36\rho^2-11\rho+74}{2(\rho-1)}.$$

The last terms of Γ are easier to calculate because they only contain expres-

sions $B_{a,\lambda}$, $\lambda \not> a+3$ in the last two cases, for which a simple expression has been found; and in the other cases the values of $B_{a,\lambda}$ are easy to obtain because $\lambda - a$ is small.

7. The generating functions up to degree 8 are

$$G_3 = \frac{1}{1-x}, \quad G_4 = \frac{1}{(1-x)^3}, \quad G_5 = \frac{1+x}{(1-x)^5}, \quad G_6 = \frac{1+3x+x^2}{(1-x)^7},$$

$$G_7 = \frac{1+6x+6x^2+x^3}{(1-x)^9}, \quad G_8 = \frac{1+10x+20x^2+10x^3+x^4}{(1-x)^{11}}.$$

The reduced generating functions up to degree 8 are

$$G'_3 = \frac{x}{1-x}, \quad G'_4 = \frac{2x^2-x^4}{(1-x)^3}, \quad G'_5 = \frac{10x^3-10x^4+x^5+x^6}{(1-x)^5}$$

$$G'_6 = \frac{5x^3+30x^4-55x^5+29x^6-3x^7-x^8}{(1-x)^7}$$

$$G'_7 = \frac{49x^4+35x^5-189x^6+176x^7-64x^8+6x^9+x^{10}}{(1-x)^9}$$

$$G'_8 = \frac{14x^4+280x^5-252x^6-364x^7+685x^8-430x^9+120x^{10}-10x^{11}-x^{12}}{(1-x)^{11}}.$$

The reduced generating functions for degrees 3, 4, 5, represent irreducible perpetuants, that for degree 6 may be further reduced by subtracting the generating function for products $(abc)^r (def)^{w-r}$; now

$$(abc)(def)^\lambda = (def)^\lambda [(abd)-(acd)+(bcd)],$$

hence we must only take account of these products for which $r \geqslant 2$ and $w-r \geqslant 2$, that is we must subtract

$$\tfrac{1}{2}\binom{6}{3}\frac{x^4}{(1-x)^2}$$

from G'_6. Thus we find a final generating function for irreducible perpetuants degree 6

$$G''_6 = \frac{5x^3+20x^4-5x^5-71x^6+97x^7-51x^8+10x^9}{(1-x)^7}.$$

8. *Forms of degree* 4.—There are only four possible determinant

factors in the symbolical product, viz. (bcd), (adc), (abd), (acb); let us call them for short u_1, u_2, u_3, u_4. Then

$$u_1+u_2+u_3+u_4=0.$$

It will now be shown that products of the u's of total degree w may be expressed in terms of such products as have a factor $u^{w/2}$ at least. In other words perpetuants of degree 4 and weight w are expressible in terms of those which are of grade $w/2$ at least.* By means of the identity all products of degree w can be expressed in terms of the products

$$u_1^{w-\alpha-\beta}u_2^{\alpha}u_3^{\beta},$$

a linearly independent set of number $\binom{w+2}{2}$.

Let w be even and equal to 2ρ. The number of independent products is

$$\binom{2\rho+2}{2}=3\binom{\rho+1}{2}+\binom{\rho+2}{2}.$$

The number of independent products having a factor u_4^a is

$$\binom{2\rho-a+2}{2}.$$

Consider those products which have a factor u_4^ρ, those which have a factor $u_1^{\rho+1}$ or a factor $u_2^{\rho+1}$, and those which have a factor $u_3^{\rho+1}$. We have thus a set of $\binom{2\rho+2}{2}$ products, if they are linearly independent, all products of degree $w=2\rho$ can be expressed in terms of them. We will write those products which have a factor u_4^ρ in the form

$$u_4^{2\rho},\ u_4^{2\rho-1}u_2,\ u_4^{2\rho-1}u_3,\ \ldots,\ u_4^\rho u_2^\rho,\ u_4^\rho u_2^{\rho-1}u_3,\ \ldots,\ u_4^\rho u_3^\rho,$$

and then express them in terms of products $u_1^\alpha u_2^\beta u_3^\gamma$, thus

$$u_4^{2\rho-\beta'-\gamma'}u_2^{\beta'}u_3^{\gamma'}=\binom{2\rho-\beta'-\gamma'}{\beta-\beta',\ \gamma-\gamma'}u_1^{2\rho-\beta-\gamma}u_2^{\beta}u_3^{\gamma}.$$

Regard these identities as a set of equations for expressing the products $u_1^{2\rho-\beta-\gamma}u_2^\beta u_3^\gamma$ for which neither of the three indices is greater than ρ in terms of members of our chosen set of products.

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* An extension of Jordan's Lemma for Binary Forms. Another proof has been given by Wood, *Proc. London Math. Soc.*, Ser. 2, Vol. 1, p. 345.

Let Δ be the determinant of $\binom{\rho+2}{2}$ rows and columns, whose general element is $\binom{2\rho-\beta'-\gamma'}{\beta-\beta',\ \gamma-\gamma'}$, the values of β, γ being the same throughout each column, and the values of β', γ' being the same throughout each row; then our set of equations is independent if $\Delta \neq 0$. This then is the condition of independence of the members of our set.

Divide each row by $(2\rho-\beta'-\gamma')!\ \beta'!\ \gamma'!$, multiply each column by $(2\rho-\beta-\gamma)!\ \beta!\ \gamma!$, and call the resulting determinant Δ'. Then the general element of Δ' is $\binom{\beta}{\beta'}\binom{\gamma}{\gamma'}$ where

$$\beta'+\gamma' \leqslant \rho, \qquad \rho \leqslant \beta+\gamma \leqslant 2\rho,$$

and neither β nor γ exceeds ρ.

It is now convenient to fix the sequence of rows and columns in Δ' thus:—

The first $\rho+1$ rows have $\gamma' = 0$ and $\beta' = 0, 1, \ldots, \rho$, in order;

the next ρ rows have $\gamma' = 1$ and $\beta' = 0, 1, \ldots, \rho-1$, in order;

... ...

the last row has $\gamma' = \rho$, $\beta' = 0$.

The first column has $\beta = 0$, $\gamma = \rho$;

the second and third columns have $\beta = 1$, $\gamma = \rho-1, \rho$, in this order;

... ...

the last $\rho+1$ columns have $\beta = \rho$, $\gamma = 0, 1, \ldots, \rho$, in order.

Now expand Δ' as a sum of products of determinants thus:—Take first a minor of $\rho+1$ rows and columns from the first $\rho+1$ rows as the first determinant in the product; take next a minor of ρ rows and columns from the next ρ rows as the second determinant and so on, the last factor in each product being a single element out of the last row.

Consider then the minor determinants formed from the first $\rho+1$ rows, every column in any one of these has a common factor $\binom{\gamma}{0}$ (the fact that it happens to be unity should be disregarded for the moment); after removing these factors any column consists of the elements

$$\binom{\beta}{\beta'},\ \beta' = 0, 1, \ldots, \rho;$$

and as the values of β are restricted to 0, 1, ..., ρ, the resulting determinants are all

$$\begin{vmatrix} \binom{0}{0} & \binom{1}{0} & \cdots & \binom{\rho}{0} \\ \binom{0}{1} & \binom{1}{1} & \cdots & \binom{\rho}{1} \\ \cdots & \cdots & \cdots & \\ \binom{0}{\rho} & \binom{1}{\rho} & \cdots & \binom{\rho}{\rho} \end{vmatrix} = D_1$$

or zero. The first column of Δ' must supply the first column of D_1, for it is the only one for which $\beta = 0$. Hence also $\binom{\rho}{0}$ is a factor of Δ', for $\gamma = \rho$ for the first column.

Take now the next ρ rows, we may remove the factors $\binom{\gamma}{1}$ from the columns of the minor determinants, and we find ourselves always left with the determinant

$$\begin{vmatrix} \binom{1}{0} & \binom{2}{0} & \cdots & \binom{\rho}{0} \\ \binom{1}{1} & \binom{2}{1} & \cdots & \binom{\rho}{1} \\ \cdots & \cdots & \cdots & \cdots \\ \binom{1}{\rho-1} & \binom{2}{\rho-1} & \cdots & \binom{\rho}{\rho-1} \end{vmatrix} = D_2$$

or zero. For we may leave out of account the first column of Δ', that being required for the first determinant if the resulting product is not to be zero. There are only two columns for which $\beta = 1$, one of these must supply the second column of D_1, the other the first column of D_2. In one case the column factors give $\binom{\rho-1}{0}\binom{\rho}{1}$ and in the other $-\binom{\rho}{0}\binom{\rho-1}{1}$. Together the second and third columns furnish the factor

$$\begin{vmatrix} \binom{\rho-1}{0} & \binom{\rho}{0} \\ \binom{\rho-1}{1} & \binom{\rho}{1} \end{vmatrix}.$$

Proceeding thus we find $\Delta' = D_1^2 D_2^2 \ldots D_{\rho+1}^2$.

None of these determinants is zero,* and therefore Δ' is not zero, and

* Cf. Grace and Young, *Algebra of Invariants*, pp. 217–218.

therefore also Δ is not zero.

Hence the members of the chosen set are linearly independent, and all perpetuants of degree 4 and even weight w can be expressed in terms of those of grade $w/2$ at least.

When w is odd, let $w = 2\rho+1$.

Then
$$\binom{2\rho+3}{2} = 3\binom{\rho+2}{2} + \binom{\rho+1}{2}$$

and we can prove in just the same way that all products of the u's of degree $2\rho+1$ can be expressed linearly in terms of those products which have one of the factors $u_1^{\rho+1}$, $u_2^{\rho+1}$, $u_3^{\rho+1}$, $u_4^{\rho+2}$.

The perpetuant type of weight 2 is of importance for our purposes. It may be written
$$(abc)(abd) \equiv \{ab, cd\}.$$

Then
$$\{(abc)-(abd)\}^2 = \{(cdb)-(cda)\}^2$$

whence
$$2\{ab, cd\} - 2\{cd, ab\} = (abc)^2+(abd)^2-(cdb)^2-(cda)^2. \qquad \text{(I)}$$

We will use the notation (ab) to denote a transposition of two letters and $\{\text{abc}\ldots\}$ to denote the sum of the substitutions of the symmetric group of the letters $a, b, c, \ldots$ and $\{\text{abc}\ldots\}'$ to denote the sum of the substitutions of the alternating group of these letters minus the sum of those substitutions which do not belong to the alternating group. To avoid confusion substitutional symbols will be printed in Roman type. Then
$$\{\text{abc}\}\{ab, cd\} = 2(abc)\{(abd)+(bcd)+(cad)\} = 2(abc)^2. \qquad \text{(II)}$$

Thus there are two independent perpetuants of this type—as the generating function indicates. They may be taken to be $\{ab, cd\}$, $\{ac, bd\}$.

9. *Forms of degree* 5.—There are 10 irreducible forms of weight 3 to be looked for, but none of weight 2.

To reduce $(abc)(ade)$:*

$$(bdc)(bde)-(bec)(bed)$$
$$= \{-(abc)+(abd)-(acd)\}\{(abd)-(abe)+(ade)\}$$
$$-\{-(abc)+(abe)-(ace)\}\{-(abd)+(abe)-(ade)\}.$$

* The reducibility of this form was proved by Turnbull, *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 115. The reduction of this form and also the corresponding form in any number of variables was proved by Gilham, *Proc. London Math. Soc.*, Ser. 2, Vol. 20, p. 326. Gilham deduces the fact that the Jacobian of a Jacobian is reducible for any number of variables; also his theorem that the product of two Jacobians can be expressed as a sum of three-term products gives a syzygy of degree 6 and weight 2.

Hence on multiplying out

$$3(abc)(ade)$$

$$= 2(abc)(ade) - (abe)(acd) + (abd)(ace)$$

$$= -\{bd, ce\} + \{be, cd\} + 2\{ab, ce\} + \{ae, bc\} - 2\{ab, cd\} - \{ad, bc\}$$

$$+ \{ae, bd\} - \{ad, be\} - \{ae, cd\} + \{ad, ce\} + (abd)^2 - (abe)^2. \quad \text{(III)}$$

It follows from this that any perpetuant of degree r which contains a pair of factors $(abc)(ade)$, such that the letters b, c, d, e occur in no other determinant factor, must be reducible.

Now a perpetuant of degree r and weight w has w determinant factors, each may be supposed to contain the letter a; there are then $2w$ places in these factors to be filled by the remaining $r-1$ letters. Let ρ letters appear more than once, then $r-1-\rho$ letters appear once each. There will then be a pair of factors $(abc)(ade)$, such that the letters b, c, d, e do not occur again unless

$$r - \rho \leqslant w + 2.$$

But
$$r - 1 - \rho + 2\rho \leqslant 2w.$$

Hence the perpetuant is reducible unless

$$2r \leqslant 3(w+1)$$

or a minimum weight for irreducibility is $\frac{2}{3}r - 1$.

Consider the forms of weight 3. We notice that there are just 10 forms

$$(abc)^2(ade),$$

for the letters d, e can be selected in 10 ways. We proceed to show that all perpetuants of this degree and weight can be expressed in terms of these and of products of perpetuants. The possible kinds of products besides that written down are

$$(abc)(abd)(abe) \text{ and } (abc)(abd)(ace).$$

Let us write $(abc)(abd)(ace) = \{a, bc, de\}$.

Then $\{a, bc, de\} = -\{a, cb, ed\}$.

Also $(abc)[(abd)(ace) - (abc)(ade) - (abe)(acd)] = 0,$

or $\{de\}' \{a, bc, de\} = (abc)^2(ade).$

Hence $$\{\text{bc}\}\{a, bc, de\} = (abc)^2(ade). \qquad \text{(IV)}$$

Also from $$\{(abc)-(abd)\}^2 = \{(cdb)-(cda)\}^2$$

we find, on multiplying each side by (ace),

$$2\{a, bc, de\}+2\{c, da, be\}$$
$$= (abc)^2(ace)-(acd)^2(ace)+(abd)^2(ace)-(bcd)^2(ace);$$

hence, using (IV),

$$\{\text{ad}\}\,2\{a, bc, de\}$$
$$= \{\text{ad}\}[(abc)^2(ace)-(acd)^2(ace)+(abd)^2(ace)-(bcd)^2(ace)-(cda)^2(cbe)]$$

and therefore also

$$\{\text{ae}\}\,2\{a, bc, de\}$$
$$= \{\text{ae}\}[-(abc)^2(abd)+(abe)^2(abd)-(ace)^2(abd)+(bce)^2(abd)+(bea)^2(bcd)].$$

Now

$$\{\text{ade}\}-(\text{ad})\{\text{ae}\}-(\text{ae})\{\text{ad}\}$$
$$= 1+(\text{ad})+(\text{ae})+(\text{de})+(\text{ade})+(\text{aed})-(\text{ad})-(\text{ad})(\text{ae})-(\text{ae})-(\text{ae})(\text{ad})$$
$$= \{\text{de}\};$$

hence

$$4\{a, bc, de\}$$
$$= 2[\{\text{de}\}+\{\text{de}\}']\{a, bc, de\}$$
$$= 2(abc)^2(ade)-2(acd)^2(cbe)-2(ace)^2(cbd)-2(bde)^2(bca)$$
$$+2(abd)^2(bce)+2(abe)^2(bcd)+2(cde)^2(cba)$$
$$+(abc)^2\{(ace)+(acd)\}-(acd)^2\{(ace)+(dce)\}-(ace)^2\{(acd)+(ecd)\}$$
$$-(bcd)^2\{(acd)+(bde)\}+(bce)^2\{(bde)-(ace)\}-(bde)^2\{(dba)+(eba)\}$$
$$+(abd)^2\{(abe)+(dbe)-(abc)-(bcd)\}+(abe)^2\{(abd)+(ebd)-(abc)-(ebc)\}$$
$$+(cde)^2\{(eca)+(dca)-(dcb)-(ecb)\} \qquad \text{(V)}$$

which is obtained after a few trifling reductions. Also

$$(abc)(abd)(abe) = (abc)(abd)\{(cbe)+(ace)+(abc)\}$$
$$= -\{b, ac, de\}+\{a, bc, de\}+(abc)^2(abd). \qquad \text{(VI)}$$

Thus all perpetuants of degree 5 and weight 3 can be expressed in terms of perpetuants of the type $(abc)^2(ade)$ and of products.

10. *Forms of degree* 6.—The generating function shows that five irreducible forms are to be expected of weight 3.

The only possible kind of symbolical product to be considered is

$$(abc)(abd)(aef) \equiv X.$$

It is easy to see that

$$\{\mathrm{ab}\}'X = (abc)(abd)(abf) - (abc)(abd)(abe),$$

$$2[1-(\mathrm{ac})(\mathrm{bd})]X = [(abc)^2+(abd)^2-(bcd)^2-(acd)^2](aef) + 2(cda)(cdb)[(cfa)-(cea)],$$

$$\{\mathrm{cef}\}'X = 0.$$

Hence we see that, if α, β, γ be any three of the six letters,

$$\{\alpha\beta\gamma\}'X = R$$

where R stands for a sum of products of perpetuants. Also, if α, β, γ, δ be any four of the letters

$$\{\alpha\beta\gamma\delta\}X = R,$$

for, if three of these letters be taken from a, b, c, d, the reduction is made by (I), otherwise two of the letters are e and f, and the transposition (ef) changes the sign of X. Thus, if we operate on X with the substitutional series*

$$1 = \Sigma A_{\alpha_1,\alpha_2}, \ldots, T_{\alpha_1,\alpha_2}, \ldots,$$

we find $$X = A_{3,3}T_{3,3}X + R,$$

since every other T contains a factor $\{\alpha\beta\gamma\}'$ or a factor $\{\alpha\beta\gamma\delta\}$.

Now $$T_{3,3} = \Sigma\{\mathrm{abc}\}\{\mathrm{def}\}\{\mathrm{ad}\}'\{\mathrm{be}\}'\{\mathrm{cf}\}'.$$

Let $$N = \Sigma\{\mathrm{ad}\}'\{\mathrm{be}\}'\{\mathrm{cf}\}',$$

then $$T_{3,3} = \Sigma\{\mathrm{abc}\}\{\mathrm{def}\}N = N\Sigma\{\mathrm{abc}\}\{\mathrm{def}\}$$

and every substitution is permutable with N.

* Grace and Young, *Algebra of Invariants*, p. 356; *Proc. London Math. Soc.*, Vol. 33, p. 133; Vol. 34, p. 361.

There are only six terms in this sum which, operating on X, give a result other than zero. Of these

$$\begin{aligned}\{\text{ace}\}\{\text{bdf}\}NX &= \tfrac{1}{2}\{\text{ace}\}\{\text{bdf}\}N\{\text{ab}\}X+R\\ &= \tfrac{1}{2}\{\text{ace}\}\{\text{bdf}\}N\{1+(\text{ba})+(\text{bc})+(\text{be})-(\text{bc})-(\text{be})\}X+R\\ &= \tfrac{1}{2}\{\text{bdf}\}[\tfrac{1}{2}\{\text{abce}\}-\{\text{ace}\}\{(\text{bc})+(\text{be})\}]NX+R\\ &= -\tfrac{1}{2}(\text{bc})\{\text{abe}\}\{\text{cdf}\}NX+R.\end{aligned}$$

Thus we find

$$X = A[1-\tfrac{1}{2}(\text{bc})-\tfrac{1}{2}(\text{bd})]\{\text{ef}\}'\{\text{abe}\}\{\text{cdf}\}NX+R$$

where A is numerical.

We may therefore consider the form

$$\{\text{abe}\}\{\text{cdf}\}N(abc)(abd)(aef) = \begin{bmatrix} abe \\ cdf \end{bmatrix} \text{ say,}$$

as the representative perpetuant of this degree and order.

There are just five independent forms. First, it is immaterial which triad of letters is placed at the top: let that one which contains a be at the top. The interchange of letters in either triad leaves the form unaltered.

It is possible to express all forms in terms of those which have a in the top row and b in the lower row. Thus

$$[1+(\text{bc})+(\text{bd})+(\text{bf})]\begin{bmatrix} abe \\ cdf \end{bmatrix} = R,$$

and hence

$$\begin{bmatrix} abe \\ cdf \end{bmatrix} = R-\begin{bmatrix} ace \\ bdf \end{bmatrix}-\begin{bmatrix} ade \\ cbf \end{bmatrix}-\begin{bmatrix} afe \\ cdb \end{bmatrix}.$$

Amongst these latter forms there is one relation,

$$\begin{aligned}R &= \{\text{cdef}\}\begin{bmatrix} acd \\ bef \end{bmatrix}\\ &= 4\left\{\begin{bmatrix} acd \\ bef \end{bmatrix}+\begin{bmatrix} ace \\ bdf \end{bmatrix}+\begin{bmatrix} acf \\ bde \end{bmatrix}+\begin{bmatrix} ade \\ bcf \end{bmatrix}+\begin{bmatrix} adf \\ bce \end{bmatrix}+\begin{bmatrix} aef \\ bcd \end{bmatrix}\right\}.\end{aligned}$$

Thus there are just five linearly independent forms. The generating function proves that no reduction has been overlooked and the number is exact.

11. *Forms of degree* 7.—It has been shown that $\frac{2}{3}r-1$ is a minimum weight for irreducibility of forms of degree r; there are, then, no irreducible forms of degree less than 4.

Let a be placed in each determinant factor, then there are eight more places to be filled up with six letters. There are four possible kinds of product—

(i) $(abc)(abd)(abe)(afg)$,

(ii) $(abc)^2(ade)(afg)$,

(iii) $(abc)(abd)(ace)(afg)$,

(iv) $(abc)(abd)(aef)(aeg)$;

(ii) is reducible at once by (III), (iii) is reducible on multiplying (V) by (afg), in the same way (i) is reduced by (VI), and we are left with (iv).

Then $$\{\mathrm{df}\}'(abc)(abd)(aef)(aeg) = (abc)(abe)(aeg)(adf) = R.$$

Similarly the result of operating with $\{\mathrm{cf}\}'$, $\{\mathrm{cg}\}'$, and $\{\mathrm{dg}\}'$, is R in each case. Hence

$$(abc)(abd)(aef)(aeg) = \tfrac{1}{24}\{\mathrm{cdfg}\}(abc)(abd)(aef)(aeg)+R.$$

But by (II) we find

$$R = \{\mathrm{bcd}\}(abc)(abd)(aef)(aeg)$$
$$= \tfrac{1}{12}[1+(\mathrm{bc})+(\mathrm{bd})]\{\mathrm{cdfg}\}(abc)(abd)(aef)(aeg)+R.$$

Hence also

$$R = \tfrac{1}{3}\{\mathrm{bcdf}\}(abc)(abd)(aef)(aeg)$$
$$= \tfrac{1}{12}[1+(\mathrm{bc})+(\mathrm{bd})+(\mathrm{bf})]\{\mathrm{cdfg}\}(abc)(abd)(aef)(aeg)+R,$$

by subtracting the first of these results from the second we find

$$R = \tfrac{1}{12}(\mathrm{bf})\{\mathrm{cdfg}\}(abc)(abd)(aef)(aeg),$$

and therefore $(abc)(abd)(aef)(aeg)$ is reducible, and there are no irreducible perpetuants of weight 4 and degree 7.

Now G_7' the reduced generating function enumerates products $P_3.P_4$ as well as irreducible perpetuants P_7. The generating function for products $P_3.P_4$ is

$$\therefore \binom{7}{3}\frac{x^2}{1-x}\cdot\frac{2x^2-x^4}{(1-x)^3}.$$

Subtracting this from G_7', we obtain

$$G_7'' = \frac{-21x^4+385x^5-854x^6+701x^7-64x^8-274x^9+176x^{10}-35x^{11}}{(1-x)^9}.$$

The first term being negative indicates a syzygy. To obtain this, we have by (III)

$$3(abc)^2(aef)(adg) = -(abc)^2[\{de, fg\} - \{df, eg\}] + R_1,$$

where R_1 stands for products of perpetuants of which one of the factors is of degree unity (*i.e.* one of the quantics).

Again, from (V)

$$2\{a, bc, de\}(afg)$$
$$= (abc)^2(ade)(afg) - (acd)^2(cbe)(afg) - (ace)^2(cbd)(afg) - (bde)^2(bca)(afg)$$
$$+ (abd)^2(bce)(afg) + (abe)^2(bcd)(afg) + (cde)^2(cba)(afg) + R_1.*$$

Hence

$$-6\{a, bc, de\}(afg)$$
$$= \{\mathrm{fg}\}'[(abc)^2\{df, eg\} - (acd)^2\{bf, eg\} - (ace)^2\{bf, dg\} + (bde)^2\{af, cg\}$$
$$+ (abd)^2\{cf, eg\} + (abe)^2\{cf, dg\} - (cde)^2\{af, bg\}] + R_1.$$

Operate on both sides with $\{\mathrm{efg}\}'$, the left-hand side vanishes identically owing to the fundamental identity, and we obtain

$$\{\mathrm{efg}\}'\{\mathrm{ad}\}\{\mathrm{bc}\}(ace)^2\{bf, dg\} = R,$$

the required syzygy.

12. *Forms of degree* 8.—The first term in G_8' is $14x^4$, but this includes products $P_4.P_4$, the generating function for which is

$$\frac{1}{2}\binom{8}{4}\left\{\frac{2x^2-x^4}{(1-x)^3}\right\}^2.$$

On subtraction the generating function is found to begin with the term $-136x^4$, and hence a syzygy must be sought. Since $\frac{3}{3}r-1$ is greater than 4, there can be no irreducible perpetuant of weight less than 5.

* A term such as $(bcd)^2(bde)(afg)$ may be included in R_1 owing to the identity

$$(afg) = (bfg) - (abf) + (abg).$$

To find the syzygy of weight 4, consider the identity

$$(abc)(efh)\{(abd)(efg)-(abg)(efd)-(abe)(fgd)-(abf)(edg)\} = 0.$$

After using (III) we obtain

$$3\{ab, cd\}\{ef, gh\} - 3\{ab, cg\}\{ef, dh\}$$

$$= \{ab, ce\}[\{fd, gh\} - \{fg, dh\}] + \{ab, cf\}[\{ed, gh\} - \{eg, dh\}] + R_1.$$

Operate with $\{\mathrm{fgd}\}'$ and we obtain

$$\{\mathrm{fgd}\}'\{ab, cd\}\{ef, gh\} = R_1.$$

It is then clear that

$$\{\alpha\beta\gamma\}'\{ab, cd\}\{ef, gh\} = R_1$$

when α, β, γ are any three of the eight letters a, b, c, d, e, f, g, h. Also, if P be a positive symmetric group containing three letters out of either of the two factors, by (II)

$$P\{ab, cd\}\{ef, gh\} = R_1.$$

Operate then on $\{ab, cd\}\{ef, gh\} \equiv X$ with the series

$$1 = \Sigma A_{a_1, a_2, \ldots}, T_{a_1, a_2, \ldots},$$

every term but $T_{4,4}$ gives R_1 as a result.

Now $T_{4,4} = \Sigma\{\mathrm{abcd}\}\{\mathrm{efgh}\}\{\mathrm{ae}\}'\{\mathrm{bf}\}'\{\mathrm{cg}\}'\{\mathrm{dh}\}'.$

Let $N = \Sigma\{\mathrm{ae}\}'\{\mathrm{bf}\}'\{\mathrm{cg}\}'\{\mathrm{dh}'\}'.$

We have only to consider substitutional expressions in which each positive symmetric group contains two letters out of each product, any other case gives R_1, by (II).

Also $\{\mathrm{abeg}\}\{\mathrm{cdfh}\}N[1+(\mathrm{eg})+(\mathrm{fg})]X = R_1,$

and hence $\{\mathrm{abeg}\}\{\mathrm{cdfh}\}NX = -\frac{1}{2}(\mathrm{fg}).\{\mathrm{abef}\}\{\mathrm{cdgh}\}NX,$

and remembering that by (I)

$$[1-(\mathrm{eg})(\mathrm{fh})]X = R_1,$$

we see that we have only one form to consider,

$$\begin{bmatrix} abef \\ cdgh \end{bmatrix} = \{\mathrm{abef}\}\{\mathrm{cdgh}\}N\{ab, cd\}\{ef, gh\};$$

and in terms of this and products R_1 every perpetuant product $P_4.P_4$ of weight 4 can be expressed.

The number of linearly independent forms

$$\begin{bmatrix} abef \\ cdgh \end{bmatrix}$$

can be obtained thus. The form is annihilated by any positive symmetric group of degree 5.

Operate with $\{bcdgh\}$; we obtain a sum of five terms, four of which have the letter b in the lower row. Thus we need only consider those forms which have a in the upper and b in the lower row. Next those forms which have both c and d in the lower row can be expressed in terms of such as have at least one of c and d in the upper row. In the case of

$$\begin{Bmatrix} aefg \\ bcdh \end{Bmatrix}$$

this is effected by operating with $\{efgcd\}$.

Lastly, amongst forms having c above and d below there is one relation to express them in terms of those which have both c and d above; and the same is true of forms having d above and c below.

Thus there are 4 forms both c and d above, 5 forms c above and d below, and 5 forms d above and c below—14 independent forms altogether. Hence there are 14 products $P_4.P_4$ linearly independent of each other and of products R_1. This is the correct number as given by G'_8. There are thus 136 linearly independent relations given by the syzygy.

13. It is possible now to obtain a higher limit to the weight of an irreducible perpetuant of degree r. We shall set before ourselves the object of introducing as many squared factors as possible in the symbolical product. The letter a is introduced into each of the factors which only appear to the first power.

Let us suppose that there are four such factors. We may for the moment regard the product of these four factors as a perpetuant: it is of degree not greater than 9. If there are two factors $(abc)(ade)$ such that the letters b, c, d, e do not appear in the other two factors, then by using (III) we may express this product as a sum of products of four factors involving fewer letters. If three of the factors are $(abc)(abd)(ace)$ then, by (V), they can be expressed in terms of products having a squared factor. By (VI) the same is true if three of the factors are $(abc)(abd)(abe)$.

There cannot, then, be more than seven letters involved, and by referring to § 11 we see that in this case the only unconsidered possibility is

$$(abc)(abd)(aef)(aeg);$$

this, by § 11, is expressible in terms of products of four factors with fewer letters and of products with a squared factor. If six letters are involved we see by § 10 that we may suppose the factors to be

$$(abc)(abd)(aef)$$

and one other. If this contains $(ab\ )$, $(ac\ )$, or $(ad\ )$ a squared factor may be introduced as we have seen above, and the only other possibility is (aef), introducing $(aef)^2$.

Thus a symbolical product which has four factors of the first degree only can be expressed in terms of products having not more than two of the first degree. Let us suppose there are λ squared factors and three factors of the first degree {which must be of the form $(abc)(abd)(aef)$ by § 10}, the maximum degree is

$$r = 2\lambda+6,$$

the weight is $w = 2\lambda+3$. In this case the degree is even and the minimum weight is $r-3$.

If there are only two factors of the first degree they must be $(abc)(abd)$. The maximum degree is

$$r = 2\lambda+4$$

and the weight is $w = 2\lambda+2$; this gives a higher weight for perpetuants of even degree.

If there is only one factor of the first degree the maximum degree is $r = 2\lambda+3$, and the weight is $w = 2\lambda+1$. Thus we see that there can be no irreducible perpetuant of even degree r of weight less than $r-3$, and of odd degree r of weight less than $r-2$.

14. The question of syzygies naturally raises the point whether it is possible to find syzygies for ternary forms in the same manner as Stroh did for binary forms. Stroh* used the symbolical identity

$$(ab)c_x d_x+(cd)a_x b_x = (ad)c_x b_x+(cb)a_x d_x,$$

* *Math. Ann.*, Vol. 33, pp. 61–107 (§§ 18–22); Vol. 34, pp. 306–320, 354–370; Vol. 36, pp. 262–288. Grace and Young, *Algebra of Invariants*, p. 140.

simply raising both sides to the power w and expanding by the binomial theorem.

It was shown by Mr. Wood and myself* that all perpetuant syzygies of degree up to 8 at least, for binary forms, may be deduced from these Stroh syzygies and the Jacobian syzygy. We proceed to prove that no such family of syzygies exists for ternary forms.

The problem resolves itself into finding an identity of the form

$$[a_1 a_2 \dots a_r b_1 b_2 \dots b_s] + [c_1 c_2 \dots c_t d_1 d_2 \dots d_u]$$
$$= [a_1 a_2 \dots a_r d_1 d_2 \dots d_u] + [c_1 c_2 \dots c_t b_1 b_2 \dots b_s] \quad \text{(VII)}$$

where

$$[a_1 a_2 \dots a_r b_1 b_2 \dots b_s]$$
$$= \Sigma A_{1,2,3}(a_1 a_2 a_3) + \Sigma B_{1,2,3}(b_1 b_2 b_3) + \Sigma E_{1,2;1}(a_1 a_2 b_1) + \Sigma F_{1;1,2}(a_1 b_1 b_2), \quad \text{(VIII)}$$

the other brackets representing similar expressions, each term containing one determinant factor only, and the coefficients A, B, &c., being numerical.

Consider the effect of operating with the substitutional series

$$1 = \Sigma A_{a_1, a_2, \dots, a_h} T_{a_1, a_2, \dots, a_h}$$

for the $r+s+t+u = n$ letters involved. Any negative symmetric group of four letters annihilates every term of (VII). Consider the group $\{\text{efgh}\}'$; if two of the letters e, f are outside the determinant factor the term contains e and f symmetrically, and will therefore be annihilated.

Otherwise it is annihilated owing to the fundamental identity, which may be written

$$\{\text{efgh}\}'(efg)h_x = 0.$$

Hence, if S be any term of (VII),

$$T_{a_1, a_2, \dots, a_h} S = 0,$$

unless $h < 4$, for T is a sum of terms each having a negative symmetric group degree h for factor.

Again, if N_3 is a negative symmetric group degree 3,

$$T_{a_1, a_2, \dots, a_h} N_3 = 0$$

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 2, p. 221.

unless $h > 2$. And since

$$(efg) = \tfrac{1}{6}\{\mathrm{efg}\}'(efg),$$

$$T_{a_1, a_2, \ldots, a_h} S = 0,$$

unless $h > 2$. That is, we need only consider those terms of the substitutional series for which $h = 3$.

Let us suppose that $a_2 > 1$, then every term of $T_{a_1, a_2, \ldots, a_h}$ has a factor of the form $\{\mathrm{efg}\}'\{\mathrm{hi}\}'$.

But $\{\mathrm{efg}\}'\{\mathrm{hi}\}'S$ is obviously zero unless S is of the form $(efh)g_x i_x, \ldots$, but even here we have by the fundamental identity

$$\{\mathrm{efg}\}'\{\mathrm{hi}\}'(efh)g_x i_x = \{\mathrm{efg}\}'\{\mathrm{hi}\}'(efg)h_x i_x = 0.$$

Thus we see that the substitutional theorem gives

$$S = A_{n-2,1,1} T_{n-2,1,1} S,$$

for every other term vanishes.

Now, $T_{n-2,1,1}$ is a sum of terms each of which has as a factor a positive symmetric group containing $n-2$ letters—that is, all the letters but two. This group may then be defined by the missing letters ; if these be e and f we may write it [ef]. Then it is clear that

$$[\mathrm{ef}]S = 0,$$

unless both e and f appear in the determinant factor of S.

Now, if Q be a sum of terms such as S, the necessary and sufficient condition for the identity $Q = 0$ to be true is that $[\mathrm{ef}]Q = 0$ for every choice of the letters e and f. It is necessary, for if $Q = 0$, obviously $[\mathrm{ef}]Q = 0$; it is sufficient because

$$Q = \Sigma ATQ = A_{n-2,1,1} T_{n-2,1,1} Q = A_{n-2,1,1} \Sigma H[\mathrm{ef}]Q$$

where H is a substitutional expression.

Let us apply this to the proposed identity,

$$[\mathrm{ef}][a_1 a_2 \ldots a_r b_1 b_2 \ldots b_s] = 0,$$

unless both e and f are found amongst the letters $a_1 \ldots a_r b_1 \ldots b_s$.

Hence, operating on (VII) with $[\mathrm{a}_\lambda \mathrm{b}_\mu]$, we find

$$[\mathrm{a}_\lambda \mathrm{b}_\mu][a_1 a_2 \ldots a_r b_1 b_2 \ldots b_s] = 0,$$

for $\lambda = 1, 2, \ldots, r$, $\mu = 1, 2, \ldots, s$. But this set of equations is the condition that

$$[a_1 a_2 \ldots a_r b_1 b_2 \ldots b_s] = [a_1 a_2 \ldots a_r] + [b_1 b_2 \ldots b_s],$$

that is, a sum of two sets of terms, one of which involves the a's only in the determinant factor, the other the b's only, as we proceed to show.

In the determinant factors $(a_\lambda a_\mu b_\nu)$ and $(a_\lambda b_\mu b_\nu)$ we may introduce the letter a_1 by means of the fundamental identity. Let us suppose this to have been done in the first place. Then, using the notation of (VIII), we see that $E_{\lambda,\mu;\nu}$ is zero unless λ or μ is unity, and that $F_{\lambda;\mu,\nu}$ is zero unless $\lambda = 1$.

Operating with $[a_\lambda b_\mu]$, we find

$$\sum_x E_{\lambda, x;\mu} - \sum_x F_{\lambda;\mu,x} = 0.$$

But $$F_{\lambda;\mu,x} = 0$$

unless $\lambda = 1$; hence $$\sum_x E_{\lambda,x;\mu} = 0$$

unless $\lambda = 1$.

Again, if $\lambda \neq 1$, $$\sum_x E_{\lambda,x;\mu} \equiv E_{\lambda,1;\mu};$$

hence all the E's are zero.

Now use the identity

$$(a_1 b_\mu b_\nu) = (a_1 b_1 b_\nu) - (a_1 b_1 b_\mu) + (b_1 b_\mu b_\nu),$$

and we find as before that all terms $(a\, b_\mu b_\nu)$ disappear, and hence

$$[a_1 a_2 \ldots a_r b_1 b_2 \ldots b_s] = \Sigma A_{1,2,3}(a_1 a_2 a_3) + \Sigma B_{1,2,3}(b_1 b_2 b_3).$$

Thus any relation of the form (VII) that can be obtained is incapable of giving syzygies, as it only yields obvious identities between products of forms.

The argument applies in the same way to quaternary forms and to forms with any higher number of variables.

SOME LIMITING CASES IN THE THEORY OF INTEGRAL EQUATIONS.

By A. C. DIXON.

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IN this paper §§ 1–12, 14–16 contain a discussion of some integral equations in which the nucleus is a homogeneous function of the two variables of degree -1, and the integration is over the whole range* of real positive quantities.

The form of solution is given and sufficient conditions of validity are investigated. The theory differs from the usual theory of integral equations in that—

(1) the solving nucleus is not fixed, but has sometimes to be altered to secure convergency (§ 4);

(2) the solution may contain arbitrary constants and yet there may be no special condition to secure solubility (§ 7);

(3) there is nothing that corresponds at all accurately to the special values of the parameter in the ordinary theory, nor is there the same sharp distinction between the two species of integral equations.

I have taken the opportunity to add a paragraph (§ 13) on some other linear functional equations which may be regarded as limiting forms of integral equations.

1. Let x, t be real positive quantities, and take the equation

$$\phi(x) = f(x) + \frac{1}{x}\int_0^\infty \phi(t) K\left(\frac{t}{x}\right) dt, \qquad (1.1)$$

* In § 17, which was added later, with §§ 14–16, an example is given of the application of the results to important cases where the range is finite and the nucleus is not homogeneous, but has a singularity of the type met with in the earlier paragraphs of the paper.

or, as it may be written,

$$\phi(x) = f(x) + \int_0^\infty \phi(xu) K(u)\, du. \tag{1.11}$$

Here ϕ is the function sought, and f is supposed a known function; K is also given.

The equation is satisfied by $\phi(x) = x^\lambda$ if $f(x)$ has the value

$$x^\lambda \{1 - \rho(\lambda)\}, \tag{1.2}$$

where

$$\rho(\lambda) = \int_0^\infty K(u) u^\lambda\, du.$$

Since, then, an arbitrary function $f(x)$ may under certain conditions be written*

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^\infty x^{i\lambda - a} y^{a - i\lambda - 1} f(y)\, dy\, d\lambda, \tag{1.20}$$

it is suggested that $\phi(x)$ may have the value

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^\infty \frac{x^{i\lambda - a} y^{a - i\lambda - 1} f(y)}{1 - \rho(i\lambda - a)}\, dy\, d\lambda \tag{1.21}$$

or

$$f(x) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^\infty \frac{x^{i\lambda - a} y^{a - i\lambda - 1} \rho(i\lambda - a)}{1 - \rho(i\lambda - a)} f(y)\, dy\, d\lambda, \tag{1.22}$$

provided the necessary conditions of convergency are satisfied. The constant a will generally be zero, but other values may be convenient in some cases.

2. An example with a good deal of generality is

$$H\phi(x) = f(x) + M\phi(x), \tag{2.1}$$

where $M\phi(x)$ stands for

$$P \int_0^\infty \phi(t) \left\{ A \left(\frac{x}{t}\right)^a + B \left(\frac{x}{t}\right)^b + \ldots \right\} \frac{dt}{x - t}; \tag{2.11}$$

the symbol P means that the principal value is to be taken; $A, B, \ldots H$, $a, b, \ldots$ are constants.

If $\phi(x) = x^\lambda$ and the indices $\lambda - a, \ldots$ are such that the integral (2.11)

* See Mellin, *Acta Math.*, Vol. 28, p. 40. The expression may be formally derived from Fourier's integral.

converges at the limits, it follows that

$$f(x) = \phi(x)\{H-\pi\Sigma A \cot \pi(\lambda-a)\}. \tag{2.2}$$

But $\{H-\pi\Sigma A \cot \pi(\lambda-a)\}^{-1}$ may generally be written also in the form $H'-\pi\Sigma A' \cot \pi(\lambda-a')$, so that

$$\phi(x) = f(x)\{H'-\pi\Sigma A' \cot \pi(\lambda-a')\} \tag{2.3}$$

$$= H'f(x)-M'f(x), \tag{2.4}$$

where $M'f(x)$ stands for

$$P\int_0^\infty f(t)\left\{A'\left(\frac{x}{t}\right)^{a'}+B'\left(\frac{x}{t}\right)^{b'}+\ldots\right\}\frac{dt}{x-t}, \tag{2.41}$$

and the indices $\lambda-a'$, $\lambda-b'$, ... are such that this integral converges at the limits. M' is of the same form as M, and from its construction contains the same number of terms.

The connexion between the two sets of constants involved in M, M' is given by the identity

$$\{H-\pi\Sigma A \cot \pi(\lambda-a)\}\{H'-\pi\Sigma A' \cot \pi(\lambda-a')\} = 1. \tag{2.5}$$

It follows that a', b', ... are the distinct roots of the equation

$$H = \pi\Sigma A \cot \pi(x-a) \tag{2.51}$$

and a, b, ... those of $\quad H' = \pi\Sigma A' \cot \pi(x-a'). \tag{2.52}$

Putting $\pm i\infty$ for λ in (2.5), we also have

$$(H\pm i\pi\Sigma A)(H'\pm i\pi\Sigma A') = 1, \tag{2.53}$$

and therefore

$$\frac{H'}{H} = -\frac{\Sigma A'}{\Sigma A} = \frac{1}{H^2+(\pi\Sigma A)^2} \tag{2.54}$$

3. We have found that the solution of the equation (2.1) for the unknown function ϕ is given by (2.4) when f has the special form x^λ and λ satisfies certain inequalities. In order to extend this result to other functions we substitute for $\phi(x)$ from (2.4) in (2.1) and test the result, which may be written

$$(H-M)(H'-M')f(x) = f(x). \tag{3.1}$$

The essential point is to transform

$$M\{M'f(x)\} \quad \text{or} \quad P\int_0^\infty\left\{P\int_0^\infty f(y)\frac{\Sigma A'(t/y)^{a'}}{t-y}dy\right\}\frac{\Sigma A(x/t)^a}{x-t}dt \tag{3.21}$$

by changing the order of integration. Since at each integration the principal value is to be taken, and since the point $y = t = x$ is within the field of integration, the change of order affects the value of the repeated integral. The matter is discussed at length by Hardy,* and the point is investigated by another method in § 5 below. The result is that

$$M\{M'f(x)\}$$

$$= P\int_0^\infty \left\{P\int_0^\infty \frac{\Sigma A'(t/y)^{\alpha'} \,.\, \Sigma A\,(x/t)^{\alpha}}{(x-t)(t-y)}\,dt\right\} f(y)\,dy - \pi^2 f(x)\,.\,\Sigma A\,.\,\Sigma A'. \tag{3.22}$$

Here, in the integration with respect to t, the subject consists of such terms as

$$\frac{t^\alpha}{(x-t)(t-y)},$$

the integral of which converges at the limits if α has a real part between ± 1. If $R(\alpha)$ is positive, and < 1,

$$P\int_0^\infty \frac{t^\alpha dt}{(x-t)(t-y)} = \frac{1}{x-y}\,P\int_0^\infty \left\{\frac{xt^{\alpha-1}}{x-t} + \frac{yt^{\alpha-1}}{t-y}\right\} dt$$

$$= \frac{x^\alpha - y^\alpha}{x-y}\,\pi\cot\alpha\pi. \tag{3.31}$$

If $R(\alpha)$ is negative, and > -1,

$$P\int_0^\infty \frac{t^\alpha dt}{(x-t)(t-y)} = \frac{1}{x-y}\,P\int_0^\infty \left\{\frac{t^\alpha}{x-t} + \frac{t^\alpha}{t-y}\right\} dt$$

$$= \frac{x^\alpha - y^\alpha}{x-y}\,\pi\cot\alpha\pi, \text{ again.} \tag{3.32}$$

Thus

$$P\int_0^\infty \frac{\Sigma A'(t/y)^{\alpha'} \,.\, \Sigma A\,(x/t)^{\alpha}}{(x-t)(t-y)}\,dt = \Sigma\Sigma A A'\pi\cot(\alpha'-\alpha)\pi\,\frac{(x/y)^\alpha - (x/y)^\alpha}{x-y}$$

$$= \{H\,.\,\Sigma A'(x/y)^{\alpha'} + H'\,.\,\Sigma A\,(x/y)^{\alpha}\}/(x-y), \tag{3.4}$$

by the use of (2.51) and (2.52).

This result holds if all the differences $\alpha-\alpha'$, $\alpha-b'$, ... have real parts between ± 1, which will be so if the real parts of α, b, ..., α', b', ... are all contained in any interval of length less than 1.

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 7, p. 193.

Hence

$$M\{M'f(x)\} = HM'f(x)+H'Mf(x)-\pi^2 f(x)\,.\,\Sigma A\,.\,\Sigma A', \qquad (3\,.\,5)$$

and since

$$\pi^2\Sigma A\,.\,\Sigma A' = HH'-1, \qquad (3\,.\,51)$$

$$(H-M)(H'-M')f(x) = f(x), \qquad (3\,.\,1)$$

if the function f is such that the operations are permissible. Similarly

$$(H'-M')(H-M)f(x) = f(x). \qquad (3\,.\,6)$$

If we pass to the associated equation by interchanging the variables in the nucleus, the effect is to change the signs of A, B, ..., a, b, ..., but not of H; the signs of A', B', ..., a', b', ... are then also to be changed, so that (2 . 5) will still be true for all values of λ.

4. Something should be added about the conditions under which the above reasoning is legitimate. The data of the problem are the constants A, B, ..., H, a, b, ..., and the function $f(x)$. Now as to a, b, ..., it is necessary for convergency in (2 . 2) that the real parts of $a-\lambda$, $b-\lambda$, ... be positive and less than unity. We shall, therefore, assume that $R(a)$, $R(b)$, ... all lie in an interval of length less than 1. Suppose $R(a)$ to be the least of them and $R(k)$ the greatest, then $R(\lambda)$ in (2 . 2) must lie between $R(k)-1$ and $R(a)$.

Similarly in (2 . 4) $R(a')$, $R(b')$, ... must lie between $R(\lambda)$ and $R(\lambda)+1$; this requirement is to be satisfied by changing a', b', ... if necessary. a', b', ... are given by (2 . 5), and any one of them may be changed by adding an integer without (2 . 5) being affected. If $R(a')$, $R(b')$, ... all have values between $R(a)$ and $R(k)$, these values will suit any admissible value of λ, but if $R(a')$, for instance, is less than $R(a)$, and $1+R(a') > R(k)$, then, when $R(\lambda)$ is between $R(k)-1$ and $R(a')$, the value a' is to be used, but when $R(\lambda)$ is between $R(a')$ and $R(a)$, $a'+1$ must be used instead.

If $R(\lambda) = R(a')$ the form (2 . 3) is good in general, but the integral in (2 . 4) is not convergent. The case $\lambda = a'$ will be discussed later (§ 8).

For the present we shall take $R(a)$, $R(b)$, ..., $R(a')$, $R(b')$, ..., all to be between 0 and 1, exclusive, and $R(a')$ to be the least, $R(k')$ the greatest of $R(a')$, $R(b')$, ...

5. With respect to the function $f(x)$ we might follow the method of Hardy* for finding sufficient conditions; the following method is not

* *Loc. cit.*, pp. 199–204.

without interest, even if of narrower application in some respects. It depends on the fact that Cauchy's principal value of an integral, where the integrand is an analytical function with simple poles on the path, is the average of the two values of the integral given by paths in the complex domain passing on different sides of the poles. For instance

$$P\int_{-1}^{1}\frac{dx}{x}=\operatorname*{Lim}_{\epsilon\to 0}\left\{\int_{-1}^{-\epsilon}\frac{dx}{x}+\int_{\epsilon}^{1}\frac{dx}{x}\right\}=0.$$

If the points $\pm\epsilon$ were joined by a small arc above (below) the origin, so as to complete the path from -1 to $+1$, the value of $\int_{-1}^{1}\frac{dz}{z}$ along this path would be $-i\pi(+i\pi)$, and the principal value is the average of these. The same thing happens in general.

If, as in (3.21), an integration along the real axis with respect to y is followed by one with respect to t, the principal value being taken in each case, a pole of the first integrand may depend on t, and may therefore be displaced from the real axis by the proposed deformations of the t-path. This displacement must not be such that the pole crosses one of the two new y-paths.

Suppose that $f(z)$ vanishes for all real positive values of z, except those between p and q $(0<p<q)$ for which it is equal to an analytical function $F(z)$, regular at all real points z such that $p<z<q$. $F(z)$ may have singularities at p, q, but we take these singularities to be such that $F(z)$ is regular at all points but p, q of the rhombus whose vertices are p, q, and $\frac{1}{2}(p+q)\pm il$, l being a positive quantity, and is bounded* throughout this rhombus.

In (3.21) and (3.22) the limits for y are p, q; let the two y-paths consist of sides of the rhombus whose vertices are p, q, $\frac{1}{2}(p+q)\pm\frac{1}{2}il$.

In (3.21) the y-integration comes first and the subject in the t-integration may have singularities at p, q; we therefore take the t-paths in three parts, 0 to p, p to q, q to ∞. The t-paths from p to q must lie within the region enclosed by the y-paths; suppose them to consist of sides of the rhombus whose vertices are p, q, $\frac{1}{2}(p+q)\pm\frac{1}{4}il$.

In (3.22), on the other hand, the t-integration comes first and the y-paths from p to q must lie between the t-paths. Suppose then that the

* The condition that $F(z)$ shall be bounded is more stringent than it need be; it is enough if the contour integrals used in the discussion are absolutely convergent. The results can be extended by the substitution of $x^{-\kappa}f(x)$ and $x^{-\kappa}\phi(x)$ for $f(x)$ and $\phi(x)$; this has the effect of increasing a, b, ..., a', b', ... each by κ.

y-paths are the same as before, but that the t-paths consist of sides of the rhombus whose vertices are p, q, $\frac{1}{2}(p+q) \pm \frac{3}{4}il$.

The subject of integration consists of such terms as

$$f(y)(x/t)^{\alpha}(t/y)^{\alpha'}/(x-t)(t-y). \qquad (5.1)$$

Take x not to have one of the values 0, p, q, ∞; then when t approaches 0 the term (5.1) is the product of $|t|^{\alpha'-\alpha}$ and a bounded factor; when t tends to infinity it is the product of $|t|^{\alpha'-\alpha-2}$ and a bounded factor. Since $R(a'-a)$ is between ± 1, the integral is absolutely convergent in both cases. When t and y approach p in the complex domain (5.1) is the quotient of a bounded function by $t-y$; now in any triangle $c^2 \sin A \sin B = ab \sin^2 C$, and therefore $c^2 > ab \sin^2 C$; thus

$$|t-y|^2 > |t-p|\,|y-p| \sin^2\theta$$

where θ is the angle at which the t-path and the y-path meet in p. Hence (5.1) is the quotient of a bounded function by

$$\sqrt{|t-p|} \cdot \sqrt{|y-p|},$$

and the integral is again absolutely convergent. Similarly when t and y approach q. Hence the order of integration may be changed, the paths being kept the same.

Hence the excess of the integral when the y-integration is first over that when the t-integration is first is due to the change in the t-paths. The two upper t-paths differ by a closed circuit which encloses all points on the upper y-path but none on the lower; the residue at y is

$$f(y)(x/y)^{\alpha}/(x-y),$$

and the excess due to this is the integral of

$$\tfrac{1}{4} \cdot 2i\pi f(y)(x/y)^{\alpha}/(x-y) \qquad (5.2)$$

along the upper y-path; the factor $\frac{1}{4}$ enters when the average is taken.

Similarly the difference of the lower t-paths contributes to the excess the integral of the same expression along the lower y-path reversed.

The whole excess is therefore 0 when $x < p$ or $x > q$ but $(i\pi)^2 f(x)$ when $p < x < q$, for it is the integral of (5.2) round the rhombus whose vertices are p, q, $\frac{1}{2}(p+q) \pm \frac{1}{2}il$ in the negative direction. This is the result assumed in (3.22).

When $p = 0$ and t, y approach 0 along straight lines inclined at the angle θ, the term (5.1) is the product of a bounded factor and

$$|t|^{\alpha'-\alpha}|y|^{-\alpha'} \div (t-y),$$

a quantity whose modulus is less than

$$|t|^{R(a')-R(a)-g}|y|^{-R(a')+g-1}\operatorname{cosec}\theta$$

where g may be any positive proper fraction, since in any triangle

$$c > (a\sin C)^g(b\sin C)^{1-g}$$

so that here

$$|t-y| > |t|^g|y|^{1-g}\sin\theta.$$

Since $R(a)$, $R(a')$ are both between 0 and 1, g may be so chosen that it is greater than $R(a')$ but less than 1 and less than $R(a')+\{1-R(a)\}$, so that the indices $R(a')-R(a)-g$ and $-R(a')+g-1$ are both greater than -1 and the repeated integral still converges absolutely.

If q is infinite each rhombus becomes a region bounded by two straight lines from p, two sides of the rhombus disappearing. When t, y tend to infinity along two lines intersecting at the angle θ, the term (5.1) is the product of a bounded factor and

$$|t|^{a'-a-1}|y|^{-a'} \div (t-y),$$

the modulus of which is less than

$$|t|^{R(a')-R(a)-g-1}|y|^{-R(a')+g-1}\operatorname{cosec}\theta,$$

g being now a positive proper fraction less than $R(a')$ but greater than $R(a')-R(a)$, so that the indices $R(a')-R(a)-g-1$, $-R(a')+g-1$ are less than -1 and the repeated integral converges absolutely.

The rest of the argument when p is zero or q is infinite is as before.

Hence, in § 3, $f(x)$ may have such values as have been assumed in this section (§ 5). By addition, the result (3.1) still holds when the real positive axis can be divided into a finite number of segments in each of which $f(x)$ is equal to a function F, regular at internal points of a rhombus having the segment for diagonal and bounded throughout the rhombus; for the segment that extends to infinity the rhombus becomes a region containing the segment and bounded by two straight lines drawn from the initial point of the segment. The functions F may be different in different segments.

6. Further, when p is zero and q is infinite, ϕ like f is regular and bounded for a certain range of positive and negative vectorial angles.

Suppose f to be regular and bounded for vectorial angles between $\pm\alpha$ and take x in (2.4) to be a complex variable with amplitude between

$\pm\gamma$ where γ is a positive angle less than α. Estimate the principal value of

$$\int_0^\infty \frac{f(t)(x/t)^{\alpha'}}{x-t}\,dt$$

as the average of the values for two infinite straight paths from the origin, inclined to the axis at angles $\pm\beta$, where $\alpha > \beta > \gamma$.

The integral along either path is a regular analytical function of x, except at 0 or ∞, and so therefore is $\phi(x)$.*

The absolute value of the integral is less than

$$gK\int_0^\infty \frac{|x/t|^{R(\alpha')}d|t|}{(|x|+|t|)\sin\frac{1}{2}(\beta-\gamma)}$$

where K is an upper bound to $|f(x)|$, g is 1 when α' is real and otherwise is $\exp\left[\{i\alpha'\sim iR(\alpha')\}(\beta+\gamma)\right]$, and $\sin\frac{1}{2}(\beta-\gamma)$ presents itself as the least possible value of $|x-t|\div(|x|+|t|)$. The value of this expression is

$$gK\operatorname{cosec}\tfrac{1}{2}(\beta-\gamma)\,.\,\pi\operatorname{cosec}\pi R(\alpha').$$

Thus $\phi(x)$ as given by (2.4) is bounded for amplitudes of x between $\pm\gamma$.

Special Points and Exceptional Cases.

7. If $R(\alpha') < R(\alpha)$ but $R(\alpha')+1 > R(k)$, then $x^{\alpha'}$ satisfies the equation

$$(H-M)\,\phi(x) = 0,$$

and the solution of (2.1) will include a term in $x^{\alpha'}$ with an arbitrary constant coefficient and other such terms if b', c', ... have real parts lying between $R(k)-1$ and $R(a)$.

The associated equation

$$(H-\bar{M})\,\psi(x) = 0,$$

in which $\bar{M}$ is derived from M by changing the signs of A, B, ..., a, b, ..., is similarly satisfied by

$$\psi(x) = x^{-\alpha'-1}.$$

Thus from the ordinary theory of integral equations one would expect that a condition of solubility of (2.1) would be

$$\int_0^\infty x^{-\alpha'-1}f(x)dx = 0.$$

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* This is also true when p, q have other values and x is not 0, p, q, or ∞.

This is not so. We have seen that $f(x)$ may be x^λ when

$$R(a) > R(\lambda) > R(k) - 1$$

and the condition is not then satisfied. If we multiply (2.1) by $x^{-a'-1}$ and seek to integrate from 0 to ∞ the integrals often fail to converge, and it is only when

$$\int_0^\infty x^{-a'-1} \phi(x)\, dx$$

is convergent that the condition can be deduced.

8. When $f(x) = x^{a'}$ the method of solution fails, whether in the form (2.3) or (2.4), but the failure is only apparent if a' satisfies certain inequalities.

We know that when $R(a) > R(\lambda) > R(k) - 1$,

$$(H - M)x^\lambda = \{H - \pi \Sigma A \cot \pi(\lambda - a)\} x^\lambda, \tag{8.1}$$

and that the operator M is commutative with $d/d\lambda$.

Hence

$$(H - M)(x^\lambda \log x) = \{H - \pi \Sigma A \cot \pi(\lambda - a)\} x^\lambda \log x + \pi^2 \Sigma A \operatorname{cosec}^2 \pi(\lambda - a) \cdot x^\lambda. \tag{8.2}$$

Thus if $f(x) = x^{a'}$, and $R(a) > R(a') > R(k) - 1$,

$$\phi(x) = x^{a'} \log x \div \pi^2 \Sigma A \operatorname{cosec}^2 \pi(a' - a), \tag{8.3}$$

but if $\Sigma A \operatorname{cosec}^2 \pi(a' - a) = 0$ also, another differentiation with respect to λ will generally give $\phi(x)$ as a constant multiple of $x^{a'}(\log x)^2$; the number of differentiations with respect to λ will be the number of times that a' occurs among $a', b', \ldots$.

When $f(x) = x^a$ the value of $\phi(x)$ given by (2.3) is 0, which is not admissible. The equation (2.1) seems to be insoluble in this case and also when $f(x) = x^\lambda$ if the real part of any one of the indices $a - \lambda$, $b - \lambda$, ... does not fall between 0 and 1.

9. The method of § 2 depends on the determination of $H', A', \ldots, a', \ldots$ by means of (2.5). Now $H - \pi \Sigma A \cot \pi(\lambda - a)$ is equal to a fraction whose denominator is $\Pi \sin \pi(\lambda - a)$ and numerator a polynomial homogeneous in $e^{i\pi\lambda}$ and $e^{-i\pi\lambda}$ and of the same degree as the denominator. The numerator can generally be written in the form $K \Pi \sin \pi(\lambda - a')$ with distinct factors, but sometimes two or more of the factors will be the

same. The reciprocal will then contain derivatives, say of $\cot \pi(\lambda - a')$ with respect to a'. The solving function in (2.4) will thus contain the same derivatives of $(x/t)^{a'}$ with respect to a', since when $0 < R(a) < 1$ the formula

$$P\int_0^\infty \frac{(t/x)^{a-1}dt}{x-t} = \pi \cot a\pi \qquad (9.1)$$

may be differentiated freely with respect to a by the ordinary rule.

A further differentiation will be necessary in § 8, but the verification of § 3 and the work of § 5 are not substantially affected.

Another special case is that in which the numerator polynomial lacks its first or last term, so that one of its factors* is $e^{\pm i\pi\lambda}$ and is not a sine. This happens when $H = \mp i\pi \Sigma A$ and the reciprocal is then of the form

$$H' + K'e^{\mp 2i\pi\lambda} - \pi \Sigma A' \cot(\lambda - a'): \qquad (9.2)$$

$f(x)$ being x^λ, the corresponding expression for $\phi(x)$ is

$$H'f(x) + K'f(e^{\mp 2i\pi}x) - M'f(x), \qquad (9.3)$$

the number of indices $a', b', \ldots$ being now one less than of $a, b, \ldots$. The result (9.3) only holds for a restricted class of functions, as will be seen from an example in § 10.

10. By the method of (3.31) we have

$$P\int_0^\infty \frac{t^a dt}{(x-t)(t+\beta)} = \frac{\pi(x^a \cot a\pi - \beta^a \operatorname{cosec} a\pi)}{x+\beta}$$

if β is not real and negative, and $R(a)$ is between ± 1.

Hence in § 2 if $f(x) = x^\kappa/(x+\beta)$ and $R(k') - 1 < R(\kappa) < R(a') + 1$, the value of $\phi(x)$ as given by (2.4) is

$$(x+\beta)^{-1}[x^\kappa\{H' - \pi \Sigma A' \cot \pi(\kappa - a')\} + \pi\beta^\kappa \Sigma A'(x/\beta)^{a'} \operatorname{cosec} \pi(\kappa - a')].$$

A similar expression with unaccented letters is given for $f(x)$ by (2.1) if $\phi(x) = x^\kappa/(x+\beta)$ and $R(k) - 1 < R(\kappa) < R(a) + 1$. Also κ may have the special values $a, b, \ldots$ in the first case or $a', b', \ldots$ in the second, and either of the two results may thus be deduced from the other.

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* Both factors may occur together and they may be repeated, but no new principle is thereby introduced.

But in the latter case of § 9, when $f(x) = x^\kappa/(x+\beta)$, the form for $\phi(x)$ given by (9 . 3) is

$$(x+\beta)^{-1}\big[x^\kappa\{H'+K'e^{\mp 2i\pi\kappa}-\pi\,\Sigma\,A'\cot\pi(\kappa-\alpha')\} \\ +\pi\beta^\kappa\,\Sigma\,A'(x/\beta)^{\alpha'}\operatorname{cosec}\pi(\kappa-\alpha')\big],$$

and by substitution in (2 . 1) this gives, after reduction,

$$f(x) = x^\kappa/(x+\beta) \mp 2i\pi K'\beta^\kappa e^{\mp i\pi\kappa}(x+\beta)^{-1}\,\Sigma\,A(x/\beta)^\alpha e^{\mp i\pi\alpha},$$

which differs from the original value by a multiple of

$$(x+\beta)^{-1}\,\Sigma\,A(x/\beta)^\alpha e^{\mp i\pi\alpha}. \qquad (10\,.\,3)$$

When (10 . 3) is put for $f(x)$, the value of $\phi(x)$ as given by (9 . 3) is zero, which is inadmissible.

The formula (9 . 3) fails therefore when

$$f(x) = x^\kappa/(x+\beta),$$

but it succeeds when

$$f(x) = (x^\kappa-\beta^\kappa e^{\mp i\pi\kappa})/(x+\beta).$$

(The latter function has no pole where $\log x = \log\beta \mp i\pi$).

11. If there is only one term $A(x/t)^\alpha$ in (2 . 11) and we write $\phi(x)$, $f(x)$ for $Hx^{-\alpha}\phi(x)$ and $x^{-\alpha}f(x)$, we find that the solution of the integral equation*

$$\phi(x)-\theta P\int_0^\infty \frac{\phi(t)}{x-t}\,dt = f(x) \qquad (11\,.\,1)$$

is

$$(1+\theta^2\pi^2)\phi(x) = f(x)+\theta P\int_0^\infty \frac{f(t)\,(x/t)^b}{x-t}\,dt, \qquad (11\,.\,2)$$

where b is connected with θ by the relation $\tan b\pi = \theta\pi$, and

$$R(\lambda) < R(b) < R(\lambda)+1$$

if $f(x)$ has the special form x^λ where λ is an index whose real part is between -1 and 0; thus two values of b may occur, namely those for which $-1 < R(b) < 1$.

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* By the substitution $t = \tan u$, $x = \tan y$, $\phi(x) = \psi(y)\cos y$, $f(x) = g(y)\cos y$, the equation (11 . 1) takes the interesting form

$$\psi(y)-\theta P\int_0^{\frac{1}{2}\pi}\psi(u)\operatorname{cosec}(y-u)\,du = g(y).$$

(Compare Hardy, *loc. cit.*, p. 196, where the same integrand occurs.)

If $f(x)$ has a more general form it may be necessary to break it up into two parts and use the two possible values of b for the respective parts in such a way that the integrals in (11.1) and (11.2) converge at the limits. If this convergency cannot be secured the method fails, and it also fails when $f(x)$ is constant, since then (11.2) gives $\phi(x) = 0$.

The solution when $\theta = \pm i/\pi$ is

$$\phi(x) = \tfrac{1}{2}f(x) - \tfrac{1}{2}f(e^{\mp 2i\pi}x),$$

and it is valid for such forms of $f(x)$ as

$$x^{\lambda}\{-1 < R(\lambda) < 0\} \quad \text{or} \quad (x^{\kappa} - \beta^{\kappa}e^{\mp i\pi\kappa})/(x+\beta)(-1 < R(\kappa) < 1),$$

but not for a constant or for $(x+\beta)^{-1}$.

12. When $H = 0$ it follows that $H' = 0$ also; the two integral equations (2.1) and (2.4) are of the first species, but the method of solution still applies.

If on the other hand $\Sigma A = 0$ it follows that $\Sigma A' = 0$, and there is no need in (2.1) and (2.4) to stipulate for the principal value.

Take the special case when M in (2.1) contains only two terms and $b = a - \frac{1}{2}$, $B = -A$, so that the equation becomes

$$H\phi(x) = f(x) + A\int_0^{\infty} \phi(t)\,\frac{x^{a-\frac{1}{2}}t^{-a}\,dt}{x^{\frac{1}{2}} + t^{\frac{1}{2}}}, \tag{12.1}$$

or, if we put

$$\begin{aligned} &y^2 \text{ for } x, \quad u^2 \text{ for } t, \\ &\phi(y) \text{ for } Hy^{1-2a}\phi(y^2), \\ &f(y) \text{ for } y^{1-2a}f(y^2), \end{aligned} \tag{12.10}$$

and

$$\theta \text{ for } 2A/H,$$

the equation is

$$\phi(y) = f(y) + \theta\int_0^{\infty} \frac{\phi(u)\,du}{u+y}. \tag{12.11}$$

Then for (2.51) we have

$$\begin{aligned} H &= \pi A\{\cot\pi(x-a) - \cot\pi(x-a+\tfrac{1}{2})\} \\ &= 2\pi A \operatorname{cosec} 2\pi(x-a). \end{aligned} \tag{12.2}$$

Thus

$$\sin 2\pi(a'-a) = \sin 2\pi(b'-a) = \pi\theta. \tag{12.3}$$

To fix the ideas, suppose that a, a', b', b are real and in descending order of magnitude.

Then from (12.3) we have $a'+b' = a+b$, and if

$$a'-b'-\tfrac{1}{2} = a,$$

it follows from (2.52) that

$$H' = \pi A' \{\cot(a-a')\pi - \cot(a-b')\pi\} = -2\pi A' \cot a\pi. \quad (12.31)$$

The solution of (12.11) then comes out as

$$\phi(y) = f(y) + \pi^{-1} y \tan a\pi \int_0^\infty f(u) \frac{(u/y)^{-a} - (u/y)^{1+a}}{y^2-u^2}\, du, \quad (12.32)$$

where a is such that

$$\sin a\pi = \theta\pi. \quad (12.33)$$

This form holds when $R(a)$ is between 0 and -1, so that θ if real must be negative.

If θ is real and positive then the two indices in (12.32) are $-a$ and $-1+a$, both having real parts between 0 and -1.

Further, for some forms of $f(y)$ it may be necessary to take the index $2-a$ instead of $-a$, or $1+a$ instead of $-1+a$, or both, so as to secure convergency. It must not be forgotten that the conditions (§ 5) to be satisfied by f are modified by the substitution (12.10); the solution now fails if $f(y)$ is a constant or a constant multiple of y^{-1}.

13. A different type of equation is

$$\phi(x) - \theta\phi(x+a) = f(x), \quad (13.1)$$

where f and ϕ are supposed to have the period π.

The nucleus is here

$$\pi^{-1}\theta \sum_{n=-\infty}^{\infty} \exp 2ni(x-t+a),$$

the limits in the integration are 0, π, and the associated equation with the nucleus

$$\pi^{-1}\theta \Sigma \exp 2ni(t-x+a)$$

is

$$\psi(x) - \theta\psi(x-a) = g(x). \quad (13.2)$$

If the periodic functions $f(x)$, $g(x)$ are expanded in the forms $\Sigma A_n e^{-2nix}$ and $\Sigma B_n e^{2nix}$, then

$$\phi(x) = \Sigma A_n e^{-2nix}/(1-\theta e^{-2nia}) \quad (13.3)$$

$$\psi(x) = \Sigma B_n e^{2nix}/(1-\theta e^{-2nia}). \quad (13.4)$$

The special values of θ are

$$e^{2ni\alpha}(n = 0, \pm 1, \pm 2, \ldots),$$

and when $\theta = e^{2ni\alpha}$ the special functions are

$$e^{-2nix} \text{ for } (13.1), \qquad e^{2nix} \text{ for } (13.2);$$

the conditions of solubility are

$$\int_0^\pi e^{2nix} f(x)\,dx = 0 \quad \text{for } (13.1)$$

and

$$\int_0^\pi e^{-2nix} g(x)\,dx = 0 \quad \text{for } (13.2).$$

When α is real the special values of θ lie on the unit circle about the origin and do not form a sequence tending to infinity; in other ways the theory follows that of ordinary integral equations.

Another form is

$$\phi(x) - \theta\phi'(x+\alpha) = f(x) \tag{13.5}$$

with the associated

$$\psi(x) + \theta\psi'(x-\alpha) = g(x), \tag{13.6}$$

all the functions being again periodic.

With the former notation we now have

$$\phi(x) = \Sigma A_n e^{-2nix}/(1 + 2ni\theta e^{-2ni\alpha}) \tag{13.7}$$

$$\psi(x) = \Sigma B_n e^{2nix}/(1 + 2ni\theta e^{-2ni\alpha}). \tag{13.8}$$

The special values of θ are

$$-e^{2ni\alpha}/2ni \quad (n = \pm 1, \pm 2, \ldots),$$

and the special functions are again e^{-2nix} and e^{2nix}, also the conditions of solubility are the same.

The special values of θ form two sequences tending to 0 if α is real.

When $\alpha = 0$, (13.5) becomes a simple differential equation whose solution is

$$\phi(x) = C \exp\frac{x}{\theta} - \frac{1}{\theta}\int_0^x f(t) \exp\frac{x-t}{\theta}\,dt \tag{13.9}$$

containing an arbitrary constant C. But C is fixed by the supposition that $\phi(x)$ is periodic, and the solution then agrees with (13.7).

[§§ 14–17, which follow, were added later.]

14. A general result in continuation of § 12 may be proved as follows.

Let $\chi_r(\delta)$ $(r = 1, 2)$ denote analytical functions of δ, without singularity in a strip containing the lateral axis and bounded by two straight lines parallel to it; the functions are to be such that $\chi_r(\delta)\, e^{-i\beta\delta}$ tends to zero when δ tends to $+i\infty$ in the strip, and $\chi_r(\delta) e^{i\gamma\delta}$ tends to zero when δ tends to $-i\infty$ in the strip, where β, γ are positive constants.

Let
$$\kappa_r(z) = \int_{-i\infty}^{i\infty} \chi_r(\delta) z^\delta \, d\delta, \tag{14.0}$$

the path of integration being the lateral axis.

Then $\kappa_r(z)$ is an analytical function of z, with no singularity so long as the vectorial angle of z is between $-\beta$ and $+\gamma$, except at 0 and ∞.

The path of integration may be displaced slightly forwards or backwards; if forwards, the real part of δ is positive and the subject of integration vanishes when $z = 0$; if backwards, the subject vanishes when z is infinite. Thus $\kappa_r(0)$ and $\kappa_r(\infty)$ are both zero.

If x, y, t are real

$$\int_0^\infty \kappa_1\left(\frac{x}{t}\right) \kappa_2\left(\frac{t}{y}\right) \frac{dt}{t} = \int_0^\infty \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} x^\delta y^{-\mu} t^{\mu-\delta-1} \chi_1(\delta) \chi_2(\mu) \, d\delta \, d\mu \, dt. \tag{14.1}$$

Now
$$\int_0^a t^{\mu-\delta-1} dt = \frac{a^{\mu-\delta}}{\mu-\delta}$$

if the real part of $\mu-\delta$ is positive, that is if the path of μ is displaced forwards, but

$$\int_a^\infty t^{\mu-\delta-1} dt = -\frac{a^{\mu-\delta}}{\mu-\delta}$$

if the real part of $\mu-\delta$ is negative, that is if the path of μ is displaced backwards. Thus, by a legitimate change in the order of integration,

$$\int_0^\infty \kappa_1\left(\frac{x}{t}\right) \kappa_2\left(\frac{t}{y}\right) \frac{dt}{t} = \int_{-i\infty}^{i\infty} \int \frac{x^\delta y^{-\mu} a^{\mu-\delta}}{\mu-\delta} \chi_1(\delta) \chi_2(\mu) \, d\mu \, d\delta, \tag{14.11}$$

the integration with respect to μ being first from $-i\infty$ to $+i\infty$ along a line in front of the origin, and then from $+i\infty$ to $-i\infty$ along a line behind the origin; the whole path for μ is thus equivalent to a circuit

about δ, and the result of the μ-integration is

$$2i\pi \times \text{the value of } x^{\delta}y^{-\mu}a^{\mu-\delta}\chi_1(\delta)\chi_2(\mu)$$

when $\mu = \delta$, that is $\qquad 2i\pi(x/y)^{\delta}\chi_1(\delta)\chi_2(\delta).$

The point a may be taken anywhere on the real positive axis, and it does not appear in the result.

Thus
$$\int_0^{\infty}\kappa_1\left(\frac{x}{t}\right)\kappa_2\left(\frac{t}{y}\right)\frac{dt}{t} = 2i\pi\int_{-i\infty}^{i\infty}\left(\frac{x}{y}\right)^{\delta}\chi_1(\delta)\chi_2(\delta)\,d\delta. \qquad (14.2)$$

In particular this result holds when

$$\chi_2(\delta) = -\chi_1(\delta) \div \{1+\chi_1(\delta)\}$$

unless $1+\chi_1(\delta)$ vanishes at a point on the lateral axis.

Thus we have
$$\chi_1(\delta)\chi_2(\delta) = -\chi_1(\delta)-\chi_2(\delta) \qquad (14.3)$$

Hence
$$\int_0^{\infty}\kappa_1\left(\frac{x}{t}\right)\kappa_2\left(\frac{t}{y}\right)\frac{dt}{t} = -2i\pi\,\kappa_1\left(\frac{x}{y}\right)-2i\pi\,\kappa_2\left(\frac{x}{y}\right). \qquad (14.4)$$

Take two functions $\phi(x)$ and $f(x)$ so related that

$$\phi(x)+f(x) = -\frac{1}{2i\pi}\int_0^{\infty}\kappa_2\left(\frac{x}{y}\right)f(y)\frac{dy}{y}. \qquad (14.5)$$

Multiply (14.4) by $f(y)/y$ and integrate with respect to y from 0 to ∞.

Thus if f is bounded and continuous, or otherwise so restricted that the order of integration on the left side may be changed,

$$-2i\pi\int_0^{\infty}\kappa_1\left(\frac{x}{t}\right)\{\phi(t)+f(t)\}\frac{dt}{t}$$

$$= -2i\pi\int_0^{\infty}\kappa_1\left(\frac{x}{y}\right)f(y)\frac{dy}{y} - 2i\pi\int_0^{\infty}\kappa_2\left(\frac{x}{y}\right)f(y)\frac{dy}{y},$$

that is
$$\int_0^{\infty}\kappa_1\left(\frac{x}{t}\right)\phi(t)\frac{dt}{t} = -2i\pi\{\phi(x)+f(x)\}. \qquad (14.51)$$

Thus (14.51) is an integral equation for ϕ when f is known and (14.5) is an integral equation for f when ϕ is known, and each of these equations presents the solution of the other when the known function is bounded and continuous.

15. The result holds under some limitations when $\phi(x)$ and $f(x)$ are constant multiples of a power of x. We have*

$$\int_0^\infty \kappa_r(z) z^{-\lambda} \frac{dz}{z} = \int_0^\infty \int_{-i\infty}^{i\infty} \chi_r(\delta) z^{\delta-\lambda-1} d\delta\, dz, \tag{15.0}$$

and again $$\int_0^a z^{\delta-\lambda-1} dz = \frac{a^{\delta-\lambda}}{\delta-\lambda} \quad \text{if } R(\delta)-R(\lambda) \text{ is positive,}$$

$$\int_a^\infty z^{\delta-\lambda-1} dz = -\frac{a^{\delta-\lambda}}{\delta-\lambda} \quad \text{if } R(\delta)-R(\lambda) \text{ is negative.}$$

Hence as at (14.11)

$$\int_0^\infty \kappa_r(z) z^{-\lambda} \frac{dz}{z} = \int \chi_r(\delta) \frac{a^{\delta-\lambda} d\delta}{\delta-\lambda}$$

along a path which goes from $-i\infty$ to $+i\infty$ on the right of λ and back again on the left of λ, provided that λ is an internal point of the strip that contains the lateral axis and does not contain any singularity of $\chi_r(\delta)$. Hence

$$\int_0^\infty \kappa_r(z) z^{-\lambda-1} dz = 2i\pi \chi_r(\lambda) \tag{15.1}$$

if λ is within the strip mentioned. This equation shows how χ_r is to be found when κ_r is a given function.

Putting x/y for z and treating y as variable we have

$$\int_0^\infty \kappa_r\left(\frac{x}{y}\right) y^\lambda \frac{dy}{y} = 2i\pi x^\lambda \chi_r(\lambda). \tag{15.2}$$

Thus if $\phi(x)$ is a multiple of x^λ, so is $f(x)$;

$$f(x) = -\phi(x)\{1+\chi_1(\lambda)\}$$

from (14.51) and $$\phi(x) = -f(x)\{1+\chi_2(\lambda)\}$$

from (14.5). These two equations agree and the theory applies to the function x^λ when λ lies in both the strips, but if λ is in the strip for χ_1 and not in that for χ_2, then the solving function for the equation (14.51) must be modified by taking the path of integration in the definition of κ_2 to be not the lateral axis itself but a parallel to it through λ. Hence the

* For the following evaluation see Mellin, *Acta Math.*, vol. 25, pp. 156–160.

form of the solving function is not altogether independent of the given function f.

The solution of (14.51) fails when $f(x)$ is a constant multiple of x^λ if λ is such that $\chi_1(\lambda)$ is infinite, for then the proposed value of $\phi(x)$ is zero; the solution of (14.5) will then contain an arbitrary multiple of x^λ if λ is in the strip for χ_2.

16. If we wish $\kappa_1(z)$ to be $\frac{2i\pi}{H}\Sigma\frac{Az^a}{1-z}$ as in § 2, we take

$$H\chi_1(\lambda) = \Sigma A\int_0^\infty \frac{z^{a-\lambda-1}}{1-z}dz$$

$$= \pi\Sigma A\cot(a-\lambda)\pi \pm i\pi\Sigma A, \qquad (16.1)$$

with the upper sign when the path of z lies above the real axis, the lower sign when it lies below.

When the vectorial angle of z is between 0 and 2π, we have, therefore,

$$H\kappa_1(z) = 2i\pi\Sigma A\int_{-i\infty}^{i\infty}\frac{z^\delta d\delta}{1-e^{2i\pi(\delta-a)}}, \qquad (16.2)$$

and when it is between 0 and -2π,

$$H\kappa_1(z) = 2i\pi\Sigma A\int_{-i\infty}^{i\infty}\frac{e^{2i\pi(\delta-a)}z^\delta d\delta}{1-e^{2i\pi(\delta-a)}}, \qquad (16.21)$$

the δ-path in either case being a straight line that passes between a and $a-1$, such as the lateral axis.

With a slight change in notation and some increase of generality suppose that

$$\kappa_r(z) = \int_{-i\infty}^{i\infty}\{\chi_r(\delta)\pm A_r\}z^\delta d\delta \quad (r=1,2) \qquad (16.3)$$

where A_1, A_2 are constants and the upper sign is to be taken when the vectorial angle of z is positive, the lower when it is negative. The supposition that A_r is constant is true if the only singularity of $\kappa_r(z)$ on the real positive axis is a simple pole at $z=1$.

If x, t, y are real the principal value of

$$\int_0^\infty \kappa_1\left(\frac{x}{t}\right)\kappa_2\left(\frac{t}{y}\right)\frac{dt}{t}$$

is the average of the values given by two t-paths, one just above the real axis, the other just below. In the former the vectorial angle of x/t is negative and that of t/y is positive, so that from (16 . 3) the value given by (14 . 2) for this path is

$$2i\pi \int_{-i\infty}^{i\infty} \left(\frac{x}{y}\right)^{\delta} \{\chi_1(\delta) - A_1\} \{\chi_2(\delta) + A_2\} d\delta. \qquad (16 . 4)$$

The path just below the real axis gives the same with the signs of A_1 and A_2 changed. Hence the average is

$$2i\pi \int_{-i\infty}^{i\infty} \left(\frac{x}{y}\right)^{\delta} \{\chi_1(\delta)\chi_2(\delta) - A_1A_2\} d\delta, \qquad (16 . 41)$$

which is

$$-2i\pi\kappa_1\left(\frac{x}{y}\right) - 2i\pi\kappa_2\left(\frac{x}{y}\right) \qquad (16 . 4\,11)$$

if

$$\chi_1(\delta)\chi_2(\delta) + \chi_1(\delta) + \chi_2(\delta) = A_1A_2 \qquad (16 . 42)$$

and if

$$A_1 + A_2 = 0; \qquad (16 . 43)$$

the latter condition expresses that the pole at $z = 1$ disappears from the sum of the functions κ_1, κ_2.

If $\chi_1(\lambda)$ is a fraction whose numerator and denominator are quantics in $e^{2i\pi\lambda}$ with constant coefficients this result is equivalent to (3 . 4) ; but $\chi_1(\lambda)$ need not be so restricted ; in particular some of the coefficients in the quantics may themselves be quantics in λ.

17. An example will show how the results may be applied to other integral equations for which the ordinary methods fail, the nucleus becoming infinite.*

Take the equation

$$\phi(x) - \theta \int_0^a \phi(t) \left\{\frac{1}{x+t} + \kappa(x, t)\right\} dt = f(x) \qquad (17 . 1)$$

where $\kappa(x, t)$ is bounded and continuous for values of x, t between 0 and a inclusive.

* When the nucleus has the singularity of $A(x-t)^{-1}$ the method of Kellogg (*Math. Annalen*, Vol. 58) may be used.

Put y for x, multiply both sides by

$$\frac{(y/x)^{\alpha-1}-(y/x)^{-\alpha}}{x^2-y^2}\,x\,dy,$$

and integrate with respect to y from 0 to a, giving α such a value that

$$\sin\alpha\pi=\pi\theta,$$

and that the real part of α is between 0 and 1, so that the real part of θ must be positive.*

If $\kappa_1(x,t)$ is written for

$$\int_a^\infty \frac{(y/x)^{\alpha-1}-(y/x)^{-\alpha}}{(y+t)(x^2-y^2)}\,x\,dy,$$

we know as in § 12 that

$$\int_0^a \frac{(y/x)^{\alpha-1}-(y/x)^{-\alpha}}{(y+t)(x^2-y^2)}\,x\,dy+\kappa_1(x,t)$$
$$=x\pi\operatorname{cosec}\alpha\pi\,\frac{(t/x)^{\alpha-1}-(t/x)^{-\alpha}}{x^2-t^2}+\frac{\pi\cot\alpha\pi}{x+t}. \tag{17.2}$$

Thus, with a change in the order of integration, we have

$$\int_0^a\phi(y)\,\frac{(y/x)^{\alpha-1}-(y/x)^{-\alpha}}{x^2-y^2}\,x\,dy$$
$$-\theta\int_0^a\phi(t)\left[x\pi\operatorname{cosec}\alpha\pi\,\frac{(t/x)^{\alpha-1}-(t/x)^{-\alpha}}{x^2-t^2}+\frac{\pi\cot\alpha\pi}{x+t}-\kappa_1(x,t)+\kappa_2(x,t)\right]dt$$
$$=\int_0^a f(y)\,\frac{(y/x)^{\alpha-1}-(y/x)^{-\alpha}}{x^2-y^2}\,x\,dy. \tag{17.3}$$

where $$\kappa_2(x,t)=\int_0^a\kappa(y,t)\,\frac{(y/x)^{\alpha-1}-(y/x)^{-\alpha}}{x^2-y^2}\,x\,dy.$$

Multiply (17.3) by $\tan\alpha\pi/\pi$ and subtract from (17.1), noting that

* If the real part of θ is negative, the real part of α is between 0 and -1 and the two indices are $\alpha+1$ and $-\alpha$ as in § 12.

$\sin \alpha\pi = \pi\theta$; thus

$$\phi(x) - \theta \int_0^a \phi(t)[\kappa(x, t) + \{\kappa_1(x, t) - \kappa_2(x, t)\} \tan \alpha\pi/\pi]\, dt$$

$$= f(x) - \frac{\tan \alpha\pi}{\pi} \int_0^a f(y)\, \frac{(y/x)^{\alpha-1} - (y/x)^{-\alpha}}{x^2 - y^2}\, x\, dy. \tag{17.4}$$

(17.4) is an integral equation to which the method of Fredholm may be applied, since κ_1, κ_2 are bounded and continuous* over the range from 0 to a.

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* For if we put $y = 1/u$ in $\kappa_1(x, t)$ we have it in the form

$$-\int_0^{1/a} \frac{(xu)^{2-\alpha} - (xu)^{\alpha+1}}{(1+tu)(1-x^2u^2)}\, du,$$

in which the subject is a continuous function of xu and tu for the values concerned, so that the continuity of the integral follows; also

$$\kappa_2(x, t) = \int_0^{a/x} \kappa(xu, t)\, \frac{u^{\alpha-1} - u^{-\alpha}}{1-u^2}\, du,$$

so that, if $x > x'$,

$$\kappa_2(x', t') - \kappa_2(x, t) = \int_0^{a/x} \{\kappa(x'u, t') - \kappa(xu, t)\}\, \frac{u^{\alpha-1} - u^{-\alpha}}{1-u^2}\, du + \int_{a/x}^{a/x'} \kappa(x'u, t')\, \frac{u^{\alpha-1} - u^{-\alpha}}{1-u^2}\, du.$$

Since κ has been supposed bounded and continuous, it follows, except when x approaches 0, that $\kappa_2(x, t)$ is continuous. When $x \to 0$, we have

$$\kappa_2(x, t) - \pi\kappa(0, t') \cot \alpha\pi$$

$$= \pi\{\kappa(0, t) - \kappa(0, t')\} \cot \alpha\pi - \kappa(0, t) \int_{a/x}^{\infty} \frac{u^{\alpha-1} - u^{-\alpha}}{1-u^2}\, du$$

$$+ \int_0^{(a/x)^{\frac{1}{2}}} \{\kappa(xu, t) - \kappa(0, t)\}\, \frac{u^{\alpha-1} - u^{-\alpha}}{1-u^2}\, du + \int_{(a/x)^{\frac{1}{2}}}^{a/x} \{\kappa(xu, t) - \kappa(0, t)\}\, \frac{u^{\alpha-1} - u^{-\alpha}}{1-u^2}\, du.$$

Each part of this expression tends to zero when $x \to 0$, $t \to t'$, and therefore $\kappa_2(x, t)$ is a continuous function of x and t even when x approaches 0, and

$$\kappa_2(0, t) = \pi\kappa(0, t) \cot \alpha\pi.$$

PLANE HOMALOIDAL FAMILIES OF GENERAL DEGREE

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1. *History and Summary.*

A homaloidal family of curves is one that can be transformed into the straight lines of another plane by a Cremona transformation. In the memoir that in 1865 laid the foundation of the plane theory, Cremona gave 49 types of homaloidal families, of which 44 are essentially distinct, for general values of the degree n of the form $x\mu+\nu$, $\nu = 0, 1, \ldots, \mu-1$, with $x = 1, 2, 3,$ and 4. We shall quote these as C. 1, ..., C. 49. Ruffini's formulæ (R. 4, ..., R. 7) contain as particular cases a few of these and five more for $x = 2$, of which Larice rediscovered two, and his general formula (R. m) gives one more for each of the forms of n with $x = 3, 4$.

By compounding two de Jonquières transformations, with all possible varieties of coincidence of the fundamental points in the intermediate space, Bianchi obtained a very general solution (B.), the form of n involving 3 parameters; the simplest case with no coincidences was given later by de Jonquières. These do not produce any new types with $x \leqslant 4$. Palatini repeated Bianchi's work and extended it. In his formula (P.) for the composition of three de Jonquières transformations, with arbitrary coincidences of the fundamental points, n involves 6 parameters. From this there can be deduced all the remaining types with $x = 3$, and many with $x = 4$. All these general formulæ will be found in § 7 below. Palatini also gives a heavy formula, with 16 parameters, for the composi-

tion of four de Jonquières transformations, which has not been analysed here.

In most of these researches the new general types are obtained by compounding simpler known types. By this method, there is no guarantee of completeness, and none of the writings quoted contain all the types of the kind considered. On the other hand, there is a guarantee of the existence of such types as are found, as geometrical transformations and not merely as arithmetical solutions of the equations of condition.

In this paper the equations are attacked from the arithmetical point of view, so as to ensure completeness; it is shown that all the types for $x = 1$, 2, and 3 are given or hidden in one or other of the memoirs. They are collected and classified in §§ 3, 4, and 5. We make use of several general conditions that the families do not break up in specified ways: the final proof of existence is to transform each family either into one of a general type already proved to exist, or into one for which $n \leqslant 15$, of which more or less accurate tables have been published in scattered places. A complete table for $n \leqslant 11$ is given by Montesano (*loc. cit. Atti*, p. 34).

This method does not lead to the pairing of conjugate solutions; but by giving this up we are able to tabulate the types, at the end of § 5, in an order much more convenient for use.

For $x = 4$, the number of types is much larger; in § 6 they are classified, and a specimen of each set is given. The same methods could be applied to any numerical value of x. In § 7 there are given the types which exist for a general value of x.

There prove to be 2 quite general solutions ($x = 1$); 6 for n even and 5 for n odd ($x = 2$); 18 for n a multiple of three, and 16 in each of the contrary cases ($x = 3$). The inequality of these numbers is remarkable.

2. *General Conditions.*

A plane homaloidal family of degree n is completely determined by its F-system, that is, the set of fundamental or F-points common to all the curves of the family together with their multiplicities. Let the F-system consist of a_i points of multiplicity i ($i = 1, 2, \ldots, n-1$), and let it be written Σa_i^i. Cremona showed that the positive integers i, a_i satisfy the two equations:

$$\Sigma i a_i = 3n-3 \quad \text{(I)}, \qquad \Sigma i^2 a_i = n^2-1 \quad \text{(II)}.$$

We have also certain inequalities expressing that the family is not wholly degenerate, so that no fixed curve of degree y, determined by the

F-points, meets one of the family in more than yn points. Thus

(straight lines) $$i_1+i_2 \leqslant n,$$

(conics) $$i_1+i_2+\ldots+i_5 \leqslant 2n,$$

(nodal cubics) $$2i_1+i_2+\ldots+i_7 \leqslant 3n,$$

(quartics with triple point) $$3i_1+i_2+\ldots+i_9 \leqslant 4n,$$

(quartics with 3 double points) $$2(i_1+i_2+i_3)+i_4+\ldots+i_8 \leqslant 4n,$$

where i_1, i_2, ... are the highest multiplicities.

More complicated tests are needed when the degrees of both components depend on n.

A set of numbers i, α_i, which satisfy I and II but violate any of these conditions, or for which any α_i is negative, form an arithmetical but not a geometrical solution; that is, a solution of the equations of condition to which there does not correspond a homaloidal family of proper curves.

The following known properties will also be used:

$$\left.\begin{array}{lll} (a) & i_1 & \leqslant n-1. \\ (b) & i_1+i_2+i_3 & > n \\ (c) & i_1 & > n/3 \end{array}\right\} \text{(Noether).}$$

$$(d) \quad \Sigma\alpha_i \leqslant 2n-1 \quad \text{(Montesano).}$$

All those solutions which differ from each other only in the part independent of n, that is, which have values of α_i differing by definite integers or zero when i is independent of n, but equal values of α_i when i involves n, can be quickly obtained from each other by the following device.

If y is any given positive integer, then on any solution of (I), (II) we can superpose any positive or negative multiple of

$$E_y \equiv 1^y\{-\tfrac{1}{2}y(y-1)\}^2\{y(y-2)\}^1,$$

and obtain another arithmetical solution, which is also a geometrical solution provided that, in the result, every $\alpha_i \geqslant 0$ and that the family does not break up. Hence, if at first we admit negative values of α_1 and α_2 we can put $\alpha_3 = \alpha_4 = \ldots 0$, in the part independent of n, in all the work until the last stage of all, when from this arithmetical solution or *base* we can deduce the corresponding set of geometrical solutions, by systematically

superposing proper multiples of

$$E_3 \equiv 1^3(-3)^2 3^1, \qquad E_4 \equiv 1^4(-6)^2 8^1,$$
$$E_5 \equiv 1^5(-10)^2 15^1, \qquad E_6 \equiv 1^6(-15)^2 24^1, \quad \&c.$$

3. $x = 1$.

In any solution for a quite general value of n, both i and a_i are polynomials in n. Since the solution is to hold for n large, the properties (a) and (d) of § 2 show that these polynomials are of degrees 0 or 1; and further, (a) shows that i has one of the forms $n-\sigma$ or σ, where σ is a positive integer, and (d) shows that a_i has one of the forms $2n-\sigma$, $n \pm \sigma$, σ. In every case where n occurs, its coefficient is positive.

Since (I) and (II) are satisfied for all values of n, we equate coefficients of its powers. From the terms in n^2 in (I), or in n^3 in (II), we see that i and a_i cannot both contain n for the same value of i. We therefore assume an F-system denoted, with a slight change of notation, by

$$\sum_{\sigma_a=1}^{\sigma_0} \alpha^{n-\sigma_a} \sum_{i=1}^{i_0} (\epsilon_i n + \eta_i)^i,$$

separating the groups of points for which i involves n from the groups in which it does not; in the latter only, a_i may involve n, if $\epsilon_i > 0$.

The upper limits σ_0, i_0 are positive integers to be determined; α, σ_a, i, ϵ_i are positive integers or zero, and $\epsilon_i \leqslant 2$; the η_i are integers which may be negative; if $\epsilon = 0$, then $\eta_i \geqslant 0$; if $\epsilon_i = 1$, then $\eta_i \gtreqless 0$; and if $\epsilon_i = 2$, then $\eta_i < 0$.

Now equate coefficients of powers of n in (I), (II), in the order most convenient for future use.

$$(\text{II}\, n^2) \qquad \Sigma\alpha = 1, \tag{1}$$

$$(\text{I}\, n) \qquad \Sigma\alpha + \Sigma i\,\epsilon_i = 3, \tag{2}$$

$$(\text{II}\, n) \qquad -2\Sigma\alpha\sigma_a + \Sigma i^2\epsilon_i = 0, \tag{3}$$

$$(\text{I}\, 1) \qquad -\Sigma\alpha\sigma_a + \Sigma i\,\eta_i = -3, \tag{4}$$

$$(\text{II}\, 1) \qquad \Sigma\alpha\sigma_a^2 + \Sigma i^2\eta_i = -1. \tag{5}$$

From (1), the first sum has only one term, with $\alpha = 1$, arising from a single F-point; from (2), the second has only one term involving n, that is, with $\epsilon_i > 0$, which arises from two or one F-points, giving the two cases:

(1) $\epsilon_1 = 2$, all other $\epsilon_i = 0$; (2) $\epsilon_2 = 1$, all other $\epsilon_i = 0$.

In either case we have the condition

(straight lines) $\quad n-\sigma_\alpha+i \leqslant n$, that is, $i \leqslant \sigma_\alpha$.

Case 1.—Since $\quad \epsilon_1 = 2$, hence $\Sigma i^2 \epsilon_i = 2$;

from (3) $\quad \sigma_\alpha = 1$, whence $\quad i = 1$.

Thus there is only one term in the second sum. Then (4), (5) are both satisfied by $\eta_1 = -2$; all the constants are now determined, and we have the well-known de Jonquières solution, which is Cremona's first general type:

C. 1 $\quad 1^{n-1}(2n-2)^1$.

Case 2.—Here $\quad \epsilon_2 = 1, \quad \Sigma i^2 \epsilon_i = 4$.

From (3) $\quad \sigma_\alpha = 2$, whence $i \leqslant 2$,

and there are just two terms in the second sum. Now (4), (5) give

$$\eta_1 = 3, \quad \eta_2 = -2,$$

and we have the second solution

C. 2 $\quad 1^{n-2}(n-2)^2 3^1$.

These two, both given by Cremona, are therefore the only types possible for a quite general value of n.

In both cases a quadratic transformation *applied to* the three highest F-points of the family, that is, with its three F-points coinciding with them, reduces it to one of the same type and of lower degree. A series of such quadratic transformations reduces either family to straight lines or conics; since these do not break up, neither do the two general families.

4. $x = 2$.

To find the solutions depending on whether n is odd or even, we write $n = 2\mu+\nu$, $\nu = 0$ or 1, and assume i, a_i to be polynomials in μ, instead of in n, with coefficients depending on ν. As before, by (a), (d)

$$i \leqslant 2\mu+\nu-1, \quad a_i \leqslant 4\mu+2\nu-1,$$

and i, a_i do not involve μ both at once. The F-system has the form

$$\Sigma \alpha^{n-\sigma_\alpha} \Sigma \beta^{\mu-\sigma_\beta} \Sigma(\epsilon_i \mu+\eta_i)^i \quad (\epsilon_i = 0, 1, 2, 3, \text{or } 4).$$

Here σ_β may be negative, but not σ_α.

Equate coefficients of powers of μ in (I), (II).

$$4\Sigma\alpha + \Sigma\beta = 4, \qquad (1)_2$$

$$2\Sigma\alpha + \Sigma\beta + \Sigma i\,\epsilon_i = 6, \qquad (2)_2$$

$$-4\Sigma\alpha(\sigma_\alpha-\nu)-2\Sigma\beta\sigma_\beta+\Sigma i^2\epsilon_i = 4\nu, \qquad (3)_2$$

$$-\Sigma\alpha(\sigma_\alpha-\nu)-\Sigma\beta\sigma_\beta+\Sigma i\,\eta_i = 3\nu-3, \qquad (4)_2$$

$$\Sigma\alpha(\sigma_\alpha-\nu)^2+\Sigma\beta\sigma_\beta^2+\Sigma i^2\eta_i = \nu^2-1. \qquad (5)_2$$

Since α, β are positive integers, $(1)_2$ gives two alternatives:

1. The first sum has one term, with $\alpha = 1$, the second sum is absent;
2. The first sum is absent, the second has not more than four terms, arising from exactly four F-points.

Case 1.— $\alpha = 1, \quad \beta = 0.$

From $(2)_2$, $(3)_2$ $\Sigma i\epsilon_i = 4, \quad \Sigma i^2\epsilon_i = 4\sigma_\alpha.$

The first of these gives five possibilities:

$$\epsilon_1 = 4; \quad \epsilon_1 = 2,\ \epsilon_2 = 1; \quad \epsilon_2 = 2; \quad \epsilon_1 = 1,\ \epsilon_3 = 1; \quad \epsilon_4 = 1.$$

In the first and third of these, the only ϵ_i is even; in the whole F-system, μ only appears in the form 2μ or 4μ, and we are shut up to types already met in § 3. In the second and fourth, $\Sigma i^2\epsilon_i$ is not a multiple of 4, so these are excluded by $(3)_2$. We are left with the last case:

$$\epsilon_4 = 1, \quad \sigma_\alpha = 4,$$

with the further condition

(straight lines) $n-4+i \leqslant n$, that is, $i \leqslant 4$.

It remains to satisfy $(4)_2$ and $(5)_2$, which become

$$\eta_1+2\eta_2+3\eta_3+4\eta_4 = 2\nu+1,$$

$$\eta_1+4\eta_2+9\eta_3+16\eta_4 = 8\nu-17.$$

We first assume $\eta_3 = \eta_4 = 0$, and obtain the base

$$\eta_1 = 19-4\nu, \quad \eta_2 = -9+3\nu.$$

Considering the last set of F-points, we see that only η_4 can be negative. We therefore superpose negative multiples of E_4, and positive multiples of E_3, for each value of ν, in every possible way that leaves $\eta_1, \eta_2 \geqslant 0$.

We can, however, exclude many cases and limit the work by another consideration. Write down the condition that the family does not break up, one component being of degree y, determined by a $(y-1)$-fold point at the $(n-4)$-fold F-point, and $2y$ simple points at the next highest F-points, that is, by $1^{y-1}(2y)^1$. Take

$$2y = \mu+\eta_4+\eta_3+\eta_2-\rho, \qquad \rho \leqslant \eta_2,$$

where ρ is chosen to make this expression even. If our type exists for both odd and even values of μ, we can take $\rho = 0$.

We have the condition

$$yn \geqslant (y-1)(n-4)+4(\mu+\eta_4)+3\eta_3+2(\eta_2-\rho),$$

whence

$$2\eta_4+\eta_3 \leqslant \nu-4.$$

This excludes all but four cases, C. 22, 18, 28, 26, valid for all values of $\mu \geqslant -\eta_4$; they are shown in the table. In certain other cases where $\eta_2 = 0$, and therefore $\rho = 0$, the value assumed for y is only possible if μ is odd (or even, as the case may be), and must be replaced in the opposite case by

$$2y = \mu+\eta_4+\eta_3-\rho, \qquad 0 < \rho \leqslant \eta_3,$$

$$yn \geqslant (y-1)(n-4)+4(\mu-\eta_4)+3(\eta_3-\rho),$$

whence

$$2\eta_4+\eta_3 \leqslant \nu-4+\rho.$$

This leads to some types valid if n is of the form $4\mu+\nu$ instead of $2\mu+\nu$; we return to these in § 6.

For each existing type, we verify that it does not break up in some other way by transforming it into a known family of lower degree, by quadratic transformations applied to the three highest F-points.

Case 2.—Let $\sigma_{\beta_1}, \ldots, \sigma_{\beta_4}$, in descending order, refer to the four points of the second sum, where these σ's can be equal to each other. We have the conditions

(straight lines) $\mu-\sigma_{\beta_1}+\mu-\sigma_{\beta_2} \leqslant 2\mu+\nu$, that is, $\sigma_{\beta_1}+\sigma_{\beta_2} \geqslant -\nu$;

(conics) $4\mu-\Sigma\sigma_{\beta_1}+i \leqslant 4\mu+2\nu$, that is, $i \leqslant 2\nu+\Sigma\sigma_{\beta_1}$.

From $(2)_2$

$$\Sigma i\epsilon_i = 2,$$

giving two sub-cases, as for n general,

$$\text{(i)} \quad \epsilon_1 = 2, \qquad \text{(ii)} \quad \epsilon_2 = 1.$$

Case 2 i.— $\alpha = 0, \quad \Sigma\beta = 4, \quad \epsilon_1 = 2.$

From $(3)_2$ $\sigma_{\beta_1}+\sigma_{\beta_2}+\sigma_{\beta_3}+\sigma_{\beta_4} = 1-2\nu$, whence $i \leqslant 1$.

But $\sigma_{\beta_1}+\sigma_{\beta_2} \geqslant -\nu$,

whence $\sigma_{\beta_3}+\sigma_{\beta_4} \leqslant 1-\nu$,

and similarly for any other pair. Sum the three inequalities of each type that involve σ_{β_1}, and use $(3)_2$:

$$-1-\nu \leqslant 2\sigma_{\beta_1} \leqslant 2-\nu,$$

and similarly for σ_{β_2}, σ_{β_3}, σ_{β_4}. The only solution of $(3)_2$ compatible with these is

$$\sigma_{\beta_1} = -\nu, \quad \sigma_{\beta_2} = \sigma_{\beta_3} = 0, \quad \sigma_{\beta_4} = 1-\nu.$$

Now, since $i \leqslant 1$, the last sum has only one term; from $(4)_2$, $(5)_2$

$$\eta_1 = \nu-2, \quad \eta_1 = -\nu^2+2\nu-2,$$

which are compatible since $\nu = 0$ or 1. We have the pair of solutions C. 3, 5.

Case 2 ii.— $\alpha = 0, \quad \Sigma\beta = 4, \quad \epsilon_2 = 1.$

A similar argument leads to $\Sigma i^2\epsilon_i = 4$, $\Sigma\sigma_{\beta_1} = 2-2\nu$, whence $i \leqslant 2$,

$$-\nu \leqslant \sigma_{\beta_1}+\sigma_{\beta_2} \leqslant 2-\nu,$$

$$-2-\nu \leqslant 2\sigma_{\beta_1} \leqslant 4-\nu.$$

Here are three possibilities:

$$\sigma_{\beta_1} = -1, \quad \sigma_{\beta_2} = \sigma_{\beta_3} = 1-\nu, \quad \sigma_{\beta_4} = 1;$$

or $$\sigma_{\beta_1} = \sigma_{\beta_2} = 0, \quad \sigma_{\beta_3} = \sigma_{\beta_4} = 1-\nu;$$

or $$\sigma_{\beta_1} = \sigma_{\beta_2} = \sigma_{\beta_3} = 0, \quad \sigma_{\beta_4} = 2-2\nu.$$

Each gives two types with $\nu = 0$ or 1; the last two are the same if $\nu = 1$, but different if $\nu = 0$. It is this circumstance that causes the numbers of solutions for n odd and n even to be unequal.

Now $i \leqslant 2$, and η_1, η_2 are given by $(4)_2$, $(5)_2$; we have Ruffini's five solutions, three for n even and two for n odd.

This exhausts the possibilities with n of the form $2\mu+\nu$.

5. $x = 3$.

Next let $n = 3\mu+\nu$, $\nu = 0$, 1, or 2. The F-system is

$$\Sigma\alpha^{n-\sigma_\alpha}\Sigma\beta^{2\mu-\sigma_\beta}\Sigma\gamma^{\mu-\sigma_\gamma}\Sigma(\epsilon_i\mu+\eta_i)^i,$$

and the constants satisfy

$$9\Sigma\alpha \quad +4\Sigma\beta \ + \ \Sigma\gamma \qquad = 9, \qquad (1)_3$$

$$3\Sigma\alpha \quad +2\Sigma\beta \ + \ \Sigma\gamma \ +\Sigma i\epsilon_i = 9, \qquad (2)_3$$

$$-6\Sigma\alpha(\sigma_\alpha-\nu) -4\Sigma\beta\sigma_\beta-2\Sigma\gamma\sigma_\gamma+\Sigma i^2\epsilon_i = 6\nu, \qquad (3)_3$$

$$-\Sigma\alpha(\sigma_\alpha-\nu) - \ \Sigma\beta\sigma_\beta- \ \Sigma\gamma\sigma_\gamma+\Sigma i\eta_i = 3\nu-3, \qquad (4)_3$$

$$\Sigma\alpha(\sigma_\alpha-\nu)^2+ \ \Sigma\beta\sigma_\beta^2+ \ \Sigma\gamma\sigma_\gamma^2+\Sigma i^2\eta_i = \nu^2-1. \qquad (5)_3$$

From $(1)_3$ there are four possibilities:

$$\alpha = 1, \ \beta = \gamma = 0; \quad \alpha = 0, \ \Sigma\beta = 2, \ \gamma = 1;$$

$$\alpha = 0, \ \beta = 1, \ \Sigma\gamma = 5; \quad \alpha = \beta = 0, \ \Sigma\gamma = 9.$$

The second is excluded (straight lines), and we have three cases.

Case 1.—From $(2)_3$, $(3)_3$ $\Sigma i\epsilon_i = 6$, $\Sigma i^2\epsilon_i = 6\sigma_\alpha$,

(straight lines) $\qquad n-\sigma_\alpha+i \leqslant n$, that is, $i \leqslant \sigma_\alpha$.

Now every $\epsilon_i \geqslant 0$; if we relax this condition, and then put $\epsilon_3 = \epsilon_4 = \ldots = 0$, we have the base for the ϵ_i,

$$\epsilon_1 = 12-6\sigma_\alpha, \quad \epsilon_2 = -3+3\sigma_\alpha.$$

It follows that the geometrical possibilities are

$$\sigma_\alpha = 1, \ \epsilon_1 = 6; \quad \sigma_\alpha = 2, \ \epsilon_2 = 3; \quad \sigma_\alpha = 3, \ \epsilon_3 = 2; \quad \sigma_\alpha = 6, \ \epsilon_6 = 1.$$

In the first two of these, μ only appears in the form 3μ, and we have types already met in § 3. There remain two sub-cases.

Case 1i.— $\alpha = 1$, $\beta = \gamma = 0$, $\sigma_\alpha = 3$, $\epsilon_3 = 2$, $i \leqslant 3$.

Here η_3 can be negative, but not η_1, η_2. From $(4)_2$, $(5)_2$ we have the base

$$\eta_1 = 10-2\nu, \quad \eta_2 = -5+2\nu, \quad \eta_3 = 0,$$

whence six types, C. 8, 6, 12, 10, 14, 16, valid for $\mu \geqslant -\eta_3$.

Case 1ii.— $\alpha = 1, \quad \beta = \gamma = 0, \quad \sigma_\alpha = 6, \quad \epsilon_6 = 1, \quad i \leqslant 6.$

Only η_6 can be negative. The base is

$$\eta_1 = 43-8\nu, \quad \eta_2 = -20+5\nu, \quad \eta_3 = \eta_4 = \eta_5 = \eta_6 = 0.$$

We can exclude many arithmetical solutions by the test of § 4, Case 1. That argument leads to

$$3\eta_6+2\eta_5+\eta_4 \leqslant -6+\nu.$$

A quadratic transformation applied to the three highest F-points reduces each of these types to a similar one with μ lowered by 2. To prove the proper existence of each we have therefore to check it for $\mu = -\eta_3$ and $-\eta_3+1$. Three are found to fail, breaking up in ways other than those specified so far. The remaining geometrical solutions are tabulated; again the number depends on the form of n, being 8 for $\nu = 0$ and 6 for $\nu = 1$ or 2.

A general formula for this set can be obtained from (P.) by putting

$$N = 6, \quad n_1 = 3, \quad 2n_3-l' = \mu;$$

the remaining parameters l'', l''', n_2 taking all sets of values that make all the $a_i > 0$.

Case 2.—Here we have the conditions:

(straight lines) $\quad 2\mu-\sigma_\beta+\ \mu-\sigma_{\gamma_1} \leqslant 3\mu+\nu,$

(conics) $\quad 2\mu-\sigma_\beta+4\mu-(\sigma_{\gamma_2}+\sigma_{\gamma_3}+\sigma_{\gamma_4}+\sigma_{\gamma_5}) \leqslant 6\mu+2\nu,$

(nodal cubics) $\quad 2(2\mu-\sigma_\beta)+5\mu-\Sigma\sigma_{\gamma_1}+i \leqslant 9\mu+3\nu,$

that is, $\quad -\sigma_\beta-\nu \leqslant \sigma_{\gamma_1} \leqslant \sigma_\beta+2\nu+\Sigma\sigma_{\gamma_1},$

$$i \leqslant 2\sigma_\beta+3\nu+\Sigma\sigma_{\gamma_1}.$$

Now, from $(2)_3$ we have $\Sigma i\epsilon_i = 2$, giving the usual two sub-cases.

Case 2i.— $\quad \alpha = 0, \quad \beta = 1, \quad \Sigma\gamma = 5, \quad \epsilon_1 = 2, \quad \Sigma i^2\epsilon_i = 2.$

From $(3)_3$ $\quad \Sigma\sigma_{\gamma_1} = 1-2\sigma_\beta-3\nu,$ whence $i \leqslant 1.$

Now $\quad -\sigma_\beta-\nu \leqslant \sigma_{\gamma_1} \leqslant 1-\sigma_\beta-\nu,$

hence each of the five numbers σ_{γ_1} is either $-\sigma_\beta-\nu$ or $1-\sigma_\beta-\nu$; from $(3)_3$ we find that $4-3\sigma_\beta-2\nu$ have the former value and $1+3\sigma_\beta+2\nu$ the latter.

Sum the inequalities for $\sigma_{\gamma_1}, \sigma_{\gamma_2}, \ldots$, substitute for $\Sigma\sigma_{\gamma_1}$ and rearrange:

$$-1-2\nu \leqslant 3\sigma_\beta \leqslant 4-2\nu.$$

For each value of ν there are two possible values of σ_β, and for each of these the σ_γ are determined. From $(4)_3$ we have $\eta_1 = -2-\sigma_\beta$, and then $(5)_3$ is also satisfied. This leads to six more types, C. 7, 9, 13, 11, 15, 17.

Case 2 ii.— $\alpha = 0, \quad \beta = 1, \quad \Sigma\gamma = 5, \quad \epsilon_2 = 1, \quad \Sigma i^2\epsilon_i = 4.$

The same argument leads to

$$\Sigma\sigma_{\gamma_1} = 2-2\sigma_\beta-3\nu, \quad i \leqslant 2,$$

$$-\sigma_\beta-\nu \leqslant \sigma_{\gamma_1} \leqslant 2-\sigma_\beta-\nu,$$

$$-2-2\nu \leqslant 3\sigma_\beta \leqslant 8-2\nu.$$

We find twenty-one sets of values for σ_γ; then η_1, η_2 are determined by $(4)_3$, $(5)_3$, and three cases must be rejected because $\eta_1 < 0$. Of the eighteen geometrical solutions, three are particular cases of (R.m). A general formula for the set arises from (P.) with $n_1 = 2$, $n_3 = 3$.

A quadratic transformation, applied to the $(2\mu-\sigma_\beta)$-fold point and any two points of the next set, reduces the degree of the family to

$$2(3\mu+\nu)-(2\mu-\sigma_\beta)-(2\mu-\sigma_{\gamma_1}-\sigma_{\gamma_2}) = 2\mu+2\nu+\sigma_\beta+\sigma_{\gamma_1}+\sigma_{\gamma_2},$$

which therefore belongs to § 4. Conversely, we can get all the types of the present Case 2 i and 2 ii by compounding those of § 4 with a quadratic transformation, letting its F-points coincide in any way with the simple and double F-points of the earlier general type. But except when there is only one value of σ_γ, the later type arises several times over.

Case 3.— $\alpha = \beta = 0, \quad \Sigma\gamma = 9.$

From $(2)_3$, $(3)_3$, $(4)_3$, $\Sigma i\epsilon_i = 0$, $\Sigma\gamma\sigma_\gamma = -3\nu$, $\Sigma i\eta_i = -3$.

The first of these requires that every $\epsilon_i = 0$, and therefore every $\eta_i \geqslant 0$, contradicting the last; there are no solutions in this case, and we have exhausted the possibilities with n of the form $3\mu+\nu$.

6. $x = 4$.

The case $x = 4$ is treated on the same lines. We assume $n = 4\mu+\nu$, $\nu = 0, 1, 2$, or 3; the F-system is

$$\Sigma\alpha^{\mu-\sigma_\alpha}\,\Sigma\beta^{3\mu-\sigma_\beta}\,\Sigma\gamma^{2\mu-\sigma_\gamma}\,\Sigma\delta^{\mu-\sigma_\delta}\,\Sigma(\epsilon_i\mu+\eta_i)^i,$$

§ 3.	n general			
Case 1.	C. 1	1^{n-1}		$(2n-2)^1$
Case 2.	C. 2	1^{n-2}	$(n-2)^2$	3^1

§ 4.	$n = 2\mu$						$n = 2\mu+1$						
			α_4	α_3	α_2	α_1			α_4	α_3	α_2	α_1	
Case 1.	C. 22 ≡ 38	1^{n-4}	$\mu-2$	0	3	3	C. 28 ≡ 46	1^{n-4}	$\mu-2$	1	3	2	
	C. 18 ≡ 36	1^{n-4}	$\mu-3$	2	3	1	C. 26 ≡ 42	1^{n-4}	$\mu-3$	3	3	0	
Case 2 i.	C. 3	$3^{\mu}\,1^{\mu-1}$				$2\mu-2$	C. 5	$1^{\mu+1}\,3^{\mu}$				$2\mu-1$	
Case 2 ii.	R. 6	$1^{\mu+1}\,3^{\mu-1}$			$\mu-2$	3	R. 7	$1^{\mu+1}\,2^{\mu}\,1^{\mu-1}$			$\mu-1$	2	
	R. 4	$2^{\mu}\,2^{\mu-1}$			$\mu-1$	1	R. 4	4^{μ}			μ	0	
	R. 7	$3^{\mu}\;\;1^{\mu-2}$			$\mu-2$	3							

§ 5.	$n = 3\mu$								$n = 3\mu+1$								$n = 3\mu+2$							
			α_6	α_5	α_4	α_3	α_2	α_1			α_6	α_5	α_4	α_3	α_2	α_1			α_6	α_5	α_4	α_3	α_2	α_1
Case 1 i.	C. 8	1^{n-3}				$2\mu-2$	1	4	C. 12	1^{n-3}				$2\mu-1$	0	5	C. 14	1^{n-3}				$2\mu-1$	2	3
	C. 6	1^{n-3}				$2\mu-3$	4	1	C. 10	1^{n-3}				$2\mu-2$	3	2	C. 16	1^{n-3}				$2\mu-2$	5	0
Case 1 ii.	P.	1^{n-6}	$\mu-2$	0	0	3	1	4	P.	1^{n-6}	$\mu-2$	0	1	3	0	4	P.	1^{n-6}	$\mu-2$	1	0	3	1	3
		1^{n-6}	$\mu-2$	0	0	2	4	1		1^{n-6}	$\mu-2$	0	1	2	3	1		1^{n-6}	$\mu-2$	1	0	2	4	0
		1^{n-6}	$\mu-3$	1	1	3	0	3		1^{n-6}	$\mu-3$	2	0	3	1	2		1^{n-6}	$\mu-2$	0	2	2	2	1
		1^{n-6}	$\mu-3$	1	1	2	3	0		1^{n-6}	$\mu-3$	1	2	2	2	0		1^{n-6}	$\mu-3$	2	1	3	0	2
		1^{n-6}	$\mu-3$	0	3	2	1	1		1^{n-6}	$\mu-3$	0	4	2	0	1		1^{n-6}	$\mu-3$	1	3	2	1	0
		1^{n-6}	$\mu-4$	3	0	3	1	1		1^{n-6}	$\mu-4$	3	1	3	0	1		1^{n-6}	$\mu-4$	4	0	3	1	0
		1^{n-6}	$\mu-4$	1	4	2	0	0																
		1^{n-6}	$\mu-5$	4	1	3	0	0																
Case 2 i.	C. 9	$1^{2\mu}$			4^{μ}	$1^{\mu-1}$		$2\mu-2$	C. 13	$1^{2\mu+1}$			5^{μ}			$2\mu-1$	C. 15	$1^{2\mu+1}$			$3^{\mu+1}$	2^{μ}		$2\mu-1$
	C. 7	$1^{2\mu-1}$		$1^{\mu+1}\,4^{\mu}$				$2\mu-3$	C. 11	$1^{2\mu}$			$2^{\mu+1}\,3^{\mu}$			$2\mu-2$	C. 17	$1^{2\mu}$			$5^{\mu+1}$			$2\mu-2$
Case 2 ii.	R. m	$1^{2\mu}$			4^{μ}		$1^{\mu-2}$	$\mu-2$	3	R. m	$1^{2\mu+1}$			4^{μ}	$1^{\mu-1}$		$\mu-1$	2	R. m	$1^{2\mu+2}$				5^{μ}		$\mu-1$	3
	P.	$1^{2\mu}$			3^{μ}	$2^{\mu-1}$		$\mu-1$	1	P.	$1^{2\mu}$			$3^{\mu+1}$	$2^{\mu-1}$		$\mu-2$	3	P.	$1^{2\mu+1}$			$3^{\mu+1}$	1^{μ}	$1^{\mu-1}$	$\mu-1$	2
		$1^{2\mu-1}$		$2^{\mu+1}\,1^{\mu}$		$2^{\mu-1}$		$\mu-2$	2		$1^{2\mu}$			$2^{\mu+1}\,2^{\mu}$	$1^{\mu-1}$		$\mu-1$	1		$1^{2\mu+1}$			$2^{\mu+1}$	3^{μ}		μ	0
		$1^{2\mu-1}$		$1^{\mu+1}\,3^{\mu}$		$1^{\mu-1}$		$\mu-1$	0		$1^{2\mu-1}$	$1^{\mu+2}\,1^{\mu+1}\,3^{\mu}$					$\mu-2$	2		$1^{2\mu}$	$2^{\mu+2}$			3^{μ}		$\mu-2$	3
		$1^{2\mu-2}$	$1^{\mu+2}$	4^{μ}				$\mu-3$	3		$1^{2\mu-1}$		$3^{\mu+1}\,2^{\mu}$				$\mu-1$	0		$1^{2\mu}$		$1^{\mu+2}\,2^{\mu+1}$		2^{μ}		$\mu-1$	1
		$1^{2\mu-2}$		$2^{\mu+1}\,3^{\mu}$				$\mu-2$	1		$1^{2\mu-2}$		$5^{\mu+1}$				$\mu-3$	3		$1^{2\mu-1}$		$1^{\mu+2}\,4^{\mu+1}$				$\mu-2$	3

and the constants satisfy

$$16\Sigma\alpha \quad +9\Sigma\beta \quad +4\Sigma\gamma \quad + \Sigma\delta \quad = 16, \qquad (1)_4$$

$$4\Sigma\alpha \quad +3\Sigma\beta \quad +2\Sigma\gamma \quad + \Sigma\delta \quad +\Sigma i\epsilon_i = 12, \qquad (2)_4$$

$$-8\Sigma\alpha(\sigma_\alpha-\nu) -6\Sigma\beta\sigma_\beta-4\Sigma\gamma\sigma_\gamma-2\Sigma\delta\sigma_\delta+\Sigma i^2\epsilon_i = 8\nu, \qquad (3)_4$$

$$-\Sigma\alpha(\sigma_\alpha-\nu) - \Sigma\beta\sigma_\beta- \Sigma\gamma\sigma_\gamma- \Sigma\delta\sigma_\delta+\Sigma i\eta_i = 3\nu-3, \qquad (4)_4$$

$$\Sigma\alpha(\sigma_\alpha-\nu)^2+ \Sigma\beta\sigma_\beta^2+ \Sigma\gamma\sigma_\gamma^2+ \Sigma\delta\sigma_\delta^2+\Sigma i^2\eta_i = \nu^2-1. \qquad (5)_4$$

From $(1)_4$, the possibilities are

$\alpha =$	1	0	0	0	0	0	0	0
$\beta =$	0	1	1	0	0	0	0	0
$\Sigma\gamma =$	0	1	0	4	3	2	1	0
$\Sigma\delta =$	0	3	7	0	4	8	12	16.

The second is excluded (straight lines), and the last by condition (*c*) of § 2. There remain six cases to examine.

Case 1.— $\alpha = 1, \quad \beta = \gamma = \delta = 0.$

From $(2)_4$, $(3)_4$ $\Sigma i\epsilon_i = 8, \quad \Sigma i^2\epsilon_i = 8\sigma_\alpha,$

(straight lines) $i \leqslant \sigma_\alpha.$

The base for the ϵ_i is $\epsilon_1 = 16-8\sigma_\alpha, \quad \epsilon_2 = -4+4\sigma_\alpha.$

We consider these for $\sigma_\alpha = 1, 2, \ldots,$ and deduce the admissible sets of values with each $\epsilon_i \geqslant 0$ and $i \leqslant \sigma_\alpha$. In every case except one, every ϵ is even, and the types have occurred already in §§ 3, 4.

Case 1 i.—In § 4, Case 1, we met types with $\sigma_\alpha = 4$, $\eta_4 = 2$ in the notation of this section, which arose there but really belong here. By the methods shown there, we find eight types, C. 24, 20, 32, 30, 40, 34, 48, 44.

A specimen of the set is

C. 24. $\qquad n = 4\mu, \quad 1^{n-4}(2\mu-2)^4 1^3 6^1,$

which is compounded of C. 1 and a quartic transformation with the *F*-system $1^3 6^1$. Since 2μ is even, this can be reduced, by a series of quadratic transformations, to the type with $\mu = 1$,

$$n = 4, \quad 1^3 6^1,$$

which is proper. But if 2μ is replaced by an odd number, $2\mu+1$, we obtain

$$n = 4\mu+2, \quad 1^{n-4}(2\mu-1)^4 1^3 6^1,$$

which can be reduced to the type with $\mu = 1$,

$$n = 6, \quad 1^4 1^3 1^2 6^1,$$

which breaks up, one component being the straight line joining the two highest F-points.

Case 1ii.—The only other sub-case to pursue is

$$\sigma_\alpha = 8, \quad \epsilon_1 = \epsilon_2 = \ldots = \epsilon_7 = 0, \quad \epsilon_8 = 1, \quad i \leqslant 8;$$

we can also obtain the condition

$$4\eta_8+3\eta_7+2\eta_6+\eta_5 \leqslant -8+\nu.$$

From $(4)_4$, $(5)_4$ the base for the η is

$$\eta_1 = 75-12\nu, \quad \eta_2 = -35+7\nu, \quad \eta_3 = \ldots = \eta_8 = 0,$$

whence a large number of geometrical solutions. All the types of this set involve μ in the same way, the F-systems beginning with $1^{n-8}\mu^8$, and differing only in the integers $\eta_8, \ldots, \eta_1$ that follow; η_8 is always negative. Each is reduced, by a quadratic transformation, to one of the same type with μ lower by 2, and the process repeated brings μ down to $-\eta_8$ or $-\eta_8+1$.

If in (P.) we put $N = 8$, $n_1 = 4$, $\mu = 2n_3-l'$, we obtain thirty-two different types of this set, in all of which, however, $\eta_2 = 0$. An example other than these, given at the end of this section, is compounded of C. 2 and a quartic transformation with $3^2 3^1$.

Case 2.—From $(2)_4$, $\Sigma i\epsilon_i = 2$, giving the usual two sub-cases.

Case 2 i.—

$$\alpha = \gamma = 0, \quad \beta = 1, \quad \Sigma\delta = 7, \quad \epsilon_1 = 2, \quad \sigma_\beta+\Sigma\delta\sigma_\delta = -4\nu+1.$$

By straight lines, nodal cubics, and quartics with a triple point, we arrive at the limits

$$-1-3\nu \leqslant 4\sigma_\beta \leqslant 6-3\nu,$$

$$-\sigma_\beta-\nu \leqslant \sigma_{\delta_1} \leqslant 1-\sigma_\beta-\nu.$$

For each of $\nu = 0, 1, 2, 3$, there are two possible values for σ_β, and for each of these, one set of values of $\sigma_{\delta_1}, \ldots, \sigma_{\delta_7}$. We find eight solutions,

C. 25, 21, 33, 31, 41, 35, 49, 45, the first being compounded of a quartic transformation with $1^3 6^1$ and C. 1.

Case 2 ii.—With $\epsilon_2 = 1$, a similar argument leads to a set with $i \leqslant 2$, which is given by (P.) when $n_1 = 2$, $n_3 = 4$; there are twenty-six types, seven for each of the even forms of n and six for each of the odd forms. The first is also contained in (R.m), and is compounded of a quartic transformation with $1^3 6^1$ and C. 2.

All the types of Case 2 i and 2 ii arise in many ways from § 4, Case 2, by compounding with a quadratic transformation.

Case 3.— $\alpha = \beta = \delta = 0, \quad \Sigma\gamma = 4, \quad \Sigma i\epsilon_i = 4, \quad \Sigma i^2\epsilon_i = 8\nu + 4\Sigma\gamma\sigma_\gamma.$

Again excluding the cases where every ϵ_i is even, we are left with $\epsilon_1 = 1$, $i \leqslant 4$. The specimen of this set is compounded of a quadratic transformation and C. 22. All these can be obtained in many ways from § 4, Case I. None of them is included in (P.).

Case 4.—Again $\Sigma i\epsilon_i = 2$, and we subdivide.

Case 4 i.—

$$\alpha = \beta = 0, \quad \Sigma\gamma = 3, \quad \Sigma\delta = 4, \quad \epsilon_1 = 2, \quad 2\Sigma\gamma\sigma_\gamma + \Sigma\delta\sigma_\delta = 1 - 4\nu.$$

Straight lines, nodal cubics, and conics give limits for σ_γ and σ_δ, and quartics with three double points give $i \leqslant 1$. We find eight types, C. 23, 19, 29, 27, 39, 37, 47, 43. Each of these is compounded of a quadratic transformation and one of § 4, Case 2 i, the simple F-points of the two coinciding in any way. The resolution is unique.

Case 4 ii.— $\alpha = \beta = 0, \quad \Sigma\gamma = 3, \quad \Sigma\delta = 4, \quad \epsilon_2 = 1.$

In the same way, each of these arises, once only, by compounding a quadratic transformation with one of § 4, Case 2 ii; there are forty such.

Case 5.— $\alpha = \beta = 0, \quad \Sigma\gamma = 2, \quad \Sigma\delta = 8.$

A quadratic transformation applied to the three highest F-points would reduce this to § 5, Case 3, which we have seen does not exist; neither does the present case.

Case 6.— $\alpha = \beta = 0, \quad \gamma = 1, \quad \Sigma\delta = 12.$

From $(2)_4$, $\Sigma i\epsilon_i = -2$, which is impossible, and Case 6 is also excluded. We have therefore discussed all the solutions with $n = 4\mu + \nu$. There are seven kinds, of which the following types are specimens, with $n = 4\mu$.

Case 1 i.	C. 24.	$1^{n-4}(2\mu-2)^4 1^3 6^1.$
1 ii.		$1^{n-8}(\mu-2)^8 3^4 3^2 3^1.$
2 i.	C. 25.	$1^{3\mu} 6^\mu 1^{\mu-1}(2\mu-2)^1.$
2 ii.	R. $m.$	$1^{3\mu} 6^\mu 1^{\mu-2}(\mu-2)^2 3^1.$
3.		$3^{2\mu} 1^{2\mu-4}(\mu-2)^4 3^2 3^1.$
4 i.	C. 23.	$3^{2\mu} 3^\mu 1^{\mu-1}(2\mu-2)^1.$
4 ii.		$3^{2\mu} 1^{\mu+1} 3^{\mu-1}(\mu-2)^2 3^1.$

7. *x general.*

Let $n = x\mu+\nu$, and let us search for solutions valid for general values of x, μ. We assume i, a_i to be multinomials in x, μ, with coefficients depending on ν. The conditions

$$i \leqslant n-1, \quad \Sigma a_i \leqslant 2n-1$$

show that the multinomials are linear in each of x, μ. The conditions

$$i_1 > n/3, \quad i_1+i_2 \leqslant n$$

show that there is one and only one point for which i has a term in $x\mu$, and the coefficient is 1. We assume provisionally an F-system of the form

$$1^{x\mu+ax+b\mu+c}\,\Sigma(ax\mu+\beta x+\gamma\mu+\delta)^{dx+e\mu+f}.$$

Here a, d, $e \geqslant 0$, and if $a = 0$, then β, $\gamma \geqslant 0$.

Equate coefficients of certain powers of x, μ in II:

$$(x^3\mu) \quad \Sigma a d^2 = 0,$$

$$(x\mu^3) \quad \Sigma a e^2 = 0,$$

$$(x^3) \quad \Sigma \beta d^2 = 0,$$

$$(\mu^3) \quad \Sigma \gamma e^2 = 0.$$

Hence for every term in which $a > 0$, we have $d = e = 0$.

,, ,, ,, $a = 0$, $\beta > 0$, ,, $d = 0$.

,, ,, ,, $a = 0$, $\gamma > 0$, ,, $e = 0$.

Slightly changing the notation, we have the F-system more closely represented by

$$1^{x\mu+ax+b\mu+c}\,\Sigma(\alpha\mu+\beta)^{dx+e}\,\Sigma(\gamma x+\delta)^{f\mu+g}\,\Sigma(\epsilon x\mu+\eta x+\xi\mu+\theta)^i.$$

Here $a, b \leqslant 0$; $d, f > 0$ (for if either $= 0$, the term would be absorbed in the last sum); $\alpha, \gamma, \epsilon \geqslant 0$; $\epsilon = 0$, 1, or 2; if $\epsilon = 0$, then $\eta, \xi \geqslant 0$.

Equate coefficients of the remaining powers of x, μ in (I), (II):

$$
\begin{array}{lllllll}
\text{(I)} & (x\mu) & 1 & +\Sigma\alpha d & +\Sigma\gamma f & +\Sigma\epsilon i & = 3, \\
 & (x) & a & +\Sigma\beta d & +\Sigma\gamma g & +\Sigma\eta i & = 0, \\
 & (\mu) & b & +\Sigma a e & +\Sigma\delta f & +\Sigma\xi i & = 0, \\
 & (1) & c & +\Sigma\beta e & +\Sigma\delta g & +\Sigma\theta i & = 3\nu-3. \\
\text{(II)} & (x^2\mu) & 2a & +\Sigma a d^2 & & & = 0, \\
 & (x\mu^2) & 2b & & +\Sigma\gamma f^2 & & = 0, \\
 & (x\mu) & 2c+2ab & +\Sigma 2ade & +\Sigma 2\gamma fg & +\Sigma\epsilon i^2 & = 2\nu, \\
 & (x^2) & a^2 & +\Sigma\beta d^2 & & & = 0, \\
 & (\mu^2) & b^2 & & +\Sigma\delta f^2 & & = 0, \\
 & (x) & 2ac & +\Sigma 2\beta de & +\Sigma\gamma g^2 & +\Sigma\eta i^2 & = 0, \\
 & (\mu) & 2bc & +\Sigma a e^2 & +\Sigma 2\delta fg & +\Sigma\xi i^2 & = 0, \\
 & (1) & c^2 & +\Sigma\beta e^2 & +\Sigma\delta g^2 & +\Sigma\theta i^2 & = \nu^2-1.
\end{array}
$$

In (I $x\mu$), every term $\geqslant 0$; also from (II $x^2\mu$) we have an even value for $\Sigma a d^2$; thus we cannot have $\Sigma a d = 1$, which would give, say, $\alpha_1 = d_1 = 1$, every other $\alpha = 0$, and therefore $\Sigma a d^2 = 1$. Similarly for the other sums. Hence one of these three sums $= 2$, and the other two $= 0$.

If $$\Sigma\epsilon i = 2, \quad \Sigma a d = \Sigma\gamma f = 0,$$

then from (II $x^2\mu$, $x\mu^2$, x^2, μ^2, I x, μ) we find

$$a = b = \beta = \delta = \eta = \xi = 0.$$

Make the substitutions $c = \nu - \sigma_a$, $\theta_i = \epsilon_i \nu + \eta_i$;

then x, μ enter only in the combination $x\mu + \nu \equiv n$, and the remaining equations are equivalent to the set of § 3. We are shut up to the two types there found, valid for any form of n.

The other two possibilities are obtained from each other by interchanging x, μ; we need only investigate one. We therefore take

$$\Sigma a d = 2, \quad \Sigma\gamma f = \Sigma\epsilon i = 0,$$

that is, every γ, $\epsilon = 0$, and therefore every δ, η, $\zeta \geqslant 0$. We have also

(II $x\mu^2$) $\quad b = 0,$

(μ^2) $\quad \Sigma\delta f^2 = 0,$ hence every $\delta = 0$ as well as every γ, and the second sum is entirely absent.

(μ) $\quad \Sigma\alpha e^2 + \Sigma\zeta i^2 = 0,$ hence $e = 0$ if $\alpha > 0$, $\Sigma\alpha e$, $\Sigma\alpha d e$, $\Sigma\alpha e^2 = 0$, and also every $\zeta = 0$. Then (I μ) is satisfied.

($x\mu$) $\quad c = \nu.$

(Straight lines) $\quad (a+d)\mu + e \leqslant 0$

for all values of μ, hence $\quad d \leqslant -a.$

Since $\Sigma\alpha d = 2$, we have the following cases:

Case 1.— $\quad a_1 = 2, \quad d_1 = 1$

Case 2.— $\quad a_1 = 1, \quad d_1 = 2$

} all other $\alpha = 0$.

The apparent possibility of two terms with $\alpha = d = 1$ does not arise; for $e = 0$ whenever $\alpha > 0$, the two terms would have the same index μ, and would coalesce into a single term $\alpha = 2$, $d = 1$, reducing this to Case 1.

Case 1.—Here $\quad \Sigma\alpha d^2 = 2.$

(II $x^2\mu$) $\quad a = -1.$

But $\quad 0 < d \leqslant -a, \quad$ hence every $d = 1.$

(II x^2) $\quad \Sigma\beta = -1.$

(I x) $\quad \Sigma\eta i = 2,$

giving two sub-cases

(i) $\quad \eta_1 = 2, \quad \Sigma\eta i^2 = 2;$ $\qquad$ (ii) $\quad \eta_2 = 1, \quad \Sigma\eta i^2 = 4,$

with every other $\eta = 0$ in each case.

Case 1 i.—

(II x) $\quad \Sigma\beta e = \nu - 1.$

Then using a curve of degree μ, determined by $1^{\mu-1}(2\mu)^1$, we find

$$i \leqslant 1;$$

hence the last sum has only one term, with $i = 1$.

Using a curve of degree $\mu-1$, we find

$$e \geqslant -1.$$

(Straight lines) $$e \leqslant 0,$$

hence $e=0$ or -1, and the first sum has only two terms. The F-system is reduced to

$$1^{n-x}(2\mu+\beta_1)^x \beta_2^{x-1}(2x+\theta_1)^1.$$

(II x^2) $$\beta_1+\beta_2=-1 \quad \text{hence } \beta_1=\nu-2,$$

(II x) $$-\beta_2=\nu-1 \quad \beta_2=1-\nu,$$

and $\nu \leqslant 1$ since $\beta_2 \geqslant 0$.

(I 1) $$\theta_1=\nu-2$$

and (II 1) is satisfied. The solution is therefore

(B) $$1^{n-x}(2\mu+\nu-2)^x(1-\nu)^{x-1}(2x+\nu-2)^1,$$

valid if $\nu \leqslant 1$, $2x \geqslant 2-\nu$. If $\nu \leqslant 0$, this is equivalent to Bianchi's formula for the composition of two de Jonquières transformations (C. 1) with $-\nu$ of the simple F-points of the two transformations coinciding in the intermediate space.

Case 1 ii.— $$\eta_2=1, \quad \Sigma\eta i^2=4.$$

The same argument gives

$$\Sigma\beta e=\nu-2, \quad i \leqslant 2, \quad e=0, -1, \text{ or } -2;$$

there are three terms in the first sum and two in the last. The F-system is

$$1^{n-x}(2\mu+\beta_1)^x \beta_2^{x-1} \beta_3^{x-2}(x+\theta_2)^2 \theta_1^1.$$

(II x^2) $$\beta_1+\beta_2+\beta_3=-1,$$

(II x) $$-\beta_2-2\beta_3=\nu-2,$$

(I 1) $$\theta_1+2\theta_2=\nu-1,$$

(II 1) $$\beta_2+4\beta_3+\theta_1+4\theta_2=-1.$$

We can express the solution in terms of one of the constants as parameter, say β_3,

$$\beta_1=-3+\nu+\beta_3, \quad \theta_1=1+\nu+2\beta_3,$$

$$\beta_2=2-\nu-2\beta_3, \quad \theta_2=-1-\beta_3.$$

All the solutions are derived from the common base

$$1^{n-x}(2\mu-3+\nu)^x(2-\nu)^{x-1}(x-1)^2(1+\nu)^1,$$

by superposing a suitable multiple β_3 of

$$1^x(-2)^{x-1}\,1^{x-2}(-1)^2 2^1.$$

The conditions for β_3 are $\beta_3 \geqslant 0$,

$\beta_2 \geqslant 0$, that is, $2\beta_3 \leqslant 2-\nu$, whence $\nu \leqslant 2$,

$\theta_1 \geqslant 0$, that is, $2\beta_3 \geqslant -1-\nu$.

If $\nu = 2$ or 1, then $\beta_3 = 0$; if $\nu \leqslant 0$, there are always two values for β_3, their form depending on whether ν is even or odd.

Since ν can take all negative values, but only two positive values, it is convenient to write

$$-\nu = \rho = 2\sigma+\tau, \quad \tau = 0, 1;$$

then $\sigma \geqslant -1$ and $\beta_3 = \sigma+1$ or σ, provided these $\geqslant 0$.

We thus have two solutions, given in the table. In the first put

$\sigma = 0$, $\tau = 0$; $n = \mu x$; $1^{n-x}(2\mu-2)^x\,1^{x-2}(x-2)^2\,3^1$.

$\sigma = -1$, $\tau = 0$; $n = \mu x+2$; $1^{n-x}(2\mu-1)^x \quad (x-1)^2\,3^1$.

$\sigma = -1$, $\tau = 1$; $n = \mu x+1$; $1^{n-x}(2\mu-2)^x\,1^{x-1}(x-1)^2\,2^1$.

These can be expressed in the following single formula, equivalent to Ruffini's, with $n = x\mu+\nu$,

(R. m) $1^{n-x}(2\mu-2)^x\,1^{x+\nu-2}\{x-\frac{1}{2}(\nu^2-3\nu+4)\}^2(\nu^2-2\nu+3)^1$, $\nu = 0, 1, 2$.

Case 2.— $a_1 = 1$, $d_1 = 2$, $\Sigma a d^2 = 4$.

(II $x^2\mu$) $a = -2$.

(Straight lines) $d = 1$ or 2, and if $d = 2$, then $e \leqslant 0$.

The first sum now consists of three sorts of terms:

one with $a_1 = 1$, $d_1 = 2$, $e = 0$, $\beta = \beta_1$, say;

a set with $a = 0$, $d = 2$, $e < 0$; let Σ_2 refer to these;

a set with $a = 0$, $d = 1$; let Σ_1 refer to these.

Using a curve of degree $\frac{1}{2}(\mu+\beta_1+\Sigma_2\beta)$, we find

$$\beta_1+\Sigma_2\beta \leqslant -2.$$

(II x^2) $\quad 4\beta_1+4\Sigma_2\beta+\Sigma_1\beta = -4,$

(I x) $\quad 2\beta_1+2\Sigma_2\beta+\Sigma_1\beta+\Sigma\eta i = 2,$

whence $\quad \beta_1+\Sigma_2\beta = -3+\frac{1}{2}\Sigma\eta i \geqslant -3.$

Therefore $\quad \beta_1+\Sigma_2\beta = -3$ or $-2.$

In the first case, every $\eta = 0$, every $\theta \geqslant 0$; further,

(II x) $\quad 4\Sigma_2\beta e+2\Sigma_1\beta e = 4\nu,$

(I 1) $\quad \Sigma_2\beta e+\ \Sigma_1\beta e+\Sigma\theta i = 2\nu-3,$

$$-\Sigma_2\beta e \qquad +\Sigma\theta i = -3.$$

But in every term of Σ_2 we have $\beta \geqslant 0$, $e < 0$; in the last equation, every term on the left $\geqslant 0$, which is impossible. Hence we must have

$$\beta_1+\Sigma_2\beta = -2, \quad \Sigma\eta i = 2, \quad \Sigma_1\beta = 4.$$

Now use a curve of degree $\mu-1$ determined by $1^{\mu-3}(\mu-3)^2 5^1$, we find that in any term of Σ_2,

$$e \geqslant -2.$$

Hence Σ_2 consists of two terms, say $\beta_2^{2\nu-1}\beta_3^{2\nu-2}$.

Now use a curve of degree $y = \frac{1}{2}(\mu-2+p)$, $p = 0,\ 1,\ 2,\ 3,$ or 4, determined by $1^{y-1}(2y)^1$. This leads to

$$\Sigma_2\beta e+\sum_{(p)} e \leqslant \nu,$$

where $\sum_{(p)} e$ is 0 or the sum of the values of e at any set of p out of the four points of Σ_1.

(II x) $\quad 4\Sigma_2\beta e+2\Sigma_1\beta e+\Sigma\eta i^2 = 4\nu.$

Use a curve of degree μ determined by $1^{\mu-2}(\mu-2)^2 5^1$:

$$i \leqslant \tfrac{1}{2}\Sigma\eta i^2.$$

Since $\Sigma\eta i = 2$, there are the usual two sub-cases.

Case 2 i.— $\qquad \eta_1 = 2, \quad \Sigma\eta i^2 = 2, \quad i = 1.$

$$\text{(II } x\text{)} \qquad -4\beta_3-2\beta_2+\Sigma_1\beta e \qquad = 2\nu-1,$$

$$\text{(I 1)} \qquad -2\beta_3- \beta_2+\Sigma_1\beta e +\theta_1 = 2\nu-3,$$

$$\text{(II 1)} \qquad 4\beta_3+ \beta_2+\Sigma_1\beta e^2+\theta_1 = -1,$$

$$2\beta_3 \qquad +\Sigma_1\beta e^2 \qquad = 1.$$

Hence $\beta_3 = 0$, and in Σ_1, for one of the four points, $e^2 = 1$, $e = \pm 1$, and for the other three, $e = 0$.

With the upper sign, the equations give

$$\beta_2 = 1-\nu, \quad \beta_1 = -3+\nu, \quad \theta_1 = -3+\nu,$$

and with the lower,

$$\beta_2 = -\nu, \quad \beta_1 = -2+\nu, \quad \theta_1 = -2+\nu.$$

The two solutions are given in the table.

Case 2 ii.— $\eta_2 = 1, \quad \Sigma\eta i^2 = 4, \quad i = 1$ or 2.

$$\text{(II } x\text{)} \qquad 2\Sigma_2\beta e+\Sigma_1\beta e = 2\nu-2.$$

Now we have shown

$$\Sigma_2\beta e \leqslant \nu, \quad \Sigma_2\beta e+\Sigma_1\beta e \leqslant \nu, \quad \text{and also} \quad \Sigma_2\beta e \leqslant 0.$$

Hence there are three possibilities:

$$\Sigma_2\beta e = \nu-y, \quad \Sigma_1\beta e = 2y-2, \quad y = 0, 1, 2,$$

provided $\nu-y \leqslant 0$.

We have also proved $\Sigma_2\beta e+\underset{(p)}{\Sigma} e \leqslant \nu,$

whence $\underset{(p)}{\Sigma} e \leqslant y, \quad p = 1, 2, 3, \text{ or } 4.$

Thus for each value of y there are two sets of values of e in Σ_1. The simultaneous values are given in the following list:

y	e_1	e_2	e_3	e_4	$\Sigma_1\beta e^2$
0	0	0	0	−2	4
0	0	0	−1	−1	2
1	1	0	0	−1	2
[1	0	0	0	0	0]
2	2	0	0	0	4
2	1	1	0	0	2.

We can now determine all the remaining constants in terms of one of them, say β_2, from the relations

$$\beta_2 + \beta_3 = -2-\beta_1,$$

$$-\beta_2 - 2\beta_3 = \nu - y,$$

(I 1) $$-\beta_2 - 2\beta_3 + \Sigma_1 \beta e + \theta_1 + 2\theta_2 = 2\nu - 3,$$

(II 1) $$\beta_2 + 4\beta_3 + \Sigma_1 \beta e^2 + \theta_1 + 4\theta_2 = -1,$$

The solutions are $\beta_1 = \nu - 2 - y + \beta_2$,

$$\beta_2 = -\nu + y - 2\beta_3,$$

$$\theta_1 = \nu - 1 - y + 2\beta_3 + \Sigma_1 \beta e^2,$$

$$\theta_2 = -\beta_3 - \tfrac{1}{2}\Sigma_1 \beta e^2.$$

Now we must have $\beta_2 \geqslant 0, \quad \theta_1 \geqslant 0,$

which give $-\nu + y \geqslant 2\beta_3 \geqslant -\nu + y + 1 - \Sigma_1 \beta e^2;$

and also, by addition, $\Sigma_1 \beta e^2 - 1 \geqslant 0,$

which excludes one line of the list.

Since $\beta_2 \geqslant 0$, we see again that $\nu - y \leqslant 0$. We can take any arbitrary value of $\nu \leqslant y$; there are then either two or one value of β_3 according as $\Sigma_1 \beta e^2$ is 4 or 2, the form of β_3 in terms of ν depending on whether ν is even or odd. As before, let

$$-\nu = \rho = 2\sigma + \tau, \quad \tau = 0 \text{ or } 1;$$

in the five surviving cases of the list we find

$$\beta_3 = \sigma \text{ or } \sigma - 1, \quad \sigma, \quad \sigma + \tau, \quad \sigma + 1 \text{ or } \sigma, \quad \sigma + 1,$$

which give the seven types tabulated below.

We can prove that they actually exist as follows.

Certain quadratic transformations have the effect of altering the values of the constants in a way depending on the multiplicities of the points used, as follows:

Multiplicities.			*Effect.*
$n-2x$	$2x$	$2x$	$\mu-2$ for μ
$n-2x$	0	0	$\mu+2$ for μ
$n-2x$	$2x$	$2x-1$	$\mu-2$ for μ, $\nu+1$ for ν
$n-2x$	1	0	$\mu+2$ for μ, $\nu-1$ for ν
$n-2x$	$2x$	$2x-2$	$\mu-2$ for μ, $\nu+2$ for ν, β_3-1 for β_3
$n-2x$	2	0	$\mu+2$ for μ, $\nu-2$ for ν, β_3+1 for β_3

provided none of these changes introduces a negative number of points. By combining these, we can independently vary

$$\mu \text{ by } \pm 2, \quad \nu \text{ by } \pm 1, \quad \beta_3 \text{ by } \pm 1,$$

and the existence of each type depends on that of the particular case in which these parameters have their extreme values,

$$\mu = 2 \text{ or } 3, \quad \nu = y, \quad \beta_3 = 0,$$

and these are ten types found in §§ 3, 4.

This exhausts the possible types with $n = x\mu+\nu$ where x, μ can take all values $>$ certain lower limits depending on ν. In every case, ν has an upper limit $\leqslant 2$, but can take an infinite number of negative values.

Many types can be constructed, with n depending on three or more parameters, by compounding two or more of the foregoing types. The parameters of the new type are those which already occur in the components and a new set determining the coincidences of F-points. Thus (R. m) is compounded of a C. 1 of degree x and a C. 2 of degree $\mu+\nu$, the $(x-1)$-fold point of the C. 1 being of multiplicity ν for the C. 2, where $\nu = 0$, 1, or 2. Since ν has just three different values, the numbers of simple and double F-points in the result can be expressed as quadratic functions of ν.

Another example is Palatini's formula for compounding three de Jonquières transformations, frequently referred to above:

$$\text{(P)} \qquad n = n_3N - l'n_1 - l''(n_1-1) - l''', \qquad N = n_1n_2 - l,$$

$$1^{n-N}(2n_3-2-l'-l''-l''')^N\, l'''^{N-1}\, l''^{N+1-n_1}(l'+1)^{N-n_1}$$

$$(2n_2-2-l-l')^{n_1}(l+l''+1)^{n_1-1}(2n_1-2-l-l''')^1.$$

Owing to its large number of parameters, this includes, as particular cases,

a great many of the types which we have been examining, but as it admits only seven different values of i other than unity, it is not exhaustive, for at least one of our types has eight.

$$n = x\mu - \rho, \quad \rho = 2\sigma + \tau.$$

1 i. $1^{n-x}\,(2\mu-\rho-2)^{x}\quad(\rho+1)^{x-1}\quad(2x-\rho-2)^{1}$

1 ii. $1^{n-x}\,(2\mu-\sigma-\tau-2)^{x}\;\tau^{x-1}\quad(\sigma+1)^{x-2}\quad(x-\sigma-2)^{2}(3-\tau)^{1}$

$1^{n-x}\,(2\mu-\sigma-\tau-3)^{x}(\tau+2)^{x-1}\;\sigma^{x-2}\quad(x-\sigma-1)^{2}(1-\tau)^{1}$

2 i. $1^{n-2x}(\mu-\rho-3)^{2x}\quad(\rho+1)^{2x-1}\quad 1^{x+1}3^{x}\quad(2x-\rho-3)^{1}$

$1^{n-2x}(\mu-\rho-2)^{2x}\quad\rho^{2x-1}\quad 3^{x}1^{x-1}\quad(2x-\rho-2)^{1}$

2 ii. $1^{n-2x}(\mu-\sigma-\tau-2)^{2x}\;\tau^{2x-1}\quad\sigma^{2x-2}\quad 3^{x}1^{x-2}\quad(x-\sigma-2)^{2}(3-\tau)^{1}$

$1^{n-2x}(\mu-\sigma-\tau-3)^{2x}(\tau+2)^{2x-1}(\sigma-1)^{2x-2}\;3^{x}1^{x-2}\quad(x-\sigma-1)^{2}(1-\tau)^{1}$

$1^{n-2x}(\mu-\sigma-\tau-2)^{2x}\;\tau^{2x-1}\quad\sigma^{2x-2}\quad 2^{x}2^{x-1}\quad(x-\sigma-1)^{2}(1-\tau)^{1}$

$1^{n-2x}(\mu-\sigma-3)^{2x}\quad(1-\tau)^{2x-1}(\sigma+\tau)^{2x-2}\,1^{x+1}2^{x}1^{x-1}(x-\sigma-\tau-1)^{2}\;\tau^{1}$

$1^{n-2x}(\mu-\sigma-\tau-3)^{2x}\;\tau^{2x-1}\quad(\sigma+1)^{2x-2}\,1^{x+2}3^{x}\quad(x-\sigma-3)^{2}(3-\tau)^{1}$

$1^{n-2x}(\mu-\sigma-\tau-4)^{2x}(\tau+2)^{2x-1}\;\sigma^{2x-2}\quad 1^{x+2}3^{x}\quad(x-\sigma-2)^{2}(1-\tau)^{1}$

$1^{n-2x}(\mu-\sigma-\tau-3)^{2x}\;\tau^{2x-1}\quad(\sigma+1)^{2x-2}\,2^{x+1}2^{x}\quad(x-\sigma-2)^{2}(1-\tau)^{1}.$

AN APPLICATION OF PFEIFFER'S METHOD TO A PROBLEM OF HARDY AND LITTLEWOOD.

By B. M. Wilson.

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1. Let $N(\eta)$ denote the number of lattice-points* inside the right-angled triangle whose sides are

$$x = 0, \quad y = 0, \quad \omega x + \omega' y = \eta \geqslant \omega + \omega',$$

where ω and ω' are two positive numbers whose ratio $\theta = \omega/\omega'$ is irrational, and it is agreed that if η is of the form $m\omega + n\omega'$, where m and n are positive integers, the one lattice-point on the hypotenuse shall count as one-half.† The approximate value of $N(\eta)$ for large values of η has been studied recently by Messrs. Hardy and Littlewood,‡ Hecke,§ and Ostrowski.‖ The present note, however, is concerned, not with such asymptotic expression of $N(\eta)$, but with its expression by means of an exact formula, valid for all values of η which exceed $\omega + \omega'$, *i.e.* a formula analogous to that of Voronoï in Dirichlet's problem of the rectangular hyperbola or to that of Hardy in Gauss's problem of the circle.¶ Such

* A lattice-point (*Gitterpunkt*) is a point whose coordinates x and y are both integral.

† A lattice-point at a vertex ($m = 0$, or $n = 0$) is, like all lattice-points on the axes, not to be counted at all.

‡ G. H. Hardy and J. E. Littlewood, "Some Problems of Diophantine Approximation: the Lattice-Points of a Right-Angled Triangle", I, *Proc. London Math. Soc.*, Vol. 20 (1922), pp. 15–36; II, *Hamburg. Math. Abh.*, Vol. 1 (1922), pp. 212–249. We refer to these two memoirs as H.L. I and H.L. II respectively.

§ E. Hecke, "Analytische Funktionen und die Verteilung von Zahlen mod. Eins", *Hamburg. Math. Abh.*, Vol. 1 (1922), pp. 54–76.

‖ A. Ostrowski, "Bemerkungen zur Theorie der Diophantischen Approximation", *ibid.*, pp. 77–98.

¶ G. H. Hardy, "On the Expression of a Number as the Sum of Two Squares", *Quarterly Journal of Pure and Appl. Math.*, Vol. 46 (1915), pp. 263–283. See also E. Landau, "Über die Gitterpunkte in einem Kreise", *Göttinger Nachrichten*, 1920, pp. 109–134. Hardy's proof depends upon properties of Dirichlet's series, and is somewhat special; Landau employs Pfeiffer's method.

an expression for $N(\eta)$ has been obtained by Hardy and Littlewood, namely

$$\text{(I)} \quad N(\eta) = D(\eta) + \frac{\omega^2 + 3\omega\omega' + \omega'^2}{12\omega\omega'}$$

$$- \frac{1}{2\pi} \lim_{R \to \infty} \left(\sum_{2m\pi < \omega R} \frac{\cos \frac{2m\pi}{\omega} (\eta - \frac{1}{2}\omega')}{m \sin \frac{m\omega'\pi}{\omega}} + \sum_{2n\pi < \omega' R} \frac{\cos \frac{2n\pi}{\omega'} (\eta - \frac{1}{2}\omega)}{n \sin \frac{n\omega\pi}{\omega'}} \right)$$

where

$$\text{(II)} \quad D(\eta) = \frac{\eta^2}{2\omega\omega'} - \frac{\eta}{2\omega} - \frac{\eta}{2\omega'},$$

and where R tends to infinity through a sequence of values R_j, all of which differ by more than δ ($\delta > 0$) from any of the numbers

$$2m\pi/\omega, \quad 2n\pi/\omega' \quad (m, n = 0, 1, 2, \ldots).^*$$

Hardy and Littlewood's proof of (I) depends upon an application of Cauchy's theorem; in the present note a proof of the equation is given which uses only "real" analysis.

2. We write

$$\phi_m(x) = 1 + 2 \sum_{\mu=1}^{m} \cos 2\mu\pi x = \frac{\sin (2m+1)\pi x}{\sin \pi x}.$$

Then it follows from theorems proved by Landau and van der Corput that *if $f(x)$ is a continuous monotone decreasing function of x, defined for $0 \leqslant x \leqslant X$, where $f(X) = 0$, and if A denote the number of lattice-points inside the curvilinear triangle bounded by*

$$x = 0, \quad y = 0, \quad y = f(x),$$

$$A = \lim_{m \to \infty} \iint_{\Delta} \phi_m(x)\, \phi_m(y)\, dx\, dy,$$

where the area of integration Δ is bounded by

$$x = \tfrac{1}{2}, \quad y = \tfrac{1}{2}, \quad y = f(x);$$

lattice-points on the axes are here disregarded in forming A, but lattice-

* H.L. II, pp. 226–231. The result only is announced in H.L. I, pp. 35–36.

points on the curve $y = f(x)$ *(but not on either axis) each contribute one-half to* A.*

Thus we have the expression

$$(2.1)\quad N(\eta) = \lim_{m\to\infty} \iint_{\Delta} \phi_m(x)\,\phi_m(y)\,dx\,dy = \lim_{m\to\infty} \sum_{\mu,\,\nu=-m}^{\mu,\,\nu=m} P(\eta,\,\mu,\,\nu)$$

where

$$(2.2)\qquad P(\eta,\,\mu,\,\nu) = \iint_{\Delta} \cos 2\mu\pi x \cos 2\nu\pi y\,dx\,dy.$$

By direct integration we find that

$$(2.3)\qquad P(\eta,\,0,\,0) = \Delta(\eta) = \frac{1}{2\omega\omega'}\left\{\eta^2 - \eta(\omega+\omega') + \tfrac{1}{4}(\omega+\omega')^2\right\},$$

and, provided that μ and ν are not both zero,

$$(2.35)\qquad P(\eta,\,\mu,\,\nu) = \frac{1}{4\pi^2}\left\{S(\mu,\,\nu;\ \omega,\,\omega') + S(\nu,\,\mu;\ \omega',\,\omega)\right\}$$

where

$$(2.37)\qquad S(\mu,\,\nu;\ \omega,\,\omega') = \frac{(-)^{\mu}\omega\omega' \cos\dfrac{2\nu\pi}{\omega'}(\eta - \frac{1}{2}\omega)}{(\mu^2\omega'^2 - \nu^2\omega^2)}.$$

If then we write

$$R(\omega,\,\omega') = R(\eta,\,m;\ \omega,\,\omega') = \sum_{\mu,\,\nu=-m}^{m}{}' S(\mu,\,\nu;\ \omega,\,\omega')$$

$$= 4 \sum_{\mu,\,\nu=0}^{m}{}' \epsilon_{\mu,\,\nu}\, S(\mu,\,\nu;\ \omega,\,\omega')$$

where

$$\epsilon_{\mu\nu} = 1 \text{ or } \tfrac{1}{2},$$

according as neither or one of the numbers μ, ν is zero, and the dash denotes that $\mu = \nu = 0$ is omitted from the summation, we have

$$(2.5)\qquad N(\eta) = \Delta(\eta) + \lim_{m\to\infty} \frac{1}{4\pi^2}\Big(R(\omega,\,\omega') + R(\omega',\,\omega)\Big).$$

3. Of the numbers ω, ω' let ω be the less; then $R(\omega,\,\omega')$ is very readily transformed into a sum of the type that appears in the right-hand

* E. Landau, "Die Bedeutung der Pfeiffer'schen Methode für die analytische Zahlentheorie", *Wiener Sitzungsber.*, Vol. 121 (1912), pp. 2195–2332; esp. pp. 2217–2219. E. Landau, "Neuere Untersuchungen über die Pfeiffer'she Methode u.s.w.", *ibid.*, Vol. 124 (1915), pp. 469–595; esp. pp. 472–478. E. Landau and J. G. van der Corput, "Über Gitterpunkte in ebenen Bereichen", *Göttinger Nachrichten*, 1920, pp. 135–171; esp. pp. 152–157.

member of (I). For

$$\frac{1}{\pi^2}\sum_{\mu,\nu=0}^{m}{}'\,\epsilon_{\mu\nu}\,S(\mu,\nu;\ \omega,\ \omega') = \frac{1}{2\pi^2}\sum_{\mu=1}^{m}\frac{(-1)^\mu\omega\omega'}{\mu^2\omega'^2} + \frac{1}{\pi^2}\sum_{\nu=1}^{m}\sum_{\mu=0}^{m}\epsilon_{\mu\nu}\,S(\mu,\nu;\ \omega,\ \omega')$$

$$= \frac{1}{2\pi^2}\frac{\omega}{\omega'}\left(-\frac{\pi^2}{12}+O(m^{-2})\right) + \frac{1}{\pi^2}\sum_{\nu=1}^{m}\cos\frac{2\nu\pi}{\omega'}(\eta-\tfrac{1}{2}\omega)$$

$$\times\left\{-\frac{\pi}{2\nu}\operatorname{cosec}\left(\nu\pi\,\frac{\omega}{\omega'}\right) - \sum_{\mu=m+1}^{\infty}\frac{(-1)^\mu\omega\omega'}{\mu^2\omega'^2-\nu^2\omega^2}\right\}.$$

Since $\omega<\omega'$ and $1\leqslant\nu\leqslant m$, we have

$$\left|\sum_{\mu=m+1}^{\infty}\frac{(-1)^\mu\omega\omega'}{\mu^2\omega'^2-\nu^2\omega^2}\right| < \frac{\omega\omega'}{(m+1)^2\omega'^2-\nu^2\omega^2} < \frac{\omega\omega'}{m^2(\omega'^2-\omega^2)}.$$

Thus
$$\left|\sum_{\nu=1}^{m}\cos\frac{2\nu\pi}{\omega'}(\eta-\tfrac{1}{2}\omega)\sum_{\mu=m+1}^{\infty}\frac{(-1)^\mu\omega\omega'}{\mu^2\omega'^2-\nu^2\omega^2}\right| < \frac{\omega\omega'}{m(\omega'^2-\omega^2)},$$

and consequently, as m increases,

$$\text{(3.5)}\qquad \frac{1}{4\pi^2}R(\omega,\omega') = -\frac{1}{24}\frac{\omega}{\omega'} - \frac{1}{2\pi}\sum_{\nu=1}^{m}\frac{\cos\dfrac{2\nu\pi}{\omega'}(\eta-\frac{1}{2}\omega)}{\nu\sin\dfrac{\nu\omega\pi}{\omega'}} + O(m^{-1}).$$

4. The discussion of $R(\omega',\omega)$ is a little more delicate ($\omega<\omega'$). We define a sequence (m_j, n_j) as a sequence of values of the pair of positive integers (m, n) which is such that m_j and n_j both tend to infinity with j, and that, if (m, n) is any pair of values selected from the sequence,

$$\text{(4.01)}\qquad (m+1)\omega-n\omega' > \delta > 0,$$

$$\text{(4.02)}\qquad (n+1)\omega'-m\omega > \delta > 0.$$

In order that a sequence (m_j, n_j) may exist it is clearly necessary that $\delta<\frac{1}{2}(\omega+\omega')$; on the other hand such a sequence will exist provided that δ is not too large. Thus, if $\delta<\frac{1}{2}\omega'$, we may take $n_j=j$, and the existence of the corresponding m_j is then assured by the fact that

$$(n+1)\omega'-\delta-(n\omega'-\omega+\delta) = \omega'+\omega-2\delta > \omega,$$

so that at least one integer m satisfies both inequalities.* For any sequence (m_j, n_j), as j increases

$$\text{(4.2)}\qquad n/m \sim \omega/\omega'.$$

* As another example: let n assume only values n for which $n\omega'/\omega$ differs from an integer by more than $\epsilon\,(0<\epsilon<\frac{1}{2})$. A corresponding m_j is then given by $m=[n\omega'/\omega]$.

In what follows the integers m and n will always be supposed to form a pair selected from a sequence (m_j, n_j). With this explanation we now commence the discussion of $R(\omega', \omega)$.

5. We have

$$\frac{1}{\pi^2}\sum_{\mu,\nu=0}^{m}\epsilon_{\mu\nu}S(\nu,\mu;\omega',\omega)$$

$$=\frac{1}{2\pi^2}\sum_{\nu=1}^{m}\frac{(-1)^\nu\omega\omega'}{\nu^2\omega^2}+\frac{1}{\pi^2}\left(\sum_{\mu=1}^{n}\sum_{\nu=0}^{m}+\sum_{\mu=n+1}^{m}\sum_{\nu=0}^{m}\right)\epsilon_{\mu\nu}S(\nu,\mu;\omega',\omega)$$

$$=S_1+S_2+S_3.$$

Just as before

$$S_1=-\frac{1}{24}\frac{\omega'}{\omega}+O(m^{-2}). \tag{5.1}$$

Also

$$|S_3|\leqslant\frac{1}{\pi^2}\sum_{\mu=n+1}^{m}\left|\sum_{\nu=0}^{m}\epsilon_{\mu\nu}\frac{(-)^\nu\omega\omega'}{\nu^2\omega^2-\mu^2\omega'^2}\right|<\frac{1}{\pi^2}\sum_{\mu=n+1}^{m}\left(\frac{\omega\omega'}{\mu^2\omega'^2-m^2\omega^2}+\frac{\omega}{2\mu^2\omega'}\right)$$

$$<\frac{1}{\pi^2}\frac{\omega\omega'}{(n+1)\omega'+m\omega}\sum_{\mu=n+1}^{m}\frac{1}{\mu\omega'-m\omega}+O(n^{-1})$$

$$<\frac{1}{\pi^2}\frac{\omega\omega'}{(n+1)\omega'+m\omega}\sum_{k=0}^{m-n-1}\frac{1}{\delta+k\omega'}+O(m^{-1})=O\left(\frac{\log m}{m}\right).$$

Finally, as in § 3,

$$\left|S_2+\frac{1}{2\pi}\sum_{\mu=1}^{n}\frac{\cos\frac{2\mu\pi}{\omega}(\eta-\frac{1}{2}\omega')}{\mu\sin\frac{\mu\omega'\pi}{\omega}}\right|$$

$$=\frac{1}{\pi^2}\left|\sum_{\mu=1}^{n}\cos\frac{2\mu\pi}{\omega}(\eta-\tfrac{1}{2}\omega')\sum_{\nu=m+1}^{\infty}\frac{(-1)^\nu\omega\omega'}{\nu^2\omega^2-\mu^2\omega'^2}\right|$$

$$<\frac{1}{\pi^2}\sum_{\mu=1}^{n}\left|\sum_{\nu=m+1}^{\infty}\frac{(-1)^\nu\omega\omega'}{\nu^2\omega^2-\mu^2\omega'^2}\right|$$

$$<\frac{1}{\pi^2}\sum_{\mu=1}^{n}\frac{\omega\omega'}{(m+1)^2\omega^2-\mu^2\omega'^2}=O\left(\frac{\log m}{m}\right),$$

as in the discussion of S_2.

6. Collection of the results obtained in §§ 3–5 now gives the following result :

$$\frac{1}{4\pi^2}\left(R(\omega, \omega')+R(\omega', \omega)\right) = -\frac{1}{24}\left(\frac{\omega}{\omega'}+\frac{\omega'}{\omega}\right)-\frac{1}{2\pi}\sum_{\mu=1}^{n}\frac{\cos\frac{2\mu\pi}{\omega}(\eta-\frac{1}{2}\omega')}{\mu\sin\frac{\mu\omega'\pi}{\omega}}$$

$$-\frac{1}{2\pi}\sum_{\nu=1}^{m}\frac{\cos\frac{2\nu\pi}{\omega'}(\eta-\frac{1}{2}\omega)}{\nu\sin\frac{\nu\omega\pi}{\omega'}}+O\left(\frac{\log m}{m}\right).$$

Thus, in virtue of (2.5) and (2.3), equation (I) will be proved if it is shown that the conditions here imposed on the members of a sequence (m_j, n_j) agree with Hardy and Littlewood's definition of a sequence (R_j). It is, in fact, readily proved that to every sequence (m_j, n_j) there corresponds a sequence (R_j), and conversely. The relation between corresponding elements is

$$n = \frac{\omega R}{2\pi}-f, \qquad m = \frac{\omega' R}{2\pi}-g,$$

wherein $$0 < \epsilon < f, \qquad g < 1-\epsilon < 1.$$

ABEL'S THEOREM AND ITS CONVERSE (II)

By G. H. Hardy *and* J. E. Littlewood.

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1. *Introduction.*

1. In an earlier paper bearing the same title as this one* we referred to two theorems which we were then unable to prove, and whose absence detracted from the completeness of our results.

In the first place, we proved† that *if*

$$a_n = O\left(\frac{1}{n}\right) \tag{1.1}$$

and

$$\Phi(x) = \frac{1}{1-x} \Sigma \frac{a_n}{n+1}(1-x^{n+1}), \tag{1.2}$$

then

$$\Sigma a_n = A \tag{1.3}$$

implies

$$\Phi(x) \to A \tag{1.4}$$

when $x \to 1$ along any path $C_\ddagger^*$ which does not pass outside the circle. It is natural to suppose that this theorem, like other theorems of its kind, is a "best possible" theorem, *i.e.* that it becomes false if (1.1) is replaced by any less exacting condition of the same kind; but this we were at the time unable to prove. We give a proof now in § 2.

A second and more important gap in our work was this. We proved§

* "Abel's Theorem and its Converse", *Proc. London Math. Soc.*, Ser. 2, Vol. 18 (1918), pp. 205–235. See p. 210. We refer to this paper as A.T.

† A.T., pp. 220–223 (Theorem **T**).

‡ See A.T., pp. 205–206, for precise definitions of *path*, *internal path*, *regular path*, *Stolz path*.

§ A.T., pp. 214–219 (Theorem **Q**).

that (1.1) and (1.4) involve (1.3), so that, if (1.1) is satisfied, (1.4) is the necessary and sufficient condition for the convergence of the series Σa_n. From this we deduced a theorem* which is a more direct extension of Tauber's Theorem, viz. that (1.1) *and*

$$f(x) = \Sigma a_n x^n \to A \tag{1.5}$$

involve the convergence of the series. But here we had to impose a restriction on C, viz. that it is what we called a regular path. We could also prove the result when C is a Stolz path† ; but we were unable to eliminate *all* restrictions on C. This we can now do, and present the proof in § 3.

There is one point which demands a word of explanation. The series (1.2) is uniformly convergent for $|x| \leqslant 1$, and, so long as we are concerned with $\Phi(x)$, it is irrelevant whether our path C is restricted to be internal or may have points in common with the circle. The situation is different when we are concerned with $f(x)$, since $f(x)$ is not generally defined at points on the circle. We therefore state and prove our theorem on the hypothesis that C is internal.

The theorem remains true, however, if this restriction is removed, provided the hypotheses are properly stated. It may be possible to define $f(x)$, at a point common to C and the circle, by an Abelian limit

$$f(x) = f(1, \theta) = \lim_{r \to 1} f(r, \theta),$$

and in this manner to complete the definition of $f(x)$ for all points of C; and it may then happen that $f(x)$, so defined, tends to a limit along C. Elementary considerations of continuity then show C may be replaced by a "sufficiently close" internal path C' along which also $f(x)$ tends to a limit. Our theorem then establishes the convergence of the series, the additional complication lying only in the interpretation of its hypotheses.

In § 4 we restate the final result of our work, which may now be regarded as completed, in the form of a comprehensive theorem.

2. *Addendum to Theorem* **T**.

2.1. THEOREM **T'**.—*It is not possible, in Theorem* **T**, *to replace the condition* (1.1) *by any less exacting condition of the same type. That is to say, if* $\phi(n)$ *is any positive function of* n, *tending steadily to infinity*

* A.T., pp. 219-220 (Theorem **R**).

† A.T., p. 220 (Theorem **S**).

with n, it is possible to find a convergent series A, for which

$$a_n = O\left(\frac{\phi(n)}{n}\right), \tag{2.11}$$

and a regular internal path C for which (1.4) *is false.*

We may obviously suppose, without loss of generality, that the a's are real, that $a_0 = 0$, and that

$$\phi(n) < \sqrt{n} \quad (n > 0);$$

and we may replace

$$\frac{a_n}{n+1}(1-x^{n+1}),$$

in (1.2), by

$$\frac{a_n}{n}(1-x^n),$$

since

$$\frac{1}{1-x}\left\{\Sigma\frac{a_n}{n}(1-x^n)-\Sigma\frac{a_n}{n+1}(1-x^{n+1})\right\} = \Sigma\frac{a_n}{n(n+1)}\frac{1-x^n}{1-x}-\Sigma\frac{a_n}{n+1}x^n,$$

and each of these series is uniformly convergent for $|x|<1$.

Let

$$\chi(\alpha) = \frac{1}{1-e^{i\alpha}}\Sigma\frac{a_n}{n}(1-e^{ni\alpha}), \tag{2.12}$$

where $0 < \alpha < 2\pi$, and

$$\chi(r,\alpha) = \frac{1}{1-re^{i\alpha}}\Sigma\frac{a_n}{n}(1-r^n e^{ni\alpha}), \tag{2.121}$$

where $0 < r < 1$. For any fixed α, $\chi(r,\alpha)\to\chi(\alpha)$ when $r\to 1$; and we can define a regular internal path C, by an equation $r = r(\alpha)$, in such a manner that $\chi(r,\alpha)-\chi(\alpha)\to 0$ when $\alpha\to 0$. The theorem will therefore be proved if we can establish the existence of a function $\chi(\alpha)$ which does not tend to a limit when $\alpha\to 0$.

Again, if

$$\Phi(\alpha) = \Sigma a_n\frac{\sin n\alpha}{n\alpha}, \tag{2.13}$$

$$\mathfrak{R}\left(e^{\frac{1}{2}i\alpha}\chi(\alpha)\right) = \frac{\alpha}{2\sin\frac{1}{2}\alpha}\Phi(\alpha), \tag{2.14}$$

and $\chi(\alpha)$ cannot tend to a limit unless $\Phi(\alpha)$ does so. It is therefore sufficient to define a $\Phi(\alpha)$ which does not tend to a limit.

2.2. LEMMA 1.—*If A is convergent, then*

$$\sum_{n\leqslant 1/\alpha} a_n \frac{\sin n\alpha}{n\alpha} = A + o(1). \tag{2.21}$$

Choose N so that

$$\left| \sum_{\nu\leqslant n} a_\nu - A \right| < \epsilon \quad (n \geqslant N). \tag{2.22}$$

Then

$$\begin{aligned} &\left| \sum_{n\leqslant 1/\alpha} a_n \frac{\sin n\alpha}{n\alpha} - A \right| \\ &\leqslant \left| \sum_{n\leqslant N} a_n \left(\frac{\sin n\alpha}{n\alpha} - 1 \right) \right| + \left| \sum_{n\leqslant N} a_n - A \right| + \left| \sum_{N<n\leqslant 1/\alpha} a_n \frac{\sin n\alpha}{n\alpha} \right| \\ &= S_1 + S_2 + S_3, \end{aligned} \tag{2.23}$$

say. We have

$$|S_2| < \epsilon, \tag{2.241}$$

by (2.22); and

$$|S_3| \leqslant \operatorname*{Max}_{N<n_1\leqslant 1/\alpha} \left| \sum_{N<n\leqslant n_1} a_n \right| < 2\epsilon, \tag{2.242}$$

since $\sin u/u$ decreases steadily from 1 in the interval $0 < u \leqslant 1$. Finally, when N is fixed,

$$|S_1| < \epsilon \tag{2.243}$$

for all sufficiently small values of α. From (2.23)–(2.243) we deduce

$$\left| \sum_{n\leqslant 1/\alpha} a_n \frac{\sin n\alpha}{n\alpha} - A \right| < 4\epsilon$$

for all sufficiently small values of α, which proves the lemma.

2.3. LEMMA 2.—*If $\phi(x)$ is any function of x which tends steadily to infinity with x, we can find a sequence of positive integers n_ν such that*

$$m_\nu = \frac{n_{\nu+1}}{n_\nu} \to \infty, \qquad m_\nu < \phi(n_\nu). \tag{2.31}$$

We may clear our ideas by a preliminary remark. The important cases are those in which $\phi(x)$ tends to infinity very slowly, *e.g.* like $\log\log x$. The first condition on the m's means that n_ν tends to infinity more rapidly than any exponential $e^{\alpha\nu}$, and the second that its increase is very little more rapid than that of such exponentials. Thus, when

$\phi(x) = \log\log x$, the conditions would be satisfied (except that the m's would not be integers) if we took

$$n_\nu = e^{\nu \log\log\log \nu}.$$

We begin by choosing non-integral m's such that

$$m_\nu \to \infty, \quad 1 < m_\nu < \tfrac{1}{2}\phi(n_\nu). \tag{2.32}$$

We can obviously find a function $\chi(x)$, with a positive, decreasing, and continuous derivative χ', such that

$$\chi \to \infty, \quad \chi > 1, \quad \chi' = o\left(\frac{1}{x}\right), \quad (x+1)\chi' < 1, \quad e^\chi < \frac{\phi(e^x)}{2e}. \tag{2.33}$$

We take

$$n_\nu = e^{\nu\chi(\nu)}. \tag{2.34}$$

Then $$e^{\chi(\nu)+\nu\chi'(\nu+1)} < e^{(\nu+1)\chi(\nu+1)-\nu\chi(\nu)} = m_\nu < e^{\chi(\nu)+(\nu+1)\chi'(\nu)},$$

and so, using the third of the conditions (2.33),

$$1 < m_\nu \sim e^{\chi(\nu)} \to \infty. \tag{2.35}$$

Also, using the fourth and fifth of (2.33),

$$m_\nu < e^{\chi+(\nu+1)\chi'} < e^{\chi+1} < \tfrac{1}{2}\phi(e^\nu) < \tfrac{1}{2}\phi(n_\nu). \tag{2.36}$$

Thus the conditions (2.32) are satisfied.

We now replace every m_ν by the integer equal to or immediately above it. This increases n_ν and therefore $\phi(n_\nu)$, while m_ν is at most doubled. Hence the new set of m's satisfy (2.31).

We note that

$$\sum_\mu^\infty \frac{1}{n_\nu^2} = O\left(\frac{1}{n_\mu^2}\right), \tag{2.37}$$

an obvious corollary of (2.31).

2.4. LEMMA 3.—*When m_ν is defined as in Lemma 2, we can find a positive integer k_ν which tends steadily to infinity with ν in such a manner that*

$$\frac{k_{\nu+1}}{k_\nu} \to 1, \quad \frac{\log m_\nu}{k_\nu} \to \infty. \tag{2.41}$$

This is obvious; we have only to make the increase of k_ν sufficiently slow.

LEMMA 4.—*If*

$$a_\nu = 2\pi k_\nu/n_\nu, \tag{2.42}$$

then

$$a_{\nu+1}/a_\nu \to 0, \tag{2.43}$$

$$a_\nu - a_{\nu'} \sim a_\nu \quad (\nu' > \nu), \tag{2.44}$$

and

$$n_{\nu-1} < 1/a_\nu < n_\nu \tag{2.45}$$

for all sufficiently large values of ν.

Of these relations (2.43) follows from (2.31), (2.41), and (2.42), and (2.44) is an obvious corollary. Also

$$n_\nu a_\nu = 2\pi k_\nu \to \infty, \qquad n_{\nu-1} a_\nu = \frac{2\pi k_\nu}{m_{\nu-1}} \sim \frac{2\pi k_{\nu-1}}{m_{\nu-1}} \to 0,$$

by (2.41); which proves (2.45).

LEMMA 5.—*If* $0 < \theta < \frac{1}{2}\pi$ *and* $0 < N < N'$, *then*

$$\sum_N^{N'} \sin n\theta = O\Big(\frac{1}{\theta}\Big), \qquad \sum_N^{N'} \frac{\cos n\theta}{n} = O\Big(\frac{1}{N\theta}\Big), \tag{2.46}$$

uniformly in the parameters. If also $0 < \phi < \frac{1}{2}\pi$, $\phi \neq \theta$, *then*

$$\sum_N^{N'} \frac{\sin n\theta \sin n\phi}{n} = O\Big(\frac{1}{N|\theta-\phi|}\Big). \tag{2.47}$$

These are familiar results, collected merely for purposes of reference.

2.5. We define our series A by

$$a_n = \frac{\sin na_\nu}{n_\nu} = \frac{1}{n_\nu} \sin \frac{2n\pi k_\nu}{n_\nu} \quad (n_\nu \leqslant n < n_{\nu+1}). \tag{2.51}$$

Then

$$|a_n| \leqslant \frac{1}{n_\nu} = \frac{m_\nu}{n_{\nu+1}} \leqslant \frac{\phi(n_\nu)}{n_{\nu+1}} \leqslant \frac{\phi(n)}{n},$$

so that a_n satisfies (2.11). It remains to prove (1) that A is convergent, and (2) that $\Phi(a)$ does not tend to a limit.

We have first

$$\sum_{n_\nu}^{n_{\nu+1}-1} \sin na_\nu = \frac{\sin \frac{1}{2}(n_{\nu+1}-n_\nu)a_\nu \sin \frac{1}{2}(n_\nu+n_{\nu+1}-1)a_\nu}{\sin \frac{1}{2}a_\nu} = 0,$$

since

$$\tfrac{1}{2}(n_{\nu+1}-n_\nu)a_\nu = (m_\nu-1)k_\nu\pi$$

is a multiple of π. Hence

$$\sum_{n_\nu}^{n_{\nu+1}-1} a_n = 0. \tag{2.52}$$

Also, if $n_\nu \leqslant \mu < n_{\nu+1}$, we have

$$\sum_{n_\nu}^{\mu} a_n = \frac{1}{n_\nu} \sum_{n_\nu}^{\mu} \sin n\alpha_\nu = O\left(\frac{1}{n_\nu \alpha_\nu}\right) = O\left(\frac{1}{k_\nu}\right) = o(1), \tag{2.53}$$

by (2.46) and (2.51), and uniformly in μ. From (2.52) and (2.53) it follows that A converges to the sum 0.

2.6. We write

$$\Phi(\alpha) = \Sigma a_n \frac{\sin n\alpha}{n\alpha} = \sum_\nu \sum_{n_\nu}^{n_{\nu+1}-1} = \sum_\nu \psi_\nu(\alpha) \tag{2.61}$$

$$= \sum_1^{\mu-1} \psi_\nu(\alpha) + \psi_\mu(\alpha) + \sum_{\mu+1}^{\infty} \psi_\nu(\alpha) = S_1(\alpha) + S_2(\alpha) + \psi_\mu(\alpha);$$

and we shall prove that

$$S_1(\alpha_\mu) \to 0, \tag{2.62}$$

$$S_2(\alpha_\mu) \to 0, \tag{2.63}$$

$$\psi_\mu(\alpha_\mu) \to \infty \tag{2.64}$$

when $\mu \to \infty$. From these relations it will follow that $\Phi(\alpha) \to \infty$ when $\alpha \to 0$ through the particular sequence α_μ, and this will complete the proof of Theorem **T**′.

We have

$$S_1(\alpha_\mu) = \sum_{n \leqslant n_{\mu-1}} a_n \frac{\sin n\alpha_\mu}{n\alpha_\mu} = \sum_{n \leqslant 1/\alpha_\mu} + \sum_{1/\alpha_\mu < n < n_\mu} \tag{2.65}$$

$$= \sum_{1/\alpha_\mu < n < n_\mu} + o(1) = S_1'(\alpha_\mu) + o(1),$$

by Lemma 1. Since $1/\alpha_\mu > n_{\mu-1}$, by (2.45), $\nu = \mu - 1$ throughout S_1'. Writing

$$M = 1 + \left[\frac{1}{\alpha_\mu}\right], \tag{2.66}$$

we have

$$S_1'(\alpha_\mu) = \frac{1}{n_{\mu-1}} - \frac{1}{\alpha_\mu} \sum_M^{n_\mu - 1} \frac{\sin n\alpha_{\mu-1} \sin n\alpha_\mu}{n} = O\left(\frac{1}{n_{\mu-1}\alpha_\mu} \cdot \frac{1}{M(\alpha_{\mu-1} - \alpha_\mu)}\right)$$

by (2.47). But

$$M(\alpha_{\mu-1} - \alpha_\mu) \sim M\alpha_{\mu-1} \sim \alpha_{\mu-1}/\alpha_\mu,$$

by (2.44) and (2.66); and so

$$S_1'(a_\mu) = O\Big(\frac{1}{n_{\mu-1}a_{\mu-1}}\Big) = O\Big(\frac{1}{k_{\mu-1}}\Big) = o(1). \tag{2.67}$$

From (2.65) and (2.67) we deduce (2.62).

Next, if $\nu > \mu$,

$$\psi_\nu(a_\mu) = \frac{1}{n_\nu a_\mu} \sum_{n_\nu}^{n_{\nu+1}-1} \frac{\sin na_\mu \sin na_\nu}{n} = \frac{1}{n_\nu a_\mu} O\Big(\frac{1}{n_\nu(a_\mu - a_\nu)}\Big) = O\Big(\frac{1}{n_\nu^2 a_\mu^2}\Big),$$

by (2.47) and (2.44); and hence, using (2.37), we obtain

$$S_2(a_\mu) = O\Big(\frac{1}{a_\mu^2} \sum_{\mu+1}^{\infty} \frac{1}{n_\nu^2}\Big) = O\Big(\frac{1}{n_{\mu+1}^2 a_\mu^2}\Big) = O\Big(\frac{1}{n_\mu^2 a_\mu^2}\Big) = O\Big(\frac{1}{k_\mu^2}\Big) = o(1),$$

which is (2.63).

Finally

$$\psi_\mu(a_\mu) = \frac{1}{n_\mu a_\mu} \sum_{n_\mu}^{n_{\mu+1}-1} \frac{\sin^2 na_\mu}{n} \tag{2.68}$$

$$= \frac{1}{2n_\mu a_\mu} \sum_{n_\mu}^{n_{\mu+1}-1} \frac{1}{n} - \frac{1}{2n_\mu a_\mu} \sum_{n_\mu}^{n_{\mu+1}-1} \frac{\cos 2na_\mu}{n}.$$

The first term in (2.68) is greater than a constant multiple of

$$\frac{1}{n_\mu a_\mu} \log \frac{n_{\mu+1}}{n_\mu} = \frac{\log m_\mu}{2\pi k_\mu},$$

which tends to infinity, by (2.41). The second, by (2.46), is

$$O\Big(\frac{1}{n_\mu a_\mu} \cdot \frac{1}{n_\mu a_\mu}\Big) = O\Big(\frac{1}{k_\mu^2}\Big) = o(1).$$

This proves (2.64) and completes the proof of the theorem.

3. *The general Tauberian theorem.*

3.1. In this section we prove

THEOREM **R'**.—*If*

$$a_n = O\Big(\frac{1}{n}\Big), \tag{1.1}$$

and

$$f(x) \to A \tag{1.5}$$

when $x \to 1$ along an internal path C, then

$$\Sigma a_n = A. \tag{1.3}$$

We may suppose $A = 0, \quad |na_n| \leqslant 1,$

and it is convenient to write e^{-z} for x, and then $z = x+iy$. We have then to prove: *if*

$$|na_n| \leqslant 1, \tag{3.11}$$

and

$$f(z) = \Sigma a_n e^{-nz} \to 0 \tag{3.12}$$

when $z \to 0$ along a Jordan curve C lying, except for the point $z = 0$, in the half-plane $x > 0$, then

$$\Sigma a_n = 0. \tag{3.13}$$

We suppose that C is

$$z = Z(t), \tag{3.14}$$

the parameter t ranging from 0 to 1, and $t = 1$ corresponding to $z = 0$. There is no loss of generality in supposing that

$$|f| < 1 \tag{3.15}$$

at all points of C.

The letter K denotes an absolute constant, whose value varies from one occurrence to another. Occasionally we must preserve the identity of a K; in this case we distinguish it by a suffix, and such constants are the same wherever they occur. The constants of the O's which occur in our argument are also absolute.

3.2. LEMMA 6.—*If*

$$|z| < 1, \quad |z'-z| < cx \quad (0 < c < 1) \tag{3.21}$$

then

$$|f(z')-f(z)| < k(c), \tag{3.22}$$

where $k(c)$ depends only on c.

It follows from (3.21) that x is positive and

$$x' > Kx. \tag{3.23}$$

Also $$|e^{-nz'}-e^{-nz}| = |n\int_z^{z'} e^{-nu}du| \leqslant n|z'-z|\,\mathrm{Max}\,(e^{-nx'}, e^{-nx})$$
$$\leqslant n|z'-z|(e^{-nx'}+e^{-nx}).$$

Hence

$$|f(z')-f(z)| = |\Sigma a_n(e^{-nz'}-e^{-nz})| \leqslant \Sigma n|a_n||z'-z|(e^{-nx'}+e^{-nx})$$

$$\leqslant |z'-z|\left(\frac{1}{1-e^{-x'}}+\frac{1}{1-e^{-x}}\right) < K_1 x\left(\frac{1}{x'}+\frac{1}{x}\right) < K,$$

which is (3 . 22).

3 . 21. Lemma 6 leads to a very simple proof of our former Theorem **S**.* This theorem is required for the proof of **R**′, and we did not write out its proof in full: we therefore insert one here.

If C is a Stolz path, it is included in an angle **A**, whose vertex is at O, and whose sides include, and make acute angles with, the real axis. The angle **A**, in its turn, is included within a slightly larger angle **A**′ of the same type. It is plain that, if z is a point of C, and z' a point in **A**′ for which $x' \leqslant x$, then z' satisfies the condition (3 . 21), and so that $f(z')$ is bounded in **A**′. Since $f(z)$ is bounded in **A**′, and tends to a limit along C, it tends to a limit when $z \to 0$, uniformly throughout **A**.† In particular, it tends to a limit along the real axis, and therefore, by Theorem **F**, the series Σa_n is convergent.

3 . 3. The next lemma requires a word of explanation. Suppose that $z \to 0$ along C, so that $z = Z(t)$; we may denote the parameter of z by $t(z)$. If we write

$$\epsilon = \operatorname*{Max}_{t(z) \leqslant t' < 1} \left| f\big(Z(t')\big) \right|, \tag{3 . 31}$$

then $\epsilon = \epsilon(t)$ is a function of t which tends steadily to zero when $t \to 1$. We write $\epsilon(t_1) = \epsilon_1$, and so on.

We denote generally by $\delta = \delta(\epsilon)$ a positive function of ϵ which tends to zero with ϵ.

LEMMA 7.—*Suppose that T is a point on the positive imaginary axis OY; TB_2B_1 a ray in the first quadrant making an angle $\theta < \frac{1}{4}\pi$ with TY; and TL the trisector, nearer to TB_1, of the angle B_1TY. Suppose that C cuts TB_2B_1 in B_1 and B_2, that t_1 and $t_2 > t_1$ are the parameters*

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* See p. 255 above, f.n. †.

† See E. Lindelöf, "Sur un principe général de l'analyse", *Acta Soc. Fennicae*, Vol. 46 (1915), pp. 1-35 (7-11); or P. Montel, "Sur les familles de fonctions analytiques qui admettent des valeurs exceptionelles dans une domaine", *Annales scientifiques de l'École Normale Supérieure* (3), Vol. 20 (1912), pp. 487-535 (519). Lemmas β–ϵ of A.T. embody a different proof of substantially the same theorem.

of B_1 and B_2, and that the part C' of C for which $t_1 < t < t_2$ lies entirely in the angle B_1TY. Then

$$|f(z')| < K \tag{3.32}$$

at all points z' of the segment B_1B_2; and

$$|f(z'')| < \delta(\epsilon_1) \tag{3.33}$$

*at all points z'' of any chord intercepted on TL by the arc B_1B_2.**

Let P' be a point of B_1B_2 (other than B_1 or B_2). We may suppose that P' is not a point of C'. The perpendicular to B_1B_2 at P' will meet TL in a point P''. It will also meet C'; we suppose that it does so first at P.

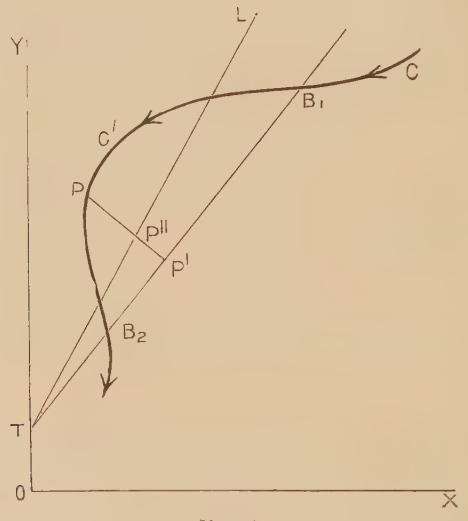

FIG. 1.

We define M as the greater of (i) ϵ_1 and (ii) the upper bound of $|f|$ on the segment B_1B_2, so that $M \geqslant \epsilon_1$. We denote by ζ a linear function of z which vanishes at T and is positive on TB_1†, so that the argument of ζ on TY is θ; and we write $F(\zeta)$ for $f(z)$. Finally we write

$$g(\zeta) = \exp\{-ik(\log\zeta - i\theta)\}, \tag{3.34}$$

where k is positive and

$$e^{k\theta} = M/\epsilon_1. \tag{3.341}$$

It may be verified at once that

$$|g| \leqslant 1 \tag{3.351}$$

* See Fig. 1. There may be no points such as z'', in which case (3.33) is naturally trivial.

† ζ is thus fixed, except for a positive multiplier.

in B_1TY, and that $|g|$ assumes the values

$$\frac{\epsilon_1}{M}, \quad \left(\frac{\epsilon_1}{M}\right)^{\frac{2}{3}}, \quad 1 \tag{3.352}$$

on TB_1, TL, and TY respectively.

If z', z'', z refer to P', P'', P, we have

$$|z''-z'| = P'P'' = \frac{\sin\frac{1}{3}\theta}{\sin\frac{2}{3}\theta}\, x'' \leqslant \tfrac{1}{2}\sec\frac{\pi}{12}\cdot x'' = K_1 x'' < K_1 x'. \tag{3.36}$$

This K_1 is less than unity, so that we may take $c = K_1$ in Lemma 6. Hence

$$|f(z'')-f(z')| < K. \tag{3.37}$$

3.3. We must now distinguish two cases.

(*a*) *P lies between P' and P''* (or at P''). Then

$$|z''-z| = PP'' \leqslant P'P'' \leqslant K_1 x'' \leqslant K_1 x. \tag{3.41}$$

In this case

$$|f(z'')| < K \tag{3.42}$$

by Lemma 6, and

$$|f(z')| < K+K < K \tag{3.43}$$

by (3.37).

(*b*) *P'' lies between P' and P.* Then P'' is a point of R, a region bounded by B_1B_2 and C' (or parts of them*). Now

$$|F| \leqslant M, \quad |g| \leqslant \epsilon_1/M$$

on B_1B_2, and

$$|F| \leqslant \epsilon_1, \quad |g| \leqslant 1$$

on C'. Hence $|Fg| \leqslant \epsilon_1$ on the boundary of R, and therefore in its interior; so that

$$|f(z'')| = |F(\zeta'')| \leqslant \frac{\epsilon_1}{|g(\zeta'')|} = M^{\frac{2}{3}}\epsilon_1^{\frac{1}{3}}, \tag{3.44}$$

by (3.352), and

$$|f(z')| < K+M^{\frac{2}{3}}\epsilon_1^{\frac{1}{3}}, \tag{3.45}$$

by (3.37) and (3.44). Comparing (3.43) and (3.45), we see that the latter holds in both cases (*a*) and (*b*).

Thus (3.45) holds for every P', and therefore

$$M < K+M^{\frac{2}{3}}\epsilon_1^{\frac{1}{3}} < K+M^{\frac{2}{3}}, \tag{3.46}$$

* R may be bounded by parts of C' only if the curve is not simple.

which is plainly impossible unless

$$M < K, \tag{3.47}$$

which proves (3.32).

To prove (3.33) we observe that, if P'' is a point of a chord intercepted by the curve on TL (which may or may not be the case), then P'' is certainly interior to an R bounded by B_1B_2 and C' (or parts of them), so that (3.44) is true. Hence

$$|f(z'')| \leqslant M^{\frac{2}{3}}\epsilon_1^{\frac{1}{3}} < K\epsilon_1^{\frac{1}{3}} < \delta(\epsilon_1), \tag{3.48}$$

which is (3.33).

3.5. LEMMA 8.—*Suppose that B is the point of C whose parameter is t; that OB makes with OY an angle $\theta < \frac{1}{4}\pi$; that B' is the last point of C, beyond B, such that no point of the arc BB' lies below OB; and that OL is the trisector, nearer OB, of the angle BOY. Then*

$$|f(z)| < K \tag{3.51}$$

at all points P of the segment BB', and

$$|f(z_1)| < \delta(\epsilon) \tag{3.52}$$

at all points P_1 of any chord intercepted on OL by the arc BB'.

It is to be observed that B' may, in extreme cases, fall at B or at O.

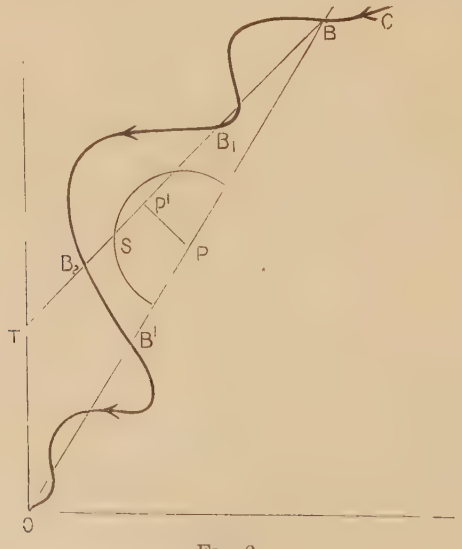

FIG. 2.

We may suppose that P is not on C'. There is then a semicircle S, with centre P and diameter along OB, which contains no point of the arc BB'. We draw a line BT, above BO, and making an angle less than $\frac{1}{4}\pi$ with OY, so as to intersect S; and we draw PP' perpendicular to BT.

There is an arc B_1B_2 of C, whose chord B_1B_2 contains P', which is related to B_1T as B_1B_2 was to B_1T in Lemma 7. Hence, by Lemma 7,

$$|f(z')| < K.$$

But BT may be drawn as close as we please to BO, so that P' is as close as we please to P. Hence, by continuity,

$$|f(z)| < K$$

which is (3 . 51).

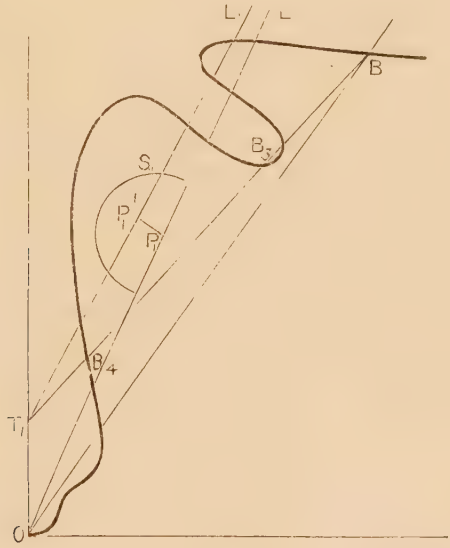

FIG. 3.

The proof of (3 . 52) is similar. We may suppose P_1 not a point of C'; there is then a semicircle S_1 related to P_1 and OL as S was to P and OB. We draw BT_1, above BO, making an angle less than $\frac{1}{4}\pi$ with OY, and so that T_1L_1, the trisector of BT_1Y nearer T_1B, intersects S_1; and P_1P_1' perpendicular to T_1L_1. There is an arc B_3B_4 of C', related to B_3T_1 in the manner of Lemma 7, and a chord intercepted by it on T_1L_1 which contains P_1'. Also $t_3 > t$, $\epsilon_3 \leqslant \epsilon$. Hence, by Lemma 7,

$$|f(z_1')| < \delta(\epsilon_3) < \delta(\epsilon).$$

But z_1' is as close as we please to z_1, and therefore

$$|f(z_1)| < \delta(\epsilon),$$

which is (3 . 52).

3·6. We can now prove the main theorem.

We denote by S, S' straight lines through the origin, in the first and fourth quadrants respectively, making with the positive or negative directions of the imaginary axis angles less than $\frac{1}{6}\pi$. We can distinguish, first, four possibilities :

(a) C cuts both an S and an S' arbitrarily near to O;

(b) C cuts an S, but no S', arbitrarily near to O;

(c) C cuts an S', but no S, arbitrarily near to O;

(d) C cuts no S or S' arbitrarily near to O.

Of these cases (b) and (c) are similar, and (a) requires only trivial changes in the argument which disposes of (b). It is sufficient then to discuss (b) and (d).

Case (b).—It is plain, first, that C lies entirely above some S', say S_1'. Let S_1 be an S cut by C arbitrarily near to O, and draw OB so that S_1 is the trisector of BOY nearer OB. Then OB makes with OY an angle less than $\frac{1}{4}\pi$.

Let C_0 be the curve derived from C by replacing every arc of C above S_1 by the corresponding chord. Then the parts of C_0 which are parts of S_1 correspond to arcs of C arbitrarily near to O; and so, by Lemma 8,

$$f(z) \to 0$$

when $z \to 0$ along C_0. But C_0 is a Stolz-path, since it lies between S_1 and S_1', and our conclusion follows from our former Theorem **S**.

Case (d).—This case falls into three, according as (d 1) C lies ultimately between an S and an S_1 or (d 2) lies ultimately above every S or (d 3) lies ultimately below every S_1. In (d 1) C is a Stolz-path, and we can use Theorem **S**. It is enough then to consider (d 2).

We can draw an S (say OB) which meets C for the last time* in B : and the bisector OB' of BOY will meet C for the last time in B', say. Then, $|f| < K$ on OB, by Lemma 8; and so $|f| < K$ in the region R bounded by OB and C. But $f \to 0$ when $z \to 0$ along C, and therefore† $f \to 0$ when $z \to 0$ along OB'. As OB' is a Stolz-path, our conclusion follows once more from Theorem **S**.

* *I.e.* for the greatest t.

† See the memoirs of Lindelöf and Montel referred to on p. 263.

4. *Summary of conclusions.*

4.1. The results of our researches may be stated comprehensively in the following theorem.

THEOREM A.—*If*

$$a_n = O\left(\frac{1}{n}\right), \tag{O}$$

then the necessary and sufficient condition that

$$\Sigma a_n = A \tag{K}$$

is that

$$\Phi(x) = \frac{1}{1-x}\,\Sigma\,\frac{a_n}{n+1}\,(1-x^{n+1}) \to A \tag{\Lambda}$$

when $x \to 1$ *along some path* C. *This condition may also be written*

$$\Phi(x) = \frac{1}{1-x}\int_x^1 f(u)\,du \to A, \tag{\Lambda'}$$

where

$$f(x) = \Sigma a_n x^n,$$

and the path of integration is the straight line $(x, 1)$ *or, if* C *is regular,* C *itself.*

If (Λ) *is satisfied for any path* C, *it is satisfied for all paths* C.

If (O) *is replaced by any condition*

$$a_n = O\left(\frac{\phi(n)}{n}\right), \tag{O'}$$

where $\phi(n)$ *tends to infinity with* n, *then* (Λ) *ceases to be either a necessary or a sufficient condition for* (K) ; *either proposition may be true and the other false.*

Assuming (O) *to be satisfied, the condition that*

$$f(x) \to A, \tag{L}$$

along some path C, *is a sufficient condition for* (K). *But it is not necessary, and does not become necessary even when* (O) *is replaced by any more exacting condition of the type*

$$a_n = O(\chi_n) \tag{O''}$$

where χ_n *is positive and* $\Sigma\chi_n$ *divergent. In any such case* (K) *may be true and* (L) *false.*

NOTE ON CONTINUOUS TRANSFORMATIONS

By B. DE KERÉKJÁRTÓ.

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1. In his paper, "On Change of Independent Variables in a Multiple Integral",* Prof. W. H. Young establishes a set of sufficient conditions for the formula

$$\iint \dots \int \phi(x_1, x_2, \dots, x_n)\, dx_1 \dots dx_n = \iint \dots \int \phi\{x_1(u_1, \dots, u_n), \dots\} \frac{\partial(x_1, \dots x_n)}{\partial(u_1, \dots u_n)}\, du_1 \dots du_n.$$

The essential advance in Prof. Young's conditions, as compared with those previously formulated by other writers, is that the transformation is not assumed explicitly to be one-to-one in both directions.

We confine ourselves, as does Prof. Young, to the two-dimensional case. Prof. Young's primary condition is then this, that

$$x = x(u, v), \qquad y = y(u, v), \tag{1}$$

where $x(u, v)$ and $y(u, v)$ are one-valued and continuous functions, and that the image of every rectangle in the (u, v)-plane, with sides parallel to the axes, is a simple Jordan curve. We prove here by direct and elementary reasoning that in these circumstances the transformation is in fact necessarily one-to-one.

2. The proof consists of a few simple remarks.

(*a*) The transformation

$$x = x(u, v), \qquad y = y(u, v)$$

* *Proc. Roy. Soc.*, (A), Vol. 96 (1919), pp. 82–91.

cannot be constant in any domain of the (u, v)-plane; that is to say, the image of that domain cannot be a single point. In fact, there is in the domain a rectangle with sides parallel to the axes whose image is a simple Jordan curve but not a single point.

(*b*) The image of a domain of the (u, v)-plane cannot be a single simple Jordan curve j. To a sufficiently small rectangle in the domain with sides parallel to the axes corresponds a small Jordan curve, which is a subset of j; but a simple Jordan curve has no true subset which is itself a simple (closed) Jordan curve.

(*c*) A straight segment

$$u = \text{const.}, \quad v_1 \leqslant v \leqslant v_2 \qquad (\text{or} \quad v = \text{const.}, \quad u_1 \leqslant u \leqslant u_2)$$

cannot be transformed into a single point. Let us suppose the segment l: $v = \text{const.} = v_0$, $u_1 \leqslant u \leqslant u_2$, is transformed into the single point P; the rectangle $(u_1, u_2; v_0, v_2)$, that is

$$v = v_0, \quad u_1 \leqslant u \leqslant u_2; \qquad v_0 \leqslant v \leqslant v_2, \quad u = u_2;$$
$$v = v_2, \quad u_2 \geqslant u \geqslant u_1; \qquad v_2 \geqslant v \geqslant v_0, \quad u = u_1;$$

has for image a simple Jordan curve j which passes through the point P. The rectangle $(u_1, u_2; v_1, v_2)$ $(v_1 < v_0)$ has for image a simple Jordan curve which contains the image of the rectangle $(u_1, u_2; v_0, v_2)$ as a subset, so that the image of the rectangle $(u_1, u_2; v_1, v_2)$ is the same Jordan curve j for every value of v_1, v_2. The image of the strip $u_1 \leqslant u \leqslant u_2$, $-\infty < v < +\infty$, is the Jordan curve j in contradiction to the proposition (*b*).

(*d*) A straight segment l: $v = v_0$, $u_1 \leqslant u \leqslant u_2$, cannot have for image a simple (closed) Jordan curve. If the image of the segment l is a simple Jordan curve, each rectangle $(u_1, u_2; v_0, v_1)$ (v_1 arbitrary) has for image the same curve j, so that the image of the strip

$$u_1 \leqslant u \leqslant u_2, \quad -\infty < v < +\infty,$$

is a single curve j.

(*e*) The image of a straight segment l: $v = v_0$, $u_1 \leqslant u \leqslant u_2$ is a simple Jordan arc. The image of l is a closed connected subset of a simple Jordan curve j; such a subset cannot be a single point, and it cannot be identical with the whole curve j; hence the image of l is a simple Jordan arc.

(*f*) The image of the two segments

$$l_1: \quad u_1 \leqslant u \leqslant u_2, \quad v = v_1,$$
$$l_2: \quad u = u_2, \quad v_1 \leqslant v \leqslant v_2,$$

is a simple Jordan arc. The two segments l_1 and l_2 form, together with the segments

$$u_2 \geqslant u \geqslant u_1, \quad v = v_2,$$

$$u = u_1, \quad v_2 \geqslant v \geqslant v_1,$$

a rectangle $(u_1, u_2; v_1, v_2)$, whose image is a simple Jordan curve j; the image of l_1+l_2 is an arc of j which cannot be a single point; further, it cannot be identical with the whole curve j; otherwise every rectangle $(u_0, u_2; v_1, v_0)$ $(u_0 \leqslant u_1, v_0 \geqslant v_2, u_0, v_0$ arbitrary) has for image the curve j, and so the image of the domain $-\infty < u \leqslant u_1, v_2 \leqslant v < +\infty$ is the single curve j in contradiction to the proposition (*b*). For the same reason the image of the three segments $l_1+l_2+l_3$:

$$\begin{cases} l_1: & u_1 \leqslant u \leqslant u_2, \quad v = v_1, \\ l_2: & u = u_2, \quad v_1 \leqslant v \leqslant v_2, \\ l_3: & u_2 \geqslant u \geqslant u_3, \quad v = v_2, \end{cases}$$

is a simple Jordan arc.

(*g*) Let $l: v = v_0, u_1 \leqslant u \leqslant u_2$, be an arbitrary segment; its image is a simple Jordan arc with the endpoints P, Q. We shall show that the endpoints P and Q correspond to the endpoints (u_1, v_0) and (u_2, v_0) of the segment l. Let $(u_1, u_2; v_1, v_2)$ be an arbitrary rectangle, $v_1 < v_0 < v_2$, such that the images of the rectangles $(u_1, u_2; v_1, v_0)$ and $(u_1, u_2; v_0, v_2)$ are not identical. We denote the sum of the three segments

$$\begin{cases} u_1 \leqslant u \leqslant u_2, & v = v_1, \\ u = u_1, & v_1 \leqslant v \leqslant v_0, \\ u = u_2, & v_1 \leqslant v \leqslant v_0, \end{cases}$$

by k_1, the sum of the three segments

$$\begin{cases} u_1 \leqslant u \leqslant u_2, & v = v_2, \\ u = u_1, & v_0 \leqslant v \leqslant v_2, \\ u = u_2, & v_0 \leqslant v \leqslant v_2, \end{cases}$$

by k_2, and the images of k_1, k_2, l by k_1', k_2', l'. The arcs k_1' and l' form a simple Jordan curve j, the arcs $k_2'+l'$ a curve j_2; let j be the image of k_1+k_2. The curves j_1 and j_2 are not identical, by our hypothesis; hence the arcs k_1' and k_2' are not identical either. If the endpoint (u_1, v_0) of l

corresponds to a point R of l' different from the endpoints P and Q of l', the arcs k_1' and k_2' pass through the point R of l'; further, both arcs k_1' and k_2' contain the points P and Q. As the arcs $k_1'+l'$ and $k_2'+l'$ form two not identical simple closed curves, $k_1'+k_2'$ cannot be a simple closed curve. This is a contradiction to our condition. We have the result that to the endpoints of the segment l correspond the endpoints of the arc (PQ) which is the image of l.

(h) By the same reasoning we obtain the following proposition:

Let l_1+l_2 denote the two segments

$$\begin{cases} u_1 \leqslant u \leqslant u_2, & v = v_1, \\ u = u_2, & v_1 \leqslant v \leqslant v_2, \end{cases}$$

and let a be the simple Jordan arc which is the image of l_1+l_2. The endpoints of a correspond to the endpoints (u_1, v_1), (u_2, v_2) of l_1+l_2.

(i) It follows that two different points (u_1, v_1), (u_2, v_2) have necessarily two different points for images. That is to say, the transformation is uniform in both senses.

3. Let us suppose that the transformation is defined only in the square C:

$$0 \leqslant u \leqslant 1, \quad 0 \leqslant v \leqslant 1$$

of the (u, v)-plane. Under the same conditions as above we obtain the result that the transformation is uniform in both senses for the interior of the square C. On the boundary of C there is a set of segments such that each of them has a single point for image, two different segments having two different points for images.

It is indeed enough to assume that the transformation (1) transforms every rectangle with sides parallel to the axes into a simple Jordan curve or into a single point. In this case there are four corner domains of the square at most in which the transformation is constant. A corner domain belonging to the corner (u_0, v_0) (u_0, v_0 are 0 or 1) contains with any one of its points (u_1, v_1) all the points in the interior and on the boundary of the rectangle $(u_0, u_1; v_0, v_1)$; the boundary of the corner domain is composed of segments $u = \text{const.}$ and $v = \text{const.}$ which form together a simple Jordan curve. The simply connected domain which remains after leaving out the points of the corner domains is bounded by a simple Jordan curve: in the interior of this curve the transformation is uniform in both senses.

4. We observe finally that the condition that the functions x and y are one-valued is also superfluous. Given a continuous * transformation of the square C which transforms every rectangle with sides parallel to the axes into a simple Jordan curve: either the transformation is uniform in both senses with the exception of four corner domains of C at most, each of them being transformed into a single point; or the image of the whole square consists of a single simple Jordan curve j, so that to each point of the square corresponds a set of points everywhere dense on this curve, and to each point of the curve corresponds a set of points everywhere dense in the square C.

* That is to say, if the points P_1, P_2, ... converge to the point P, their images, the sets $\{P_1'\}$, $\{P_2'\}$, ... converge to the set $\{P'\}$, which is the image of P, or to a subset of $\{P'\}$.

FUNCTIONS OF INTERVALS

By J. C. BURKILL.

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Introduction.

A fundamental concept of analysis is that of a function of one or more (say q) independent variables. A set of values of the independent variables is conveniently represented by a *point* in a space of q dimensions. We shall accordingly often speak of such a function as a *function of points.*

A function of points is defined by a set of rules associating a definite number with each point of a certain aggregate of points. The idea of a function has been extended by supposing rules to be formulated which associate a definite number with each *set of points* of a certain aggregate of sets of points. We have then a *function of sets.*

In the particular case when the sets of points are restricted to be intervals, we have a *function of intervals.* The term *interval* is used in the general sense of an interval in a space of q dimensions, namely, the set of points

$$a_1 < x_1 < b_1, \ldots, \quad a_q < x_q < b_q;$$

it will usually be immaterial for the purposes of this paper whether any or all of the frontier points are included. To the best of the writer's knowledge, the only functions of intervals which have hitherto been studied have been restricted to be *additive.** No account seems to have been given of even the elementary properties of general functions of intervals. This omission is the more remarkable in that almost every process of

* See, for example, de la Vallée Poussin: "Sur l'intégrale de Lebesgue", *Trans. Amer. Math. Soc.*, Vol. 16 (1915), p. 458.

analysis involves the manipulation of functions of intervals, which are not usually additive.

It is sufficient to give two examples to support this statement. In the first place, the whole of the differential calculus is based on limits of functions of intervals of the type

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}.$$

Again, the Riemann integral is defined by means of sums of functions of intervals

$$M_i(x_i-x_{i-1})$$

and

$$m_i(x_i-x_{i-1}),$$

where M_i and m_i are the upper and lower bounds of $f(x)$ in the interval (x_{i-1}, x_i).

In these examples the functions of intervals are defined by operations on a function of points. The mode of definition is, of course, irrelevant to the general concept of a function of intervals. We lay down the following formal definition.

DEFINITION.—*If rules exist which associate a unique number $g(I)$ with each interval I of a set S of intervals, then $g(I)$ is said to be a function of intervals defined in the field S.*

In this paper we develop some elementary properties of functions of intervals (nowhere assumed to be additive). Many of our results will be recognized as extensions of well known theorems for functions of points.

In § 2 the *integral* of $g(I)$ over an interval is defined and some of its properties investigated. In § 3 we introduce the idea of an *absolutely continuous* function of intervals, and in § 4 we define the integral of such a function over a measurable set; this development is a generalization of the classical theory of measure.

In § 6 the *derivative* of a function of intervals is introduced. In § 7 we prove our main result, that *the integral over a measurable set of an absolutely continuous function of intervals is equal to the Lebesgue integral of its derivative.* This includes as a special case the fundamental theorem of Lebesgue integration.

In another paper, the result of § 7 will be applied to effect the representation of the area of a surface as a double integral.

We shall use the following abbreviations:—

a.e. meaning *almost everywhere* (viz. except at a set of points of measure zero).

mI = the *measure* of I, *i.e.* the length, area ... of I in the usual sense (the product of $b_1-a_1, \ldots, b_q-a_q$).

nI = the *norm* of I = the length of the greatest side of I (*i.e.* the greatest of $b_1-a_1, \ldots, b_q-a_q$).

square meaning an interval (in any number of dimensions) having all its sides equal.

$\subset$ meaning *contained in.*

1. *Continuity.*

We now extend to functions of intervals some of the ideas used in the classification of functions of points.

DEFINITIONS.

Bounded function of intervals.

$g(I)$ is said to be bounded in R if constants H, K exist such that, for every $I \subset R$,

$$H < g(I) < K.$$

Oscillation of $g(I)$ at the point x.

Let $\omega(\delta, x)$ be the upper bound of $|g(I)|$ for every $I \subset$ a square with centre x and side δ. Then, for fixed x, $\omega(\delta, x)$ decreases as δ decreases; therefore $\lim_{\delta \to 0} \omega(\delta, x)$ exists.

The value of this limit we define to be $\omega(x)$, the oscillation of $g(I)$ at x.

Continuity at a point.

$g(I)$ is said to be continuous at x if $\omega(x) = 0$.

Continuity in a closed interval R.

$g(I)$ is said to be continuous in R if it is continuous at each point of R.

THEOREMS.—*Continuity of sum and product.*

1.1. *If $g_1(I)$ and $g_2(I)$ are continuous at x, so is $g_1(I)+g_2(I)$.*

1.2. *If $g_1(I)$ is continuous at x, and $g_2(I)$ is bounded, then $g_1(I)\,g_2(I)$ is continuous at x.*

Proofs.—1.1. Let ω_1, ω_2, ω be the oscillation functions of $g_1(I)$, $g_2(I)$, $g_1(I)+g_2(I)$ respectively. Then, for each $\delta > 0$,

$$0 \leqslant \omega(\delta, x) \leqslant \omega_1(\delta, x)+\omega_2(\delta, x).$$

Let $\delta \to 0$. Then $\quad 0 \leqslant \omega(x) \leqslant \omega_1(x)+\omega_2(x).$

But $\omega_1(x) = \omega_2(x) = 0$, and so $\omega(x) = 0$, which is the required result.

1.2. Let ω_1, ω be the oscillation functions of $g_1(I)$, $g_1(I)\,g_2(I)$ respectively, and let $|g_2(I)| < K$. Then

$$0 \leqslant \omega(\delta, x) \leqslant K\omega_1(\delta, x).$$

Let $\delta \to 0$. Hence $\quad 0 \leqslant \omega(x) \leqslant K\omega_1(x).$

But $\omega_1(x) = 0$, and so $\omega(x) = 0$, which is the result.

THEOREM 1.3.—*If at each point x of the closed interval R, $\omega(x) \leqslant k$, then, given ϵ, we can find $\lambda(\epsilon)$ such that $|g(I)| < k+\epsilon$, provided that $I \subset R$ and $nI \leqslant \lambda$.*

Proof.—Suppose the theorem untrue. Then there exists a sequence of intervals $I_1, \ldots, I_r, \ldots, \subset R$, such that

$$|g(I_r)| \geqslant k+\epsilon,$$

and $$nI_r \to 0 \quad \text{as} \quad r \to \infty.$$

The intervals I_r have at least one limit point ξ. Since R is closed, ξ is a point of R. By our definition of oscillation,

$$\omega(\xi) \geqslant k+\epsilon,$$

which contradicts the hypothesis that $\omega(x) \leqslant k$ in R.

COROLLARY. — *The Uniform Continuity Theorem for functions of intervals.*

Put $k = 0$ in Theorem 1.3, and we have :—

If $g(I)$ is continuous in the closed interval R, then, given ϵ, we can find $\lambda(\epsilon)$ such that $|g(I)| < \epsilon$, provided that $I \subset R$ and $nI \leqslant \lambda$.

We deduce the classical theorem of uniform continuity for a function of points $f(x_1, \ldots, x_q)$ by writing $g(I)$ equal to the oscillation of $f(x_1, \ldots, x_q)$ in the q-dimensional interval I.

2. *The integral.*

We may illustrate by means of the Riemann integral the idea of the integral of a function of intervals. Writing I_i for the interval (x_{i-1}, x_i) and $g(I_i) = M_i(x_i - x_{i-1})$, then the Riemann $\overline{\int}_a^b f(x)dx$ is simply the limit of $\sum_{i=1}^{n} g(I_i)$ taken as the length of the greatest I_i tends to zero. This limit we denote by $\int_a^b g(I)$.

It will be convenient to make our statements in terms of two-dimensional intervals. The extension to higher dimensions and the specialisation to one dimension will be alike clear.

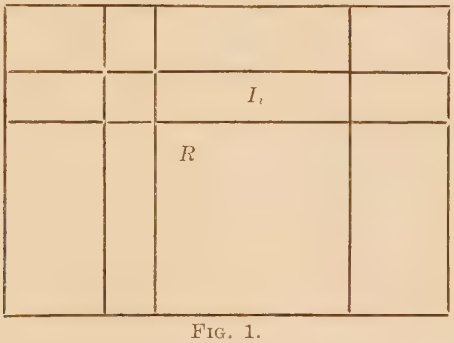

FIG. 1.

DEFINITION: $\int_R g(I)$.—*Divide R into a system of meshes $I_1, \ldots, I_n$ by lines parallel to the sides of R* (Fig. 1). *If $\sum_{i=1}^{n} g(I_i) \to$ a limit l for any sequence of mesh-systems such that max. norm $I_i \to 0$, then we say that $\int_R g(I)$ exists and $= l$.*

We have the further definitions:—

$$\overline{\int}_R g(I) = \overline{\lim} \sum_{i=1}^{n} g(I_i) \text{ as max. norm } I_i \to 0;$$

$$\underline{\int}_R g(I) = \underline{\lim} \sum_{i=1}^{n} g(I_i) \text{ as max. norm } I_i \to 0.$$

If, as max. norm $I_i \to 0$, $\sum_{i=1}^{n} g(I_i) \to +\infty$ (or $-\infty$), it is sometimes convenient to say that $\int_R g(I)$ exists and $= +\infty$ (or $-\infty$)—and similarly for $\overline{\int}_R g(I)$, $\underline{\int}_R g(I)$.

It will be observed (Fig. 2) that we may employ a more general method of subdivision by lines parallel to the sides of R, in which a dividing line need not be continued right across R. Such a system is called an *extended mesh system.*

FIG. 2.

The corresponding limits when this more general method of subdivision is allowed we call *extended integrals*, written

$$E\int g(I), \qquad E\overline{\int} g(I), \qquad E\underline{\int} g(I).$$

Clearly
$$E\underline{\int} g(I) \leqslant \underline{\int} g(I) \leqslant \overline{\int} g(I) \leqslant E\overline{\int} g(I).$$

It will generally be more convenient to work with the integral as first defined; corresponding results will hold for extended integrals. In *one dimension*, the distinction between the two kinds of integral does not occur.

The following examples illustrate the frequency with which integrals of functions of intervals present themselves in analysis.

(1) Let $f(x)$ be defined in R, $a \leqslant x \leqslant b$. Define

$$g(I_i) \doteq |f(x_i) - f(x_{i-1})|.$$

Then $\overline{\int_R} g(I)$ is the *total variation* of $f(x)$ in R.

(2) Let $y = f(x)$ be bounded and measurable in R, $a \leqslant x \leqslant b$. Suppose $H \leqslant y \leqslant K$. Let I_i be the interval (y_{i-1}, y_i). Define $g(I_i) = y_{i-1} \times$ the measure of the subset of R in which

$$y_{i-1} \leqslant f(x) < y_i.$$

Then $\int_H^K g(I)$ is the *Lebesgue integral* $\int_a^b f(x)\,dx$.

(3) Let a closed plane curve be defined by $x = x(u)$, $y = y(u)$, for $a \leqslant u \leqslant b$. Define

$$g(I_i) = g(u_{i-1}, u_i) = \tfrac{1}{2}\{x(u_{i-1})\,y(u_i) - x(u_i)\,y(u_{i-1})\}.$$

Then $\int_a^b g(I)$ is the *area* bounded by the curve.

(4) Suppose $f(x)$ bounded and $\phi(x)$ increasing in (a, b). Define

$$g(I_i) = M_i\{\phi(x_i) - \phi(x_{i-1})\},$$

where M_i is the upper bound of $f(x)$ in (x_{i-1}, x_i). Then $\int_a^b g(I)$ is the Riemann-Stieltjes* integral $\int_a^b f(x)\,d\phi(x)$.

(5) Let $h(x)$ be continuous and non-decreasing in (a, b). Let $f(x)$ be summable, and constant in any sub-interval of (a, b) in which $h(x)$ is constant. Define

$$g(I_i) = g(x_{i-1}, x_i) = \frac{\{f(x_i) - f(x_{i-1})\}^2}{h(x_i) - h(x_{i-1})}.$$

Then $\int_a^b g(I)$ is the Hellinger† integral $\int_a^b \frac{\{df(x)\}^2}{dh(x)}$.

We now proceed to the elementary properties of the integral. The following notation will be used :—

A script letter $\mathscr{E}$, $\mathscr{F}$, ... denotes a set of a finite number of intervals, not overlapping but possibly abutting.

If $\mathscr{E} = I_1 + \ldots + I_r$, $g(\mathscr{E})$ is defined to be $g(I_1) + \ldots + g(I_r)$.

The norm of $\mathscr{E}$, $n\mathscr{E}$, is the greatest of the norms of $I_1, \ldots I_r$.

$p(I)$ denotes a function of intervals which is always positive (or zero):

$$p(I) \geqslant 0.$$

THEOREMS.

2.1. If $g(I)$ is additive, $\int_R g(I) = g(R)$.

2.2. *The general principle of convergence.*—The necessary and sufficient condition that $\int_R g(I)$ exists is that given $\epsilon > 0$, there exists δ such that $|g(\mathscr{E}) - g(\mathscr{F})| < \epsilon$, where $\mathscr{E}$, $\mathscr{F}$ are any mesh-systems of R having norms $< \delta$.

* Pollard, "The Stieltjes Integral and its Generalisations," *Quarterly Journal*, Vol. 49 (1921), p. 95.

† Hobson, *The Theory of Functions of a Real Variable*, 2nd ed., Vol. 1, p. 609.

2.3. If $\int_R g(I)$ exists, so does $\int_R cg(I)$, where c is a constant, and

$$\int_R cg(I) = c\int_R g(I).$$

2.4. If $g(I) \geqslant 0$, then $\underline{\int}_R g(I) \geqslant 0$.

2.5. *The mean value theorem.*—If, for $I \subset R$,

$$HmI \geqslant g(I) \geqslant KmI,$$

then $$HmR \geqslant \overline{\int}_R g(I) \geqslant \underline{\int}_R g(I) \geqslant KmR.$$

2.6. *Integral of the sum of two functions.*—Let

$$g(I) = g_1(I)+g_2(I).$$

Then

$$\underline{\int}_R g_1(I)+\underline{\int}_R g_2(I) \leqslant \underline{\int}_R g(I) \leqslant \underline{\int}_R g_1(I)+\overline{\int}_R g_2(I) \leqslant \overline{\int}_R g(I) \leqslant \overline{\int}_R g_1(I)+\overline{\int}_R g_2(I).$$

COR.—If $\int_R g_1(I)$ and $\int_R g_2(I)$ exist, then $\int_R g(I)$ exists and

$$= \int_R g_1(I)+\int_R g_2(I).$$

2.7. *Schwarz's inequality.*

$$(1) \quad \left\{\overline{\int}_R g_1(I)\, g_2(I)\right\}^2 \leqslant \left\{\overline{\int}_R g_1^2(I)\right\}\left\{\overline{\int}_R g_2^2(I)\right\}.$$

$$(2) \quad \left\{\underline{\int}_R p_1(I)\, p_2(I)\right\}^2 \leqslant \left\{\underline{\int}_R p_1^2(I)\right\}\left\{\overline{\int}_R p_2^2(I)\right\},$$

where $p_1(I) \geqslant 0$, $p_2(I) \geqslant 0$.

2.8. *Integral over the sum of two abutting intervals.*—Let

$$R = R_1+R_2.$$

Then statements (1), (2) are *not true* :—

(1) If $\int_{R_1} g(I)$ and $\int_{R_2} g(I)$ exist, then $\int_R g(I)$ exists.

(2) If $\int_R g(I)$ exists, then $\int_{R_1} g(I)$ exists.

Statement (2) becomes true if we use the extended integral: it is true that

(3) If $E\int_R g(I)$ exists, then $E\int_{R_1} g(I)$ exists. Further, if S is any sub-interval of R, then $E\int_S g(I)$ exists.

(4) If $\int_{R_1} g(I)$, $\int_{R_2} g(I)$, $\int_R g(I)$ exist, then

$$\int_{R_1} g(I)+\int_{R_2} g(I)=\int_R g(I).$$

Proofs of Theorems 2.1–2.8.

2.1. If $\mathfrak{E}$ is any mesh-system of R, $g(\mathfrak{E})=g(R)$. Hence $\int_R g(I)$ exists and $=g(R)$.

2.1. *Necessity.*—It follows from the existence of $\int_R g(I)$ that, given ϵ, there exists δ such that

$$\left|\int_R g(I)-g(\mathfrak{E})\right|<\frac{\epsilon}{2},$$

where $\mathfrak{E}$ is any mesh-system of R having norm $<\delta$. Therefore

$$|g(\mathfrak{E})-g(\mathfrak{F})|\leqslant\left|\int_R g(I)-g(\mathfrak{E})\right|+\left|\int_R g(I)-g(\mathfrak{F})\right|<\epsilon,$$

provided that $\mathfrak{E}$, $\mathfrak{F}$ have norms $<\delta$.

Sufficiency.—Suppose that

$$\overline{\int}_R g(I)>\underline{\int}_R g(I).$$

Choose
$$\epsilon<\tfrac{1}{3}\left\{\overline{\int}_R g(I)-\underline{\int}_R g(I)\right\}. \tag{A}$$

Then, by definition of the upper and lower integrals, we can find mesh-systems $\mathfrak{E}$, $\mathfrak{F}$ of R, having norms $<\delta$, such that

$$g(\mathfrak{E})>\overline{\int}_R g(I)-\epsilon \tag{B}$$

and
$$g(\mathfrak{F})<\underline{\int}_R g(I)+\epsilon. \tag{C}$$

Combining (A), (B) and (C) we have $g(\mathfrak{E})-g(\mathfrak{F})>\epsilon$, which is a contradiction. Therefore $\int_R g(I)$ exists.

2.3. $$\sum_i g(I_i)\to\int_R g(I) \text{ as max. } nI_i\to 0.$$

Therefore $$\sum_i cg(I_i)\to c\int_R g(I) \text{ as max. } nI_i\to 0.$$

That is to say, $\int_R cg(I)$ exists, and equals $c\int_R g(I)$.

2.4. If $\mathfrak{E}$ is any mesh-system of R, $g(\mathfrak{E})\geqslant 0$. Hence

$$\underline{\lim}\, g(\mathfrak{E})\geqslant 0,$$

which is the result.

2.5. $$g(I)\leqslant HmI.$$

If $\mathfrak{E}$ is any mesh-system of R, $g(\mathfrak{E})\leqslant HmR$. Therefore

$$\bar{\int}_R g(I)=\overline{\lim}\, g(\mathfrak{E})\leqslant HmR,$$

and similarly for the other inequality.

2.6. If $\mathfrak{E}$ is any mesh-system of R, $g_1(\mathfrak{E})+g_2(\mathfrak{E})=g(\mathfrak{E})$.

(1) Given ϵ, there exists δ such that

$$g_1(\mathfrak{E})>\underline{\int}_R g_1(I)-\epsilon \quad\text{and}\quad g_2(\mathfrak{E})>\underline{\int}_R g_2(I)-\epsilon,$$

provided that $n\mathfrak{E}<\delta$.

Choose $\mathfrak{E}$ of norm $<\delta$ such that $g(\mathfrak{E})<\underline{\int}_R g(I)+\epsilon$. Then

$$\underline{\int}_R g_1(I)+\underline{\int}_R g_2(I)<\underline{\int}_R g(I)+3\epsilon.$$

But ϵ is arbitrary. Hence

$$\underline{\int}_R g_1(I)+\underline{\int}_R g_2(I)\leqslant\underline{\int}_R g(I).$$

(2) Given ϵ, there exists δ such that

$$g(\mathfrak{E})>\underline{\int}_R g(I)-\epsilon \quad\text{and}\quad g_2(\mathfrak{E})<\bar{\int}_R g_2(I)+\epsilon$$

provided that $n\mathfrak{E}<\delta$.

Choose $\mathcal{E}$ of norm $< \delta$, such that $g_1(\mathcal{E}) < \underline{\int}_R g_1(I) + \epsilon$. Then

$$\underline{\int}_R g(I) < \underline{\int}_R g_1(I) + \overline{\int}_R g_2(I) + 3\epsilon.$$

But ϵ is arbitrary. Therefore

$$\underline{\int}_R g(I) \leqslant \underline{\int}_R g_1(I) + \overline{\int}_R g_2(I),$$

and similarly for the other inequalities.

2.7 We have the inequality

$$(a_1 b_1 + \ldots + a_n b_n)^2 \leqslant (a_1^2 + \ldots + a_n^2)(b_1^2 + \ldots + b_n^2).$$

Divide R into meshes $I_1, \ldots, I_n$ and write $a_i = g_1(I_i)$ and $b_i = g_2(I_i)$. Choose a sequence of mesh-systems such that

$$\sum_{i=1}^{n} a_i b_i \to \overline{\int}_R g_1(I)\, g_2(I).$$

Therefore $$\left\{\overline{\int}_R g_1(I)\, g_2(I)\right\}^2 \leqslant \left\{\overline{\int}_R g_1^2(I)\right\}\left\{\overline{\int}_R g_2^2(I)\right\},$$

which is inequality (1).

We obtain (2) by choosing a sequence of mesh-systems such that

$$\sum_{i=1}^{n} a_i^2 \to \underline{\int}_R p_1^2(I).$$

2.8. (1) We give an example in one dimension showing that the statement is untrue.

Let R_1, R_2 be the intervals $(-1, 0)$, $(0, 1)$.

If I is an interval $(-1/n, 1/n)$ for $n = 1, 2, \ldots$, define $g(I) = (-1)^n$.

If I is any other interval in $(-1, 1)$, define $g(I) = 0$.

Then $\underline{\int}_{R_1} g(I) = \underline{\int}_{R_2} g(I) = 0$, but $\overline{\int}_R g(I) = 1$, $\underline{\int}_R g(I) = -1$.

(2) Take R_1, R_2 to be adjacent unit squares $AEFD$, $EBCF$. Let $\{\mathcal{E}_n\}$ be the class of mesh-systems of which the dividing lines parallel to AB are drawn through the points of division of AD into 2^n equal segments. Define $g(I)$ by the following rules:—

If I belongs to a mesh-system $\{\mathcal{E}_n\}$ and if $I \subset R_1$, then $g(I) = mI$.

If I belongs to a mesh-system $\{\mathcal{E}_n\}$ and if $I \subset R_2$, then $g(I) = -mI$.

If I is any other interval $\subset R$, then $g(I) = 0$.

Then $\int_R g(I)$ exists and $= 0$, but $\overline{\int}_{R_1} g(I) = 1$, $\underline{\int}_{R_1} g(I) = 0$.

(3) It is now true that *any* mesh-system $\mathfrak{E}$ of R_1 combined with *any* mesh-system $\mathfrak{F}$ of R_2 gives a mesh-system of R [this was not true in (2)].

Then $$g(\mathfrak{E})+g(\mathfrak{F}) = g(\mathfrak{E}+\mathfrak{F}).$$

If $\overline{\lim}\, g(\mathfrak{E}) > \underline{\lim}\, g(\mathfrak{E})$, then, since $\mathfrak{F}$ is independent of $\mathfrak{E}$, it would follow that $$\overline{\lim}\, g(\mathfrak{E}+\mathfrak{F}) > \underline{\lim}\, g(\mathfrak{E}+\mathfrak{F}).$$

Hence, if $E\int_R g(I)$ exists, so does $E\int_{R_1} g(I)$.

The second part of (3) is proved by repeated application of this result.

(4) Consider a mesh-system of R for which the line separating R_1 from R_2 is a line of division.

Let $\mathfrak{E}$ be the meshes in R_1 and $\mathfrak{F}$ those in R_2.

Then $$g(\mathfrak{E})+g(\mathfrak{F}) = g(\mathfrak{E}+\mathfrak{F}).$$

Let the norm of $(\mathfrak{E}+\mathfrak{F})$ tend to zero. Then, since the integrals exist,

$$\int_{R_1} g(I)+\int_{R_2} g(I) = \int_R g(I).$$

3. *Absolute continuity.*

The concept of an absolutely continuous function $f(x)$ was introduced by Vitali*; the idea is here applied to functions of intervals.†

DEFINITIONS.

Absolute continuity.

$g(I)$ is said to be absolutely continuous in R if $g(\mathfrak{E}) \to 0$ as $m\mathfrak{E} \to 0$ for all $\mathfrak{E} \subset R$.

Lipschitz condition.

$g(I)$ is said to satisfy the Lipschitz condition in R if there exists a constant K such that $|g(I)| < KmI$ for every $I \subset R$.

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* See, for example, de la Vallée Poussin, *Cours d'Analyse Infinitésimale*, 3rd edition, tome 1, p. 75.

† Cf. Carathéodory, *Vorlesungen über reelle Funktionen*, p. 502, where *additive* functions of intervals only are contemplated.

The following results are evident from the definitions.

3.1. *If $g(I)$ satisfies the Lipschitz condition in R, it is absolutely continuous in R.*

3.2. *If $g(I)$ is absolutely continuous in R, so is $|g(I)|$.*

3.3. *If $g_1(I)$ and $g_2(I)$ are absolutely continuous in R, so is*

$$g_1(I)+g_2(I).$$

3.4. *If $g_1(I)$ is absolutely continuous and $g_2(I)$ is bounded in R, then $g_1(I)g_2(I)$ is absolutely continuous in R.*

The following expression of $g(I)$ as the difference of two *positive* functions of intervals is often useful.

Write $$p(I) = \tfrac{1}{2}\{|g(I)|+g(I)\}$$

and $$n(I) = \tfrac{1}{2}\{|g(I)|-g(I)\}.$$

Then $$g(I) = p(I)-n(I)$$

and $$|g(I)| = p(I)+n(I).$$

We have from 3.2 and 3.4:

3.5. *If $g(I)$ is absolutely continuous in R, so are $p(I)$ and $n(I)$.*

The next result is important:—

THEOREM 3.6.—*If $g(I)$ is absolutely continuous in R, then each of*

$$E\bar{\int}_R g(I), \quad E\underline{\int}_R g(I), \quad E\bar{\int}_R |g(I)|$$

is finite.

Proof.—Write as above

$$g(I) = p(I)-n(I).$$

We have, using throughout the extended definition of the integral,

$$\bar{\int}_R g(I) \leqslant \bar{\int}_R p(I),$$

$$\underline{\int}_R g(I) \geqslant -\bar{\int}_R n(I),$$

and $$\bar{\int}_R |g(I)| \leqslant \bar{\int}_R p(I)+\bar{\int}_R n(I).$$

Hence it is sufficient to prove that $\overline{\int}_R p(I)$, $\overline{\int}_R n(I)$ are finite. We give the proof for the former.

Suppose that $\overline{\int}_R p(I)$ is not finite. By 3.5, $p(I)$ is absolutely continuous. Hence, given ϵ, there exists η such that

$$p(\mathfrak{E}) < \epsilon \quad \text{if} \quad m\mathfrak{E} < \eta.$$

Choose
$$N > 1 + \frac{\epsilon}{\eta} mR. \tag{A}$$

Since $\overline{\int}_R p(I)$ is not finite, we can find an extended mesh-system $\mathfrak{E}$ of arbitrarily small norm, such that

$$p(\mathfrak{E}) > N + \epsilon. \tag{B}$$

We shall suppose that each mesh of $\mathfrak{E}$ has measure $< \eta/N$. Group the meshes of $\mathfrak{E}$ to form sets $\mathfrak{E}_i$ of measure $< \eta$ and, as far as possible, $> (N-1)\eta/N$. Thus there may be *one* group of meshes having measure

$$< (N-1)\eta/N.$$

The number of these groups $\mathfrak{E}_i$ is then

$$< \frac{NmR}{(N-1)\eta} + 1.$$

Since $m\mathfrak{E}_i < \eta$, it follows that $p(\mathfrak{E}_i) < \epsilon$. Therefore

$$p(\mathfrak{E}) = \sum_i p(\mathfrak{E}_i) < \frac{NmR}{(N-1)\eta}\epsilon + \epsilon < N + \epsilon, \quad \text{by (A).}$$

But this contradicts (B); therefore the theorem is true.

COROLLARY.*—*If $f(x)$ is absolutely continuous in (a, b), it has bounded variation in (a, b).*

Write
$$g(I_i) = f(x_i) - f(x_{i-1}).$$

Then
$$\overline{\int}_a^b |g(I)| = \text{the total variation of } f(x) \text{ in } (a, b).$$

4. *The integral over a measurable set.*

We shall use the following notation:—

$$G(I) = \int_I g(I).$$

* See de la Vallée Poussin, *loc. cit.*

If $\mathfrak{E} = I_1 + \ldots + I_n$, $\qquad G(\mathfrak{E}) = \sum_{i=1}^{n} G(I_i)$.

We prove two theorems which lead up to a definition of $G(E)$, *the integral of an absolutely continuous* $g(I)$ *over a measurable set* E.

THEOREM 4 . 1.—*If*

(1) $g(I)$ *is absolutely continuous in* R,

(2) $G(I)$ *exists for every* $I \subset R$;

then $G(I)$ *is absolutely continuous in* R.

Proof.—Suppose the theorem untrue.

Then we can find $\eta > 0$ and a sequence $\mathfrak{E}_1, \ldots, \mathfrak{E}_n, \ldots$ such that

$$m\mathfrak{E}_n \to 0 \quad \text{and} \quad |G(\mathfrak{E}_n)| > 2\eta.$$

The intervals of $\mathfrak{E}_n$ may be subdivided into a set of intervals $\mathfrak{F}_n$ such that

$$|G(\mathfrak{E}_n) - g(\mathfrak{F}_n)| < \eta.$$

Then $$|g(\mathfrak{F}_n)| \geqslant |G(\mathfrak{E}_n)| - |G(\mathfrak{E}_n) - g(\mathfrak{F}_n)| > \eta,$$

which contradicts the hypothesis (1). The theorem is therefore true.

THEOREM 4 . 2.—*Suppose*

(1) $g(I)$ *is absolutely continuous in* R,

(2) $G(I)$ *exists for every* $I \subset R$,

(3) E *is a measurable set* $\subset R$.

Let ϵ_r *be a decreasing sequence* $\to 0$; *corresponding to each* r *decompose* E *into* $\mathfrak{E}_r + e'_r - e''_r$, *where* me'_r, me''_r *are* $< \epsilon_r$. *Then, as* $r \to \infty$, $G(\mathfrak{E}_r) \to$ *a limit which is independent of the particular sequence* ϵ_r, *and the particular decomposition of* E *for any* r.

Proof.—By Theorem 4 . 1, $G(I)$ is absolutely continuous and so $|G(I)|$ is absolutely continuous, by Theorem 3 . 2.

Given ϵ, we can find η such that

$$\sum_{i=1}^{n} |G(I_i)| < \epsilon \quad \text{if} \quad m \sum_{i=1}^{n} I_i < \eta.$$

Choose q so that $\epsilon_q < \frac{1}{4}\eta$, and let $s > r \geqslant q$.

Denote by $\mathfrak{E}_r \sim \mathfrak{E}_s$ the *complete difference* of $\mathfrak{E}_r$ and $\mathfrak{E}_s$, namely, the set of points which belong to one of $\mathfrak{E}_r$, $\mathfrak{E}_s$ and not to the other. Then

$$\mathfrak{E}_r \sim \mathfrak{E}_s \subset e'_r + e''_r + e'_s + e''_s.$$

Hence $$m(\mathfrak{E}_r \sim \mathfrak{E}_s) < \eta.$$

Now $\mathfrak{E}_r \sim \mathfrak{E}_s$ consists of a finite number of intervals, say $I_1, \ldots, I_n$.

Then $$|G(\mathfrak{E}_r) - G(\mathfrak{E}_s)| \leqslant \sum_{i=1}^{n} |G(I_i)| < \epsilon.$$

Therefore $G(\mathfrak{E}_r) \to$ a limit as $r \to \infty$. Q.E.D.

We are now in a position to define $G(E)$.

DEFINITION.—$G(E) = \lim_{r\to\infty} G(\mathfrak{E}_r)$, *where* $\mathfrak{E}_r$ *is defined as in Theorem* 4.2.

If E is an interval or a set of a finite number of intervals, it is clear that the value of $G(E)$ given by the definition is consistent.

Properties of $G(E)$.

THEOREM 4.3.—$G(E)$ *is an absolutely continuous function of measurable sets.*

Proof.—Suppose the theorem untrue.

Then we can find $\eta > 0$, and a sequence $E_1, \ldots, E_n, \ldots$ of measurable sets, such that $mE_n \to 0$ as $n \to \infty$, for which $|G(E_n)| > 2\eta$.

For each n choose $\mathfrak{E}_n$, such that

$$|G(E_n) - G(\mathfrak{E}_n)| < \eta$$

and $$|m\mathfrak{E}_n - mE_n| < \tfrac{1}{2} mE_n.$$

Then we have a sequence $\mathfrak{E}_n$, such that $m\mathfrak{E}_n \to 0$, for which $|G(\mathfrak{E}_n)| > \eta$, and this contradicts Theorem 4.1. Q.E.D.

THEOREM 4.4.—$G(E)$ *is a completely additive function of measurable sets, i.e. if* $E_1, \ldots, E_n, \ldots$ *are measurable and* $\subset R$, *and if no two have a common point, then*

$$G(E_1 + \ldots + E_n + \ldots) = G(E_1) + G(E_2) + \ldots + G(E_n) + \ldots.$$

Proof.—We first prove the result for two sets E_1, E_2.

Let $$E_1 = \mathfrak{E}_1 + e_1' - e_1'',$$

$$E_2 = \mathfrak{E}_2 + e_2' - e_2''$$

be decompositions of E_1, E_2 such that $me_1', \ldots, me_2''$ are $< \epsilon$.

Since E_1 and E_2 have no common point, $\mathfrak{E}_1\mathfrak{E}_2 \subset e_1'' + e_2''$. Hence

$$m\mathfrak{E}_1\mathfrak{E}_2 < 2\epsilon.$$

By Theorem 2.8 (4),

$$G(\mathfrak{E}_1 + \mathfrak{E}_2) = G(\mathfrak{E}_1) + G(\mathfrak{E}_2) - G(\mathfrak{E}_1\mathfrak{E}_2).$$

Let $\epsilon \to 0$. Then the limits of these four terms are respectively $G(E_1+E_2)$, $G(E_1)$, $G(E_2)$, 0. Thus our result is proved for two sets and hence at once for any finite number.

To extend to a denumerable infinity of sets, write

$$E_{n+1}+E_{n+2}+\ldots = R_n.$$

Then R_n is measurable and $mR_n \to 0$ as $n \to \infty$.*

Hence by Theorem 4.3, $G(R_n) \to 0$ as $n \to \infty$.

But $$G(E_1+E_2+\ldots) = G(E_1)+\ldots+G(E_n)+G(R_n).$$

Therefore $$G(E_1+E_2+\ldots) = G(E_1)+\ldots+G(E_n)+\ldots.$$ Q.E.D.

COROLLARY.—*If E_n is measurable and $\to E$, then $G(E_n) \to G(E)$.*

For E is measurable† and $mE_n \to mE$.

The theory which we have here developed, leading to the absolutely continuous completely additive function of measurable sets $G(E)$ is a generalisation of the familiar theory which leads to the measure function mE having the same two properties.

To establish the theory of measure (say in one dimension) we start with the additive function of intervals $g(I_i) = x_i - x_{i-1}$. Similarly for the variation of a function $f(x)$ we start with the additive function of intervals $g(I_i) = f(x_i) - f(x_{i-1})$. In the present section we have not restricted our basic function of intervals to be additive but only to be absolutely continuous.

We notice that the integral over an interval of an absolutely continuous function of intervals has the same value, whether the frontier points are included or not, since the set of frontier points has zero measure.

5. *Functions which increase by subdivision.*

We now define a specially simple type of non-additive function of intervals which often presents itself in practice.

DEFINITION.—If I is any interval $\subset R$, and if $I = I_1+I_2$, then $g(I)$ is said to *increase by subdivision* in R if

$$g(I) \leqslant g(I_1)+g(I_2).$$

Similarly $g(I)$ *decreases by subdivision* if $g(I) \geqslant g(I_1)+g(I_2)$.

* See de la Vallée Poussin: *Intégrales de Lebesgue, fonctions d'ensemble, classes de Baire*, p. 26.

† *Ibid.*, *loc. cit.*, p. 27.

It will be noticed that if $g(I)$ increases, then $-g(I)$ decreases by subdivision; also that a function which both increases and decreases by subdivision is additive, and conversely.

THEOREM 5.1.—*If, in* R, $g(I)$

(1) *increases by subdivision,*

(2) *is absolutely continuous,*

then $\int_R g(I)$ *exists and is finite.*

In the particular case when R *is a one-dimensional interval, if we replace condition* (2) *by*

(2a) $g(I)$ *is continuous,*

then $\int_R g(I)$ *exists, where the integral may be* $+\infty$.

Proof.—We give first a proof for a rectangle. The argument is of the well-known Darboux type.

By Theorem 3.6, $\overline{\int}_R g(I)$ is finite. Let l be its value.

Given ϵ, we can find a division of R into meshes $\mathfrak{E}_1$ such that

$$g(\mathfrak{E}_1) > l - \frac{\epsilon}{2}.$$

Let $\mathfrak{E}_2$ be a division of R into meshes of norm $\leqslant \lambda$, where $\lambda <$ the least side of a mesh of $\mathfrak{E}_1$.

Let $\mathfrak{E}_3$ denote the set of meshes formed by the aggregate of the dividing lines of the mesh-systems $\mathfrak{E}_1$, $\mathfrak{E}_2$. Since $g(I)$ increases by subdivision, it follows that

$$g(\mathfrak{E}_3) \geqslant g(\mathfrak{E}_1) \quad \text{and} \quad g(\mathfrak{E}_3) \geqslant g(\mathfrak{E}_2).$$

Also

$$g(\mathfrak{E}_3) - g(\mathfrak{E}_2) = \Sigma_1\{g(I_1)+g(I_2)-g(I_1+I_2)\}$$

$$+\Sigma_2\{g(I_1)+g(I_2)+g(I_3)+g(I_4)-g(I_1+I_2+I_3+I_4)\},$$

where Σ_1, Σ_2 are summed respectively over those meshes of $\mathfrak{E}_2$ which are divided into two and into four parts by the dividing lines of the mesh-system $\mathfrak{E}_1$.

Now the total measure of these divided meshes is $\leqslant (pb+qa)\lambda$, where p, q are the numbers of dividing lines of the system $\mathfrak{E}_1$ which are parallel to the sides b, a of R respectively. This measure $\to 0$ with λ.

Since $g(I)$ is absolutely continuous, there exists λ_0 such that

$$\Sigma_1+\Sigma_2 < \tfrac{1}{2}\epsilon,$$

provided that $\lambda \leqslant \lambda_0$. Hence

$$g(\mathfrak{E}_2) > g(\mathfrak{E}_3)-\tfrac{1}{2}\epsilon \geqslant g(\mathfrak{E}_1)-\tfrac{1}{2}\epsilon > l-\epsilon,$$

provided that $\lambda \leqslant \lambda_0$.

Therefore $\int_R g(I)$ exists and $= l$.

If R is one-dimensional, we have, by the same reasoning,

$$g(\mathfrak{E}_3)-g(\mathfrak{E}_2) = \Sigma\{g(I_1)+g(I_2)-g(I_1+I_2)\}$$

summed over those intervals of $\mathfrak{E}_2$ which are divided into two parts by the points of division of the mesh-system $\mathfrak{E}_1$. Suppose that the number of these points of division is p.

By (2a), $g(I)$ is continuous.

Hence, by Theorem 1.3, corollary, $g(I)$ is uniformly continuous, *i.e.* λ_0 can be found, such that

$$|g(I)| < \epsilon/6p \quad \text{if} \quad mI \leqslant \lambda_0.$$

Therefore $$g(\mathfrak{E}_3)-g(\mathfrak{E}_2) < \epsilon/2 \quad \text{if} \quad \lambda \leqslant \lambda_0,$$

and the argument proceeds as above (l may now be $+\infty$, in which case an obvious modification is necessary). Q.E.D.

Similarly, if $g(I)$ decreases by subdivision and satisfies the appropriate condition (2) or (2a), we can prove that $\int_R g(I)$ exists, where the value of the integral may be $-\infty$ in the second case.

COROLLARY. —*Darboux's theorem in the theory of the Riemann integral.*

If $f(x)$ is bounded, the upper approximative sums S tend to $\overline{\int} f(x)dx$ as the norms of the mesh-systems tend to zero.

Write $g(I_i) = M_i \cdot mI_i$, where M_i is the upper bound of $f(x)$ in I_i; then the conditions of our theorem are satisfied.

6. *Derivates.*

The definition of the derivative of $g(I)$ at the point x, which first suggests itself is simply the limit of $g(I)/mI$, for $I \supset x$, as the norm of $I \to 0$. But it soon becomes clear that the derivative so defined does not have the properties that we should expect, and we are led to place a restriction on the class of intervals eligible in the passage to the limit. This restriction, which we proceed to explain, is substantially the same as

that introduced by Lebesgue* in his definition of the derivative of a function of measurable sets.

DEFINITIONS.

The parameter of regularity (ρ) *of an interval* I *with respect to a point* P.

Let S be the smallest square, centre P, which $\supset I$.

Then $\rho = mI/mS$ is called the parameter of regularity of I with respect to P. (Clearly $0 < \rho \leqslant 1$).

The ρ*-derivates* $u(\rho, x)$ *and* $l(\rho, x)$.

$$u(\rho, x) = \overline{\lim} \frac{g(I)}{mI} \quad \text{as} \quad mI \to 0,$$

where the parameter of regularity with respect to x of an admissible interval I is $> \rho$.

$$l(\rho, x) = \underline{\lim} \frac{g(I)}{mI}$$

under the same conditions.

Then, for fixed x, $u(\rho, x)$ increases (in the wide sense) as ρ decreases to zero and therefore tends to a limit, finite or infinite; again, $l(\rho, x)$ decreases with ρ and so tends to a limit.

The upper and lower derivates $u(x)$ *and* $l(x)$.

We define $$u(x) = \lim_{\rho \to 0} u(\rho, x)$$

and $$l(x) = \lim_{\rho \to 0} l(\rho, x).$$

The derivative $g'(x)$.

If $u(x) = l(x)$, we say that $g(I)$ has a derivative at x, written $g'(x)$, whose value is the common value of $u(x)$ and $l(x)$.

It is to be noticed that the point x may be *external* to the intervals I admitted in our definitions.

THEOREMS.

6.1. *The necessary and sufficient condition that* $g'(x)$ *exists is that* $u(\rho, x) = l(\rho, x)$ *for every* ρ *in* $0 < \rho \leqslant 1$.

6.2. *The derivates of the sum of two functions of intervals.*

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* "Sur l'intégration des fonctions discontinues," *Annales de l'école normale supérieure*, Ser. 3, tome 27 (1910), pp. 387 *et seq.*

Let $g(I) = g_1(I) + g_2(I)$, and let the derivates of a function be distinguished by the same suffix. Then

$$l_1(x) + l_2(x) \leqslant l(x) \leqslant l_1(x) + u_2(x) \leqslant u(x) \leqslant u_1(x) + u_2(x).$$

COROLLARY.—If $g_1'(x)$ and $g_2'(x)$ exist so does $g'(x)$ and its value is

$$g_1'(x) + g_2'(x).$$

6.3. Let $F(t_1, \ldots, t_n)$ be a continuous function of the set of variables $(t_1, \ldots, t_n)$, which is homogeneous and of degree one.

If $g_1(I), \ldots, g_n(I)$ have at x derivatives $g_1'(x), \ldots, g_n'(x)$, then

$$F\{g_1(I), \ldots, g_n(I)\}$$

has at x a derivative $\quad F\{g_1'(x), \ldots, g_n'(x)\}$.

Proofs.

6.1. *Necessity.*— $\quad u(x) = l(x).$

But, for any ρ, $\quad u(x) \geqslant u(\rho, x) \geqslant l(\rho, x) \geqslant l(x).$

Therefore $\quad u(\rho, x) = l(\rho, x).$

Sufficiency.—For each ρ,

$$u(\rho, x) = l(\rho, x).$$

But, as ρ decreases, $u(\rho, x)$ increases and $l(\rho, x)$ decreases (in the wide sense).

Hence each of $u(\rho, x)$, $l(\rho, x)$ is constant for all ρ in $0 < \rho \leqslant 1$.

Therefore $\quad u(x) = l(x).$

6.2. For any I, $\quad \dfrac{g_1(I)}{mI} + \dfrac{g_2(I)}{mI} = \dfrac{g(I)}{mI}.$

Fix $\rho > 0$ and take limits as $mI \to 0$. Then

$$l_1(\rho, x) + l_2(\rho, x) \leqslant l(\rho, x) \leqslant l_1(\rho, x) + u_2(\rho, x) \leqslant u(\rho, x) \leqslant u_1(\rho, x) + u_2(\rho, x).$$

Let $\rho \to 0$ and we have the required chain of inequalities.

6.3. By the definition of a homogeneous function,

$$\frac{F\{g_1(I), \ldots, g_n(I)\}}{mI} = F\left\{\frac{g_1(I)}{mI}, \ldots, \frac{g_n(I)}{mI}\right\}.$$

Fix $\rho > 0$ and let $mI \to 0$.

Since F is continuous, the limit of the right-hand side is

$$F\{g_1'(x), \ldots, g_n'(x)\}.$$

This is true for each value of ρ.

Therefore $F\{g_1(I), \ldots, g_n(I)\}$ has at x a derivative whose value is

$$F\{g_1'(x), \ldots, g_n'(x)\}. \qquad \text{Q.E.D.}$$

THEOREM 6.4.—*For any function $g(I)$, $u(x)$ and $l(x)$ are measurable* (B).

Proof.—We give the proof for $u(x)$.

Let $\psi(x, \alpha, \rho)$ be the upper bound of $g(I)/mI$ for I's of parameter $> \rho$ with respect to x, strictly interior to the square, centre x, of side α.

Fix $\rho > 0$, α, and a point x_0.

Given ϵ, there exists an interval I_0, of parameter $> \rho$ with respect to x_0, strictly interior to the square, centre x_0, of side α, such that

$$\frac{g(I_0)}{mI_0} > \psi(x_0, \alpha, \rho) - \epsilon.$$

We can find a square S, centre x_0, such that, if x is any point of S,

(1) the parameter of regularity of I_0 with respect to x is $> \rho$,

(2) I_0 is strictly interior to the square, centre x, of side α.

Then $$\psi(x, \alpha, \rho) \geqslant \frac{g(I_0)}{mI_0} > \psi(x_0, \alpha, \rho) - \epsilon.$$

Hence, for fixed α and ρ, $\psi(x, \alpha, \rho)$ is a lower semi-continuous function of x and therefore is measurable (B).

Then $u(x, \rho) = \lim_{\alpha \to 0} \psi(x, \alpha, \rho)$, being the limit of functions measurable (B), is measurable (B).*

And $u(x) = \lim_{\rho \to 0} u(x, \rho)$ is measurable (B) for the same reason.

It is interesting to classify the derivates $u(x)$, $l(x)$ according to Young's scheme.† With his notation

$\psi(x, \alpha, \rho)$, being lower semi-continuous, is of type l;

$u(x, \rho)$, being the limit of a decreasing sequence of l's, is of type ul;

$u(x)$, being the limit of an increasing sequence of ul's, is of type lul.

Similarly $l(x)$ is of type ulu.

* See de la Vallée Poussin, *Intégrales de Lebesgue*, &c., pp, 30, 32.

† See W. H. Young, "On a New Method in the Theory of Integration", *Proc. London Math. Soc.*, Vol. 9 (1910), p. 15.

If $g'(x)$ exists, then $g'(x)$, being equal to $u(x, \rho)$ and to $l(x, \rho)$, is of type ul and also of type lu, and hence by a theorem of Young's is a function of Baire's first class.

When $g(I)$ is the increment over I of a function $f(x)$, *i.e.*

$$g(I_i) = f(x_i) - f(x_{i-1}),$$

let us compare the derivates of $g(I)$ as defined in this section with the ordinary derivates of $f(x)$.

THEOREM 6.5.—*If $\bar{u}(x)$ is an upper derivate of $f(x)$, either on the right or on the left, then*

$$u(x) \geqslant \bar{u}(x),$$

and similarly

$$l(x) \leqslant \bar{l}(x).$$

Proof.—In the limiting process by which $\bar{u}(x)$ is defined, only intervals with end-point x and parameter of regularity $\frac{1}{2}$ are admitted, and these form a sub-class of the intervals by which $u(x)$ is defined.

Therefore

$$u(x) \geqslant \bar{u}(x),$$

and similarly

$$l(x) \leqslant \bar{l}(x).$$

Example.—Define

$$f(x) = x \cos(\pi/x) \quad \text{if} \quad x \neq 0,$$

$$f(0) = 0.$$

Then $\bar{u}(0) = 1$, $\bar{l}(0) = -1$, whether the derivates are taken on the right or on the left.

We can prove that

$$u(0) = +\infty.$$

Given $\rho > 0$, $\epsilon > 0$, choose an even integer μ and an odd integer ν, arbitrarily large, satisfying

$$\frac{1}{\rho} > \frac{2\mu}{\nu} + 2 > \frac{1}{\rho} - \epsilon,$$

Let I denote the interval $\left(\frac{1}{\mu+\nu}, \frac{1}{\mu}\right)$.

Then the parameter of regularity of I with respect to the origin is

$$\frac{\nu}{2(\mu+\nu)} > \rho.$$

Now $$\frac{g(I)}{mI} = \left(\frac{1}{\mu} + \frac{1}{\mu+\nu}\right) \Big/ \left(\frac{1}{\mu} - \frac{1}{\mu+\nu}\right) = 1 + \frac{2\mu}{\nu} > \frac{1}{\rho} - 1 - \epsilon.$$

As $\rho \to 0$,

$$1/\rho \to +\infty.$$

Therefore $u(0) = +\infty$, and similarly $l(0) = -\infty$.

THEOREM 6.6.—*The existence for a particular value of x of either $f'(x)$ or $g'(x)$ implies the existence of the other and the two are equal.*

Proof.—From Theorem 6.5, if $g'(x)$ exists, so does $f'(x)$ and

$$f'(x) = g'(x).$$

Suppose now that $f'(x)$ exists.

Then, given ϵ, we can find δ such that

$$\left|\frac{f(x+h)-f(x)}{h} - f'(x)\right| < \epsilon \quad \text{if} \quad |h| < \delta.$$

Fix ρ such that $0 < \rho \leqslant 1$, and take h_1, h_2 to satisfy

$$|h_1| < \delta, \quad |h_2| < \delta, \quad |h_1 - h_2| > 2\rho h,$$

where $$h = \text{max.}\,|h_1|, |h_2|.$$

We have $$f(x+h_1)-f(x)-h_1 f'(x) = \epsilon_1 h_1,$$

$$f(x+h_2)-f(x)-h_2 f'(x) = \epsilon_2 h_2,$$

where $$|\epsilon_1| < \epsilon \quad \text{and} \quad |\epsilon_2| < \epsilon.$$

Therefore $$\frac{f(x+h_1)-f(x+h_2)}{h_1-h_2} - f'(x) = \frac{\epsilon_1 h_1 - \epsilon_2 h_2}{h_1 - h_2}.$$

The right-hand side is numerically less than $2\epsilon h/2\rho h$, that is ϵ/ρ.

Hence, for any fixed ρ in $0 < \rho \leqslant 1$,

$$u(\rho, x) = l(\rho, x) = f'(x).$$

Therefore $g'(x)$ exists and equals $f'(x)$. Q.E.D.

For a function $f(x, y)$ of more than one variable, the next theorem establishes a relation between the ordinary repeated derivative and the derivative of the associated function of intervals

$$g(I) = f(x+h, y+k)-f(x+h, y)-f(x, y+k)+f(x, y),$$

where I is the rectangle whose opposite corners are (x, y) and $(x+h, y+k)$.

THEOREM 6.7.—*If $\partial^2 f/(\partial x\,\partial y)$ exists in the neighbourhood of (x_0, y_0) and if $M(x, y)$, $m(x, y)$ denote the maximum and minimum at (x, y) of $\partial^2 f/(\partial x\,\partial y)$, then*

$$M(x_0, y_0) \geqslant u(x_0, y_0) \geqslant l(x_0, y_0) \geqslant m(x_0, y_0).$$

Proof.—By the definition of $M(x, y)$, given ϵ, we can find a neighbourhood N of (x_0, y_0) such that

$$\frac{\partial^2 f}{\partial x\, \partial y} < M(x_0, y_0) + \epsilon$$

if (x, y) is any point of N.

Let I be a rectangle contained in N whose opposite corners are

$$(x, y), \quad (x+h, y+k).$$

Then, by applying twice the mean-value theorem,

$$g(I) = hk \left(\frac{\partial^2 f}{\partial x\, \partial y}\right)_{x+\theta h,\, y+\theta' k}$$

where $$0 < \theta < 1, \ 0 < \theta' < 1.$$

Hence $$\frac{g(I)}{mI} < M(x_0, y_0) + \epsilon,$$

Therefore $u(x_0, y_0) \leqslant M(x_0, y_0)$, and similarly $l(x_0, y_0) \geqslant m(x_0, y_0)$.

COROLLARY.—If $\partial^2 f/(\partial x\, \partial y)$ is continuous at (x_0, y_0), then $g'(x_0, y_0)$ exists and equals $\partial^2 f/(\partial x\, \partial y)$.

For we have $$M(x_0, y_0) = m(x_0, y_0).$$

7. *The integral of a function of intervals and the Lebesgue integral of its derivative.*

We shall in this section use the following theorem on the covering of sets of points by intervals; the statement is a particular case of that given by Carathéodory, *Vorlesungen über reelle Funktionen*, pp. 299–307.

Vitali's covering theorem.

Let E be a set of points of finite measure.

Suppose associated with each point P of E an infinite sequence of intervals

(1) whose norms $\to 0$,

and (2) whose parameters of regularity with respect to P are

$$> \rho(P) > 0.$$

Then, given ϵ, we can choose a set $\mathfrak{E}$ of a finite number of these intervals, non-overlapping, such that

$$m(E - E\mathfrak{E}) < \epsilon \quad \text{and} \quad m(\mathfrak{E} - E\mathfrak{E}) < \epsilon.$$

Notes.—(1) The intervals of $\mathfrak{E}$ may be taken to have their norms less than any positive number given in advance.

(2) In the special case when E is an interval or a set of a finite number of intervals, $\mathfrak{E}$ may be chosen so that $\mathfrak{E} \subset E$.

(3) Carathéodory remarks (footnote, p. 304) that the regularity condition (2) is essential to the truth of the theorem. This helps to explain the necessity of a similar restriction in the definition of the derivates.

(4) A point P may be external to any or all of the intervals associated with it.

We now prove theorems connecting the integral of an absolutely continuous function of intervals with the Lebesgue integral of its derivative. The results take a different form according as we use the extended or the original definition of the integral given in § 2.

Using the extended definition we obtain the very general results of Theorem 7.2, which include many of the classical theorems of Lebesgue integration.

THEOREM 7.1.—*If* $p(I) \geqslant 0$ *and* $E\int_R p(I)$ *is finite, then* $u(x)$ *and* $l(x)$ *are finite a.e. in* R.

Proof.—Let $$E\overline{\int}_R p(I) = H.$$

There exists $\epsilon > 0$ such that, if R is divided into any extended mesh-system $\mathfrak{F}$ of norm $< \epsilon$, then $p(\mathfrak{F}) < H+1$.

It is sufficient to prove that the set E in which $u(x) = +\infty$ has measure zero.

Let E_n be the subset of R in which $u(x) > n$.

Then E_n is measurable (Theorem 6.4), $E_n \supset E_{n+1}$ and $E = \lim_{n\to\infty} E_n$.

If $mE > 0$, choose k such that $mE > k > 0$.

Then, for each n, $$mE_n > k.$$

Choose $$n > (H+1)/k.$$

To each point x of E_n corresponds $\rho(x) > 0$ such that $u(\rho, x) > n$, and therefore a sequence of intervals I of parameter $> \rho$, whose norms $\to 0$, for which $$p(I) > n \cdot mI.$$

By Vitali's theorem, we can pick out a set of a finite number of intervals $\mathfrak{E} \subset R$, having norm $< \epsilon$, and such that $m\mathfrak{E} > k$.

Then $$p(\mathfrak{E}) > n \cdot m\mathfrak{E} > nk > H+1.$$

Hence there exists a division of R into an extended mesh-system $\mathfrak{F}$, of norm $< \epsilon$, such that

$$p(\mathfrak{F}) > H+1.$$

This contradicts our original hypothesis about ϵ, and so $mE = 0$.

COROLLARY 1.—*If $g(I)$ is absolutely continuous in R, then $u(x)$ and $l(x)$ are finite a.e. in R.*

Write, as in § 3, $g(I) = p(I) - n(I)$. Then, by Theorem 3.6, $E\overline{\int}_R p(I)$ and $E\overline{\int}_R n(I)$ are finite, and the result follows.

COROLLARY 2.—Putting $p(I_i) = f(x_i) - f(x_{i-1})$, where I_i is the interval (x_{i-1}, x_i), we have the familiar result:—

If $f(x)$ has bounded variation in (a, b), then $u(x)$ and $l(x)$ are finite a.e. in (a, b).

For a function of bounded variation can be expressed as the difference of two increasing functions.

THEOREM 7.2.—*If $g(I)$ is absolutely continuous in R, then*

$$E\underline{\int}_R g(I) \leqslant \int_R l(x)dx \leqslant \int_R u(x)dx \leqslant E\overline{\int}_R g(I)$$

(the integrals being Lebesgue integrals).

Hence if $E\int_R g(I)$ exists, then $g'(x)$ exists a.e. in R and

$$E\int_R g(I) = \int_R g'(x)dx$$

(any arbitrary value being assigned to $g'(x)$ at points of R where it does not exist).

Further, if E is measurable and $\subset R$, then

$$G(E) = \int_E g'(x)dx.$$

Proof.—Define $u_{hk}(x)$ by these rules:—

$u_{hk}(x) = u(x)$ if $x \subset$ the subset E of R at which $-h < u(x) \leqslant k$.

$u_{hk}(x) = -h$,, ,, H ,, $-h \geqslant u(x)$.

$u_{hk}(x) = k$,, ,, K ,, $u(x) > k$.

Then $u_{hk}(x)$ is bounded and measurable, and therefore has a Lebesgue integral in R. We shall prove that for all sufficiently large values of h, k,

$$\int_R u_{hk}(x)\,dx \leqslant E\int_R g(I). \qquad (A)$$

Theorem 7.1 shows that $\lim_{k\to\infty} mK = 0$; we shall now prove the sharper result that

$$\lim_{k\to\infty} k\,.\,mK = 0.$$

If this is untrue, there exists $\alpha > 0$ such that

$$k\,.\,mK > \alpha \text{ for arbitrarily large values of } k.$$

To each point x of K corresponds $\rho(x)$ such that $u(\rho, x) > k$, and therefore a set of intervals I, of parameter $> \rho$ with respect to x, whose norms $\to 0$, such that

$$g(I) > k\,.\,mI.$$

By Vitali's theorem, we can pick out a set $\mathfrak{E}$ of a finite number of intervals $\subset R$, such that $2mK > m\mathfrak{E} > \frac{1}{2}mK$ and

$$g(\mathfrak{E}) > k\,.\,m\mathfrak{E} > \tfrac{1}{2}k\,.\,mK > \tfrac{1}{2}\alpha.$$

But, as $k\to\infty$, $mK\to 0$, so that $m\mathfrak{E}\to 0$.
This contradicts the hypothesis that $g(I)$ is absolutely continuous.
Therefore $\lim_{k\to\infty} k\,.\,mK = 0$, and similarly $\lim_{h\to\infty} h\,.\,mH = 0$.
Given ϵ we can choose η to satisfy (a) and (b) :—

(a) If $\mathfrak{E}$ is any extended mesh-system of R having norm $< \eta$, then

$$g(\mathfrak{E}) \leqslant E\int_R g(I) + \epsilon.$$

(b) $$|\,g(\mathfrak{E})\,| < \epsilon \quad \text{if} \quad m\mathfrak{E} < \eta.$$

Choose h, k so large that $mH < \frac{1}{3}\eta$, $mK < \frac{1}{3}\eta$, and $k\,.\,mK < \epsilon$.
Take a scale $-h = l_0 < l_1 < \ldots < l_{i-1} < l_i < \ldots < l_n = k$.
Let E_i be the subset of R at which $l_{i-1} < u(x) \leqslant l_i$. Then

$$E = \sum_{i=1}^{n} E_i.$$

By the definition of the Lebesgue integral, we can choose the scale so that

$$\sum_{i-1}^{n} l_{i-1}\,mE_i > \int_E u(x)\,dx - \epsilon.$$

But $$\int_E u(x)\,dx = \int_R u_{hk}(x)\,dx - k.mK + h.mH > \int_R u_{hk}(x)\,dx - \epsilon.$$

Therefore $$\sum_{i=1}^{n} l_{i-1}\, mE_i > \int_R u_{hk}(x)\,dx - 2\epsilon. \qquad \text{(B)}$$

Let δ be the smaller of $\epsilon/2n^2k$ and $\eta/6n^2$.

We can find sets of intervals $\mathfrak{E}'$ for $i = 1, 2, \ldots, n$, such that

$$E_i = \mathfrak{E}'_i + e'_i - e''_i,$$

where me'_i, me''_i are $< \delta$.

Then $$m\mathfrak{E}'_i E_i > mE_i - \delta.$$

Remove any intervals formed by points common to two or more of the $\mathfrak{E}'_i$. We thus obtain n sets of non-overlapping intervals, $\mathfrak{E}_1, \mathfrak{E}_2, \ldots, \mathfrak{E}_n$, where

$$m\mathfrak{E}_i E_i > m\mathfrak{E}'_i E_i - 2(n-1)\delta.$$

Therefore $$m\mathfrak{E}_i E_i > mE_i - (2n-1)\delta.$$

Apply Vitali's theorem to each set $\mathfrak{E}_i E_i$. At each point x of $\mathfrak{E}_i E_i$,

$$l_{i-1} < u(x).$$

Therefore there exists $\rho(x)$ such that

$$l_{i-1} < u(\rho, x).$$

Hence we can find a sequence of intervals I, of parameter $> \rho$ with respect to x, whose norms tend to zero, for each of which

$$l_{i-1}\, mI < g(I).$$

By Vitali's theorem we can pick out a finite set $\mathfrak{F}_i$ of these intervals, having norm $< \eta$, such that

$$\mathfrak{F}_i \subset \mathfrak{E}_i$$

and $$m\mathfrak{F}_i > mE_i\mathfrak{E}_i - \delta.$$

Hence $$m\mathfrak{F}_i > mE_i - 2n\delta.$$

The $\mathfrak{F}_i$ do not overlap. Let $\mathfrak{F} = \sum_{i=1}^{n} \mathfrak{F}_i$.

Then $$m\mathfrak{F} > mE - 2n^2\delta$$
$$\geqslant mR - mH - mK - \tfrac{1}{3}\eta$$
$$> mR - \eta.$$

Hence, by (b), for any division of $R-\mathfrak{F}$ into meshes,

$$|g(R-\mathfrak{F})|<\epsilon.$$

So that, by (a), $$g(\mathfrak{F}) \leqslant E\overline{\int}_R g(I)+2\epsilon. \tag{C}$$

But
$$\begin{aligned} g(\mathfrak{F}) &= \sum_{i=1}^{n} g(\mathfrak{F}_i) \\ &> \sum_{i=1}^{n} l_{i-1} m\mathfrak{F}_i \\ &> \sum_{i=1}^{n} l_{i-1} mE_i - \sum_{i=1}^{n} l_{i-1}.2n\delta \\ &> \sum_{i=1}^{n} l_{i-1} mE_i - 2n^2 k\delta \\ &\geqslant \sum_{i=1}^{n} l_{i-1} mE_i - \epsilon. \end{aligned}$$

Combining this with (B) and (C) we have

$$\int_R u_{hk}(x)\,dx < E\overline{\int}_R g(I)+5\epsilon.$$

Since ϵ is arbitrary, $\int_R u_{hk}(x)\,dx \leqslant E\overline{\int}_R g(I)$, which is (A).

Similar arguments show that

$$\int_R l_{hk}(x)\,dx \geqslant E\underline{\int}_R g(I).$$

By the definition of the Lebesgue integral of an unbounded function it follows that $\int_R u(x)\,dx$, $\int_R l(x)\,dx$ exist and

$$E\underline{\int}_R g(I) \leqslant \int_R l(x)\,dx \leqslant \int_R u(x)\,dx \leqslant E\overline{\int}_R g(I).$$

We have finally to prove that, if E is measurable,

$$G(E) = \int_E g'(x)\,dx.$$

Let $E = \mathfrak{E}+e'-e''$, where me', me'' are $<\eta$.

By our theorem $$G(\mathfrak{E}) = \int_{\mathfrak{E}} g'(x)\,dx.$$

Let $\eta \to 0$. Then $G(\mathfrak{E}) \to G(E)$.

Also $$\int_{\mathfrak{E}} g'(x)\,dx \to \int_E g'(x)\,dx,$$

since the Lebesgue integral is absolutely continuous.

Therefore $$G(E) = \int_E g'(x)\,dx.$$ Q.E.D.

Note.—The theorem is, of course, true for intervals in any finite number (q) of dimensions, provided that x is interpreted as the set of variables $x_1, \ldots, x_q$.

We now exhibit some familiar results in the theory of integration as special cases of Theorem 7.2.

COROLLARY 1.—*If $F(x)$ is absolutely continuous in (a, b), then $F'(x)$ exists a.e. in (a, b), and*

$$F(b)-F(a) = \int_a^b F'(x)\,dx.$$

Write in the theorem, $g(I_i) = F(x_i)-F(x_{i-1})$, where I_i is (x_{i-1}, x_i).

This result may be extended to more than one dimension; we have, for example, in two dimensions:—

If $F(x, y)$ is absolutely continuous in the rectangle R, $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$, then $F''_{xy}(x, y)$ exists a.e. in R, and

$$F(b, d)-F(b, c)-F(a, d)+F(a, c) = \iint_R F''_{xy}(x, y)\,dx\,dy,$$

where $F''_{xy}(x, y)$ is the derivative of the associated function of intervals

$$g(I) = F(x+h, y+k)-F(x, y+k)-F(x+h, y)+F(x, y).$$

This statement is substantially the same as that of de la Vallée Poussin.*

COROLLARY 2.—*The relation between the Riemann and Lebesgue integrals.*†

If $M(x)$, $m(x)$ are the maximum and minimum at x of a bounded $f(x)$, then

$$(\mathrm{R})\overline{\int_R} f(x)\,dx = (\mathrm{L})\int_R M(x)\,dx$$

* *Intégrales de Lebesgue*, &c., p. 82.

† de la Vallée Poussin, *loc. cit.*, p. 54.

and $$(\mathrm{R})\underline{\int}_R f(x)\,dx = (\mathrm{L})\int_R m(x)\,dx.$$

Proof.—Let $g(I) = M.mI$, where M is the upper bound of $f(x)$ in I.

Then $g(I)$ satisfies the Lipschitz condition and therefore is absolutely continuous.

Also $g'(x) = M(x)$, so that our theorem gives

$$(\mathrm{R})\overline{\int}_R f(x)\,dx = E\int_R g(I) = (\mathrm{L})\int_R g'(x)\,dx = (\mathrm{L})\int_R M(x)\,dx,$$

and similarly for the second equation.

We now turn to the problem of connecting $\int g(I)$ with $\int g'(x)\,dx$ when the original definition of the integral is employed. The main result is given in Theorem 7.5.

THEOREM 7.3.—*If* (1) $g(I)$ *is absolutely continuous in* R,

$$(2)\ l(x) \geqslant 0 \ \textit{a.e. in}\ R,$$

then $$\overline{\int}_R g(I) \geqslant 0.$$

Proof.—Suppose that $\overline{\int}_R g(I) < 0$, say

$$\overline{\int}_R g(I) = -3k.mR,$$

where $k > 0$.

Denote by $\mathfrak{E}_r$ the mesh-system formed by dividing R by bisections into 2^{rq} equal parts (where q is the number of dimensions of R).

Take r_0 such that $\quad g(\mathfrak{E}_r) < -2k.mR \quad$ if $\quad r > r_0.$ (A)

Let $\mathfrak{F}_r$ be the set of meshes I of $\mathfrak{E}_r$ for which

$$g(I) < -k.mI.$$

Since $g(I)$ is absolutely continuous, we can find $\eta(k)$ such that

$$g(\mathfrak{E}) > -k.mR \quad \text{if} \quad m\mathfrak{E} < \eta.$$

We shall prove that, for $r > r_0$,

$$m\mathfrak{F}_r \geqslant \eta. \tag{B}$$

For if not,
$$\begin{aligned} g(\mathfrak{E}_r) &= g(\mathfrak{E}_r - \mathfrak{F}_r) + g(\mathfrak{F}_r) \\ &> -k.m(R - \mathfrak{F}_r) - k.mR \\ &\geqslant -2k.mR, \end{aligned}$$

which contradicts (A).

Let $\mathfrak{F} = \overline{\lim}\, \mathfrak{F}_r$, where the meshes of $\mathfrak{F}_r$ are regarded as open.

Then* from (B) $\qquad m\mathfrak{F} \geqslant \eta.$

Any point x of $\mathfrak{F}$ is contained in an infinity of the $\mathfrak{F}_r$.

Since the norm of $\mathfrak{F}_r \to 0$, we can find a sequence $I_1, \ldots, I_s, \ldots$ of meshes of the $\mathfrak{F}_r$'s such that

$$I_1 \supset I_2 \supset \ldots \supset I_s \supset \ldots \supset x \quad \text{and} \quad I_s \to x.$$

Moreover the parameter of regularity of each I_s with respect to x is greater than some positive constant, depending on the shape of R.

Therefore $\qquad l(x) \leqslant \underline{\lim} \dfrac{g(I_s)}{mI_s} \leqslant -k,$

i.e. $l(x) \leqslant -k$ in a set of measure $\geqslant \eta > 0$.

This contradicts (2) of the hypothesis and the theorem is proved.

THEOREM 7.4.—*If*

(1) *$g(I)$ is absolutely continuous in R,*

(2) *$G(I)$ exists for every $I \subset R$,*

(3) *$l(x) \geqslant 0$ in a measurable set $E \subset R$,*

then $\qquad G(E) \geqslant 0.$

Proof.—From (1), given ϵ, there exists η such that

$$|g(\mathfrak{E})| < \epsilon \quad \text{if} \quad m\mathfrak{E} < \eta.$$

Decompose E into $\mathfrak{E} + e' - e''$, where me', me'' are $< \eta$.

Enclose e'' in a denumerable set S of intervals such that $mS < \eta$.

Define a function of intervals $g_1(I)$

$$g_1(I) = 0 \text{ if } I \subset S,$$

$$g_1(I) = g(I) \text{ for every other } I \subset R.$$

Then $g_1(I)$ is absolutely continuous and $l_1(x) \geqslant 0$ in $\mathfrak{E}$.

Therefore from Theorem 7.3,

$$\overline{\int_{\mathfrak{E}}} g_1(I) \geqslant 0.$$

* See, for example, Carathéodory, *Vorlesungen über reelle Funktionen*, p. 255.

But for any division of $\mathfrak{E}$ into a system of meshes $\mathfrak{E}'$,

$$|g(\mathfrak{E}')-g_1(\mathfrak{E}')|<\epsilon.$$

Hence $$G(\mathfrak{E})-\overline{\int}_{\mathfrak{E}} g_1(I) \geqslant -\epsilon,$$

which gives $$G(\mathfrak{E}) \geqslant -\epsilon.$$

Let $\eta \to 0$. Then $G(E) = \lim_{\eta \to 0} G(\mathfrak{E}) \geqslant -\epsilon.$

But ϵ is arbitrary; therefore $G(E) \geqslant 0.$ Q.E.D.

Theorems 7.3, 7.4 form the extension to the general function of intervals of de la Vallée Poussin's "fundamental lemma" in the theory of additive functions of sets.*

THEOREM 7.5.—*If*

(1) $g(I)$ *is absolutely continuous in* R,

(2) $G(I)$ *exists for every* $I \subset R$,

(3) E *is a measurable set* $\subset R$,

then $$\int_E l(x)dx \leqslant G(E) \leqslant \int_E u(x)dx.$$

Hence, if $g'(x)$ *exists a.e. in* E,

$$G(E) = \int_E g'(x)dx$$

Proof.—Applying Theorem 7.2 to $G(I)$, we deduce that $G'(x)$ exists a.e., and

$$G(E) = \int_E G'(x)dx.$$

Hence it is sufficient to prove that $l(x) \leqslant G'(x)$ a.e. in R.

Suppose that this is untrue; then we can find k and a set E of measure > 0 such that, for $x \subset E$,

$$l(x)-G'(x)-k \geqslant 0.$$

Now $l(x)-G'(x)-k$ is the lower derivate of $g(I)-G(I)-k.mI$.

Therefore, by Theorem 7.4,

$$\int_E \{g(I)-G(I)-k.mI\} \geqslant 0,$$

i.e. $$G(E)-G(E)-k.mE \geqslant 0,$$

which is untrue.

* "Sur l'intégrale de Lebesgue", *Trans. American Math. Soc.*, Vol. 16 (1915), p. 455.

Hence $l(x) \leqslant G'(x)$ a.e. and similarly $G'(x) \leqslant u(x)$ a.e. Q.E.D.

Finally we show that, in one dimension, the inequalities of Theorem 7.2 become equalities.

THEOREM 7.6.—*If*

(1) *R is one-dimensional,*

(2) *$g(I)$ is absolutely continuous in R,*

then
$$\overline{\int}_R g(I) = \int_R u(x)dx$$

and
$$\underline{\int}_R g(I) = \int_R l(x)dx.$$

Proof.—By Theorem 7.2,
$$\int_R u(x)dx \leqslant \overline{\int}_R g(I).$$

Suppose that
$$\int_R u(x)dx < \overline{\int}_R g(I),$$

say
$$\overline{\int}_R g(I) - \int_R u(x)dx = 3k.mR,$$

where $k > 0$.

Let $\epsilon_1, \epsilon_2, \ldots$ be a decreasing sequence tending to zero.

Corresponding to ϵ_r, we can find a mesh-system $\mathfrak{E}_r$ of R, having norm $< \epsilon_r$, such that
$$g(\mathfrak{E}_r) - \int_R u(x)dx > 2k.mR. \tag{A}$$

Let $\mathfrak{F}_r$ be the set of meshes I of $\mathfrak{E}_r$ for which
$$g(I) - \int_I u(x)dx > k.mI. \tag{B}$$

Since $g(I)$ and $\int_I u(x)dx$ are absolutely continuous functions of intervals, we can find η such that
$$g(\mathfrak{E}) - \int_{\mathfrak{E}} u(x)dx < k.mR,$$

provided that $m\mathfrak{E} < \eta$.

We shall prove that $m\mathfrak{F}_r \geqslant \eta$.

For if not,

$$g(\mathscr{E}_r)-\int_{\mathscr{E}_r} u(x)\,dx = \left\{g(\mathscr{E}_r-\mathscr{F}_r)-\int_{\mathscr{E}_r-\mathscr{F}_r} u(x)\,dx\right\}+\left\{g(\mathscr{F}_r)-\int_{\mathscr{F}_r} u(x)\,dx\right\}$$

$$\leqslant k.m(R-\mathscr{F}_r)+k.mR$$

$$\leqslant 2k.mR,$$

which contradicts (A).

As in Theorem 7.3, let $\mathscr{F}=\overline{\lim}\,\mathscr{F}_r$. Then $m\mathscr{F}\geqslant\eta$.

The Lebesgue integral $\int_I u(x)\,dx$ has derivative $u(x)$ a.e.

Let x be a point of $\mathscr{F}$ at which $\int_I u(x)\,dx$ has derivative $u(x)$.

As in Theorem 7.3, we can find a sequence $I_1, \ldots, I_s, \ldots$ of meshes of the $\mathscr{F}_r$ such that

$$I_1\supset I_2\ldots\supset I_s\supset\ldots\supset x \quad\text{and}\quad I_s\to x.$$

Moreover the parameter of regularity of each I_s with respect to x is greater than $\frac{1}{2}$.

Then $\overline{\lim}\,\dfrac{g(I)}{mI_s}\leqslant u(x)$ and $\lim\dfrac{1}{mI_s}\int_{I_s}u(x)\,dx=u(x)$.

Therefore we have a contradiction in (B), and the theorem is true.

COROLLARY.—*If*

(1) *R is one-dimensional,*

(2) *$g(I)$ is absolutely continuous in R,*

then the existence of either $\int_R g(I)$ or $\int_R g'(x)\,dx$ implies the existence of the other, and the two are equal.

THE EXPRESSION OF AREA AS AN INTEGRAL

By J. C. BURKILL.

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Introduction.

The purpose of this paper is to develop a theory of the area of curved surfaces. There are four conditions which such a theory should satisfy:

(1) its definition of area must apply to a large class of surfaces;

(2) for simple surfaces the definition must lead to the familiar value for the area;

(3) the theory should provide necessary and sufficient conditions (or, at least, general sufficient conditions) in terms of the functions defining the surface, that the area should exist;

(4) if a surface has an area, an analytical formula should be given from which it can be calculated.

Let us view the known theory of the length of a curve in the light of conditions analogous to those which we have laid down.

Suppose the curve defined by $x = x(u)$, $y = y(u)$, where $a \leqslant u \leqslant b$. Then the necessary and sufficient conditions that the curve should have a finite length are that $x(u)$ and $y(u)$ should have bounded variation in (a, b), so that (3) is satisfied. If, further, $x(u)$ and $y(u)$ are absolutely continuous, the length is given by the Lebesgue integral

$$\int_a^b \sqrt{[\{x'(u)\}^2 + \{y'(u)\}^2]}\, du;$$

thus condition (4) is satisfied under wide hypotheses.

It is a familiar fact that, although this adequate solution of the simpler problem of the length of a curve has been known for some years, yet no satisfactory extension to higher dimensions has been found.

Quite recently W. H. Young, by introducing the idea of the *area of a skew curve*, has proposed a solution of the problem of the area of a curved surface.

His method involves a preliminary investigation of the conditions that the curve in the (x, y)-plane defined by $x = x(u, v)$, $y = y(u, v)$ for values of (u, v) on the perimeter of a rectangle R should bound an area equal to $\iint_R J\,du\,dv$, where J is the ordinary Jacobian.

Let us consider the nature of the conditions that he obtains. A typical set is the following* :—

(i) $x(u, v)$ is an integral with respect to u, and an integral with respect to v.

(ii) $y(u, v)$ is an integral with respect to u, and an integral with respect to v.

(iii) $\dfrac{\partial y}{\partial v}$ and $\dfrac{\partial x}{\partial v}$ are, except for a set of values of v of content zero, numerically less than summable functions of v alone, $\mu(v)$ and $M(v)$ say.

(iv) the same as (iii) with u and v interchanged, or, more generally, $\iint \dfrac{\partial x}{\partial u}\mu(v)\,du\,dv$, $\iint \dfrac{\partial y}{\partial u}M(v)\,du\,dv$, exist as absolutely convergent integrals.

The most striking feature of these conditions is that they concern the independent variables (u, v) *separately*, and also the dependent variables (x, y) *separately*. But in the representation of R in the (u, v)-plane as a region in the (x, y)-plane, the fundamental entity in each plane is a *point*, that is a set of values of a *pair* of variables, (u, v) or (x, y). Taking this view, we seek a set of conditions in which (x, y) shall enter as an *unseparated pair* and (u, v) as another unseparated pair.

These considerations lead us to enunciate the following principle which shall guide us in seeking the proper generalisation to n functions of n real variables of known methods and results for a single function of a real variable.

In dealing with n functions $x_1, \ldots, x_n$ of n real variables $u_1, \ldots, u_n$ we use only those methods in which all the independent variables $u_1, \ldots, u_n$ appear on an equal footing and similarly all the dependent variables $x_1, \ldots x_n$ appear on an equal footing.

In the light of this *principle of unseparated variables* we see that the ordinary Jacobian J, constructed as it is of partial derivatives $\partial x/\partial u \ldots$ in which x is separated from y and u is separated from v, is not suitable for the analytical representation of area.

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* W. H. Young, "On a Formula for an Area", *Proc. London Math. Soc.*, Vol. 18 (1919), p. 344. In future we shall refer to this paper as F.A.

We are thus led to define a *modified Jacobian* of (x, y) with respect to (u, v). It is the aspect of the Jacobian as the generalisation of the derivative dx/du of a function x of a single variable u which gives us the clue to the new definition. For, to define the derivative dx/du, we take an interval on the u-line and the corresponding increment of x and find the limit of

$$\frac{\text{increment of } x \text{ over interval on } u\text{-line}}{\text{length of interval on } u\text{-line}}.$$

By analogy, therefore, we define the modified Jacobian as the limit of

$$\frac{\text{increment of } (x, y) \text{ over rectangle in } (u, v)\text{-plane}}{\text{area of rectangle in } (u, v)\text{-plane}},$$

where we have to give an appropriate meaning to the numerator.

The increment of (x, y) *over a rectangle* R in the (u, v)-plane is, in fact, defined in § 1 as

$$\Delta(x, y) = \tfrac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_4 - x_4y_3 + x_4y_1 - x_1y_4),$$

where 1, 2, 3, 4 denote the corners of R in anti-clockwise order, and x_1 is the value of x at the point 1, etc. (This expression is simply the area of the quadrilateral in the(x, y)-plane formed by the points corresponding to 1, 2, 3, 4.)

This increment $\Delta(x, y)$ is a *function of rectangles* in the (u, v)-plane; we notice that it is not *additive*. The writer has recently investigated some properties of functions of intervals and of rectangles, which are not assumed to be additive.*

In § 3 we apply a theorem of the paper quoted to prove that *the area of a closed plane curve is equal to the Lebesgue integral of the modified Jacobian, provided that* $\Delta(x, y)$ *is an absolutely continuous function of rectangles.*

The simplicity and generality of this result seem to justify our use of the modified Jacobian. It may be noticed, in particular, that we have to introduce no conditions involving *derivatives*, whereas the ordinary formula $\iint_R J\,du\,dv$ holds only if restrictions are imposed on the partial derivatives $\partial x/\partial u$ ....

In §§ 4 and 5 we develop the parallel theories of the length of a curve and the area of a surface. We obtain the formula for the area of a surface

* "Functions of Intervals," *Proc. London Math. Soc.*, Vol. 22, p. 275. We shall in future refer to this paper as F.I.

defined by $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, in terms of modified Jacobians, provided that $\Delta(y, z)$, $\Delta(z, x)$, $\Delta(x, y)$ *are absolutely continuous functions of rectangles.*

These conditions, independent as they are of *derivatives*, are seen to be strictly analogous to those mentioned above for the formula for the length of a curve.

1. *Increments.*

The following notation will be used :—

u, v are independent variables.

x, y, z are dependent variables, functions of u, v.

Numbers 1, 2, ... represent points in the space of the independent variables. The values of u, v at the point 1 will be written u_1, v_1, and the corresponding values of $x, \ldots$, namely, $x(u_1, v_1), \ldots$ as $x_1, \ldots$.

We recall the well known definitions of the increment of a single function x.

One function $x(u)$ of one variable.

Definition.—The increment of x over the interval I whose end-points are 1, 2 is $x_2 - x_1$.

We write the increment $\Delta_I(x\,;\,u)$, or, if there can be no misunderstanding, as $\Delta_I(x)$ or $\Delta(x)$.

One function $x(u, v)$ of two variables.

Definition.—The increment $\Delta_I(x\,;\,u, v)$ of x over the rectangle I whose corners are 1, 2, 3, 4 is

$$x_1 - x_2 + x_3 - x_4.$$

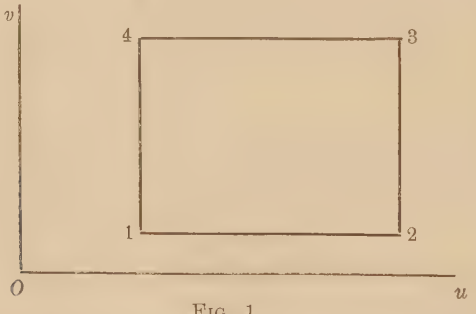

FIG. 1.

These definitions extend to the increment of a single function of n independent variables.*

* See, for example, Daniell, "Functions of Limited Variation in an Infinity of Dimensions", *Annals of Mathematics*, Vol. 21 (1919), p. 30.

The new idea that we introduce is that of *the increment of two functions of two independent variables over a rectangle.*

Definition.—Let $x = x(u, v)$, $y = y(u, v)$ and let I be a rectangle in the (u, v) plane whose corners are 1, 2, 3, 4.

Then $\Delta_I(x, y\,;\ u, v)$, the increment of (x, y) over I, is

$$\tfrac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_4 - x_4y_3 + x_4y_1 - x_1y_4).$$

We often omit I and (u, v) and write simply $\Delta(x, y)$.

This expression represents the (directed) area of the quadrilateral in the (x, y)-plane formed by the points which are the images of the points 1, 2, 3, 4 of the (u, v)-plane.

The following properties follow immediately from the definition :—

1.1. If c is a constant,

$$\Delta(x, c) = 0 \quad \text{and} \quad \Delta(c, y) = 0.$$

1.2. $\Delta(x, y)$ is skew-symmetric in x, y,

i.e. $$\Delta(x, y) = -\Delta(y, x).$$

1.3. $\Delta(x, y)$ is linear in x and in y :—

$$(a) \qquad \Delta(cx, y) = c\Delta(x, y).$$

$$(b) \qquad \Delta(x+y, z) = \Delta(x, z) + \Delta(y, z).$$

1.4. If $x = x(u)$ is independent of v

and $y = y(v)$ is independent of u,

then $$\Delta(x, y\,;\ u, v) = \Delta(x\,;\ u) \times \Delta(y\,;\ v).$$

We have thus associated with a pair of functions of points $x(u, v)$, $y(u, v)$ a function of intervals $\Delta_I(x, y)$.

We shall make statements such as

"(x, y) is absolutely continuous with respect to (u, v)"

and "(x, y) satisfies the Lipschitz condition with respect to (u, v),"

meaning that the function $\Delta_I(x, y)$ has the corresponding property.

THEOREM 1.5.—*If, in a rectangle R of the (u, v)-plane, each of $x(u, v)$, $y(u, v)$ satisfies the Lipschitz condition with respect to each of u, v, then (x, y) satisfies the Lipschitz condition with respect to (u, v).*

Proof.—If 1 2 3 4 is a rectangle contained in R, we have

$$|x_2-x_1| < K(u_2-u_1),$$

$$|y_2-y_1| < K(u_2-u_1),$$

and similar inequalities for each of the other sides.

Now

$$\begin{aligned}\Delta(x, y) &= \tfrac{1}{2}(x_1y_2-x_2y_1+x_2y_3-x_3y_2+x_3y_4-x_4y_3+x_4y_1-x_1y_4)\\ &= \tfrac{1}{2}\{(x_2-x_1)(y_3-y_2)+(x_3-x_4)(y_4-y_1)\\ &\qquad -(x_4-x_1)(y_3-y_4)-(x_3-x_2)(y_2-y_1)\}.\end{aligned}$$

Therefore $$|\Delta(x, y)| < 2K^2(u_2-u_1)(v_4-v_1),$$

which is the required result.

2. *Jacobians.*

We have in the introduction explained how the Principle of Unseparated Variables leads us to define a modified Jacobian, which is more suitable than the ordinary Jacobian for the analytical expression of area.

If $x = x(u, v)$, $y = y(u, v)$, we denote the ordinary Jacobian

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

by $J(x, y\,;\ u, v)$ or more shortly, if no misunderstanding is possible, by $J(x, y)$ or $J(u, v)$ or J.

Definition of the modified Jacobian $L(x, y\,;\ u, v)$.

We define the *upper modified Jacobian* $L^*(x, y\,;\ u_0, v_0)$ to be the upper derivate† at (u_0, v_0) of the function of intervals $\Delta_I(x, y\,;\ u, v)$.

Similarly the *lower modified Jacobian* $L_*(x, y\,;\ u_0, v_0)$ is the lower derivate at (u_0, v_0) of $\Delta_I(x, y\,;\ u, v)$.

If $L^*(x, y\,;\ u_0, v_0) = L_*(x, y\,;\ u_0, v_0)$, we say that the *modified Jacobian* $L(x, y\,;\ u_0, v_0)$ exists, and its value is the common value of L^* and L_*.

The advantages attained by the generalisation of the idea of the unique derivative of a function $x(u)$ to that of an upper or lower derivate

† See "Functions of Intervals," § 6.

are well known. It is to be observed that the introduction of L^*, L_* affords similar advantages in working with two functions $x(u, v)$, $y(u, v)$; while, if we operate with J, we have to assume that the unique J exists—there is no upper J or lower J.

The following elementary properties correspond to those of § 1.

2.1. $$L(x, c) = 0 \quad \text{and} \quad L(c, y) = 0.$$

2.2. $$L(x, y) = -L(y, x).$$

2.3. (*a*) $$L(cx, y) = cL(x, y),$$

(*b*) $$L(x+y, z) = L(x, z)+L(y, z).$$

2.4. If $x = x(u)$ is independent of v

and $y = y(v)$ is independent of u,

then $$L(x, y;\ u, v) = \frac{dx}{du}\frac{dy}{dv}.$$

Theorem 6.4 of the paper on "Functions of Intervals" gives the following important result.

2.5. *L^*, L_* and L, if it exists, are measurable* (B).

We show by examples that the existence at a particular point (u_0, v_0) of one of J, L does not imply the existence of the other; and they may both exist and be unequal.

Example 1.—Let $x = y = 0$ for $u = v = 0$,

and $x = y = 1$ for all other values of u, v.

Then $L = 0$ everywhere, but J does not exist at (0, 0).

Example 2.—Let

$$x = u\left(1+\frac{uv}{u^2+v^2}\right) \quad \text{unless} \quad u = v = 0; \quad x = 0 \quad \text{if} \quad u = v = 0;$$

and

$$y = v\left(1+\frac{uv}{u^2+v^2}\right) \quad \text{unless} \quad u = v = 0; \quad y = 0 \quad \text{if} \quad u = v = 0.$$

Then $$J = \left(1+\frac{uv}{u^2+v^2}\right)^2 \quad \text{unless} \quad u = v = 0,$$

$$J = 1 \quad \text{if} \quad u = v = 0.$$

We shall show that L does not exist at (0, 0).

If I is the rectangle whose opposite corners are $(0, 0)$ and (u, v), we have

$$\Delta_I(x, y) = uv\left(1+\frac{uv}{u^2+v^2}\right).$$

Hence plainly $L^*(0, 0) \geqslant \frac{3}{2}$ and $L_*(0, 0) \leqslant \frac{1}{2}$.

Example 3.—Let

$$x = u \quad \text{if} \quad v = 0; \qquad x = 0 \quad \text{if} \quad v \neq 0;$$

and

$$y = v \quad \text{if} \quad u = 0; \qquad y = 0 \quad \text{if} \quad u \neq 0.$$

Then $$J(0, 0) = 1 \quad \text{but} \quad L(0, 0) = 0.$$

We should expect, of course, that the results exhibited in these examples are exceptional, and that in general (that is, for most ordinary functions and most points in their domain of definition) J and L will be equal. We give simple sufficient conditions for this to be true.

THEOREM 2.6.—*If* $\frac{\partial x}{\partial u}$, $\frac{\partial y}{\partial u}$, $\frac{\partial x}{\partial v}$, $\frac{\partial y}{\partial v}$ *are continuous functions of* (u, v) *at* (u_0, v_0), *then* $L(u_0, v_0)$ *exists and equals* $J(u_0, v_0)$.

Proof.—Represent the point (u_0, v_0) shortly by 0.

Let 1 2 3 4 be a rectangle I in the neighbourhood of 0 in which $\frac{\partial x}{\partial u}$, $\frac{\partial y}{\partial u}$, $\frac{\partial x}{\partial v}$, $\frac{\partial y}{\partial v}$ exist. We have then

$$\Delta(x, y) = \tfrac{1}{2}\{(x_2-x_1)(y_3-y_2)+(x_3-x_4)(y_4-y_1) - (x_4-x_1)(y_3-y_4)-(x_3-x_2)(y_2-y_1)\} \quad \text{(A)}$$

By the mean-value theorem,

$$x_2-x_1 = (u_2-u_1)\left(\frac{\partial x}{\partial u}\right)_5,$$

where 5 is some point of 12; and we have similar equations for the other increments y_3-y_2, etc.

As the norm of I tends to zero, its parameter of regularity with respect to 0 remaining greater than some positive ρ, each of the points 5, ... tends to the point 0.

Therefore $$\left(\frac{\partial x}{\partial u}\right)_5 \to \left(\frac{\partial x}{\partial u}\right)_0, \quad \text{etc.}$$

Hence $$\frac{\Delta(x, y)}{mI} \to J(u_0, v_0),$$

i.e. $L(u_0, v_0)$ exists and equals $J(u_0, v_0)$. Q.E.D.

THEOREM 2.7.—*If* $x(u, v)$, $y(u, v)$ *are differentiable (in the sense of Stolz) at* (u_0, v_0), *then* $L(u_0, v_0)$ *exists and equals* $J(u_0, v_0)$.

Proof.—With the notation of the last theorem, suppose that the parameter of regularity with respect to 0 of 1 2 3 4 is greater than a fixed positive ρ. Let (u_0+h_1, v_0+k_1) and (u_0+h_2, v_0+k_2) be the coordinates of the points 1, 3 respectively.

Let $\quad \frac{1}{2}h = \text{max.}\,|h_1|, |h_2| \quad \text{and} \quad \frac{1}{2}k = \text{max.}\,|k_1|, |k_2|.$

Then by the definition of ρ,

$$h_2 - h_1 > \rho h$$

and

$$k_2 - k_1 > \rho k.$$

Also $\qquad h > \rho k \quad \text{and} \quad k > \rho h.$

By the definition of differentiability,

$$x_1 = x_0 + h_1\left(\frac{\partial x}{\partial u}\right)_0 + k_1\left(\frac{\partial x}{\partial v}\right)_0 + \sqrt{(h_1^2+k_1^2)}\,R(h_1 k_1),$$

where $R(h, k) \to 0$ as $h^2+k^2 \to 0$.

Hence

$$\frac{x_2-x_1}{h_2-h_1} - \left(\frac{\partial x}{\partial u}\right)_0 = \frac{R(h_2, k_1)\sqrt{(h_2^2+k_1^2)} - R(h_1, k_1)\sqrt{(h_1^2+k_1^2)}}{h_2-h_1} \qquad \text{(B)}$$

Let $\qquad R = \text{max.}\,|R(h_2, k_1)|, |R(h_1, k_1)|.$

Then using the above inequalities, we find the right-hand side of (B) to be numerically less than

$$\frac{2R}{\rho}\sqrt{\left(1+\frac{1}{\rho^2}\right)},$$

which $\to 0$ as $h^2+k^2 \to 0$.

Treating similarly the other terms of $\Delta(x, y)$ in the expansion (A) of the last theorem, we have for fixed ρ,

$$\frac{\Delta(x, y)}{mI} \to J(u_0, v_0).$$

Hence $L(u_0, v_0)$ exists and equals $J(u_0, v_0)$. Q.E.D.

We can make an important extension of Theorem 2.7 by combining with it a result due to Rademacher.* In summarising Rademacher's work, we are obliged to modify his notation, replacing his L by Λ, on account of having already assigned a conventional meaning to L.

Let $\omega_x(u, v\,;\ r)$ be the upper bound of

$$\frac{|x(u+h, v+k)-x(u, v)|}{\sqrt{(h^2+k^2)}}$$

for $$0 < h^2+k^2 \leqslant r^2.$$

Then $\omega_x(u, v\,;\ r)$ decreases as r decreases and we define

$$\Lambda_x(u, v) = \lim_{r\to 0} \omega_x(u, v\,;\ r).$$

Rademacher's theorem then is (*loc. cit.*, p. 347):

If $\Lambda_x(u, v)$ is finite and summable in a bounded domain G of the (u, v)-plane, then $x(u, v)$ is differentiable almost everywhere in G.

In particular, if x has *bounded partial derivates* in G (which is equivalent to saying that x satisfies the Lipschitz condition with respect to each of u, v), then $\Lambda_x(u, v)$ is bounded and the hypothesis of the theorem is satisfied.

Applying Rademacher's theorem, we obtain:

THEOREM 2.8.—*If $\Lambda_x(u, v)$, $\Lambda_y(u, v)$ are finite and summable in G, then at almost every point of G, $L(u, v)$ and $J(u, v)$ exist and are equal.*

In particular, the conclusion is true if $x(u, v)$, $y(u, v)$ have bounded partial derivates in G.

3. *The area of a plane curve.*

In this section our aim is to construct a theory of the area bounded by a closed plane curve, which shall satisfy the conditions (1)–(4) laid down in the introduction. We work with W. H. Young's definition of the *directed area* of a plane curve (F.A., p. 343). Prof. Young explains that it was this idea which led him to the elegant concept of the area of a skew curve, upon which he bases his theory of the area of a curved surface.

* H. Rademacher, "Über partielle und totale Differenzierbarkeit von Funktionen mehrerer Variabeln und über die Transformation der Doppelintegrale", *Math. Annalen*, Band 79 (1919), p. 340.

The equations of a closed plane curve may be given in either of two forms :—

(*a*) $x = x(u)$, $y = y(u)$ for $a \leqslant u \leqslant b$, where $x(u)$ and $y(u)$ are continuous and $x(a) = x(b)$, $y(a) = y(b)$.

(*b*) $x = x(u, v)$, $y = y(u, v)$ for points (u, v) on the perimeter of a rectangle R of the (u, v)-plane.

In the special case when the curve is simple (*i.e.* does not cut itself), we can define its area (when it exists) by two classes of approximative polygons, lying respectively inside and outside the curve; but such a definition does not satisfy our condition (4) of leading to an analytical expression for the area.

We therefore adopt Young's definition (F.A., p. 346); here the angular points of the approximative polygons lie *on* the curve.

Definition.—Let the curve be defined by $x = x(u)$, $y = y(u)$ for $a \leqslant u \leqslant b$, where $x(u)$, $y(u)$ are continuous and $x(a) = x(b)$, $y(a) = y(b)$.

Divide the range (a, b) into meshes of norm less than ϵ by points

$$a = u_0 < u_1 \ldots < u_{i-1} < u_i < \ldots < u_m = b.$$

Join the corresponding points A, P_1, ..., P_{m-1}, A of the curve to form an inscribed polygon.

Imagine forces represented in magnitude, line of action and sense by the sides of this polygon in order, supposed determined in the sense of u increasing.

These forces reduce to a couple; let F be its moment, counted positive when anti-clockwise in sense. If, as $\epsilon \to 0$, the number F has a unique limit $2A$, then A is called the area of the curve.

This definition may be framed very concisely in terms of a function of intervals and its integral :—

Let $$g(I_i) = \tfrac{1}{2}\{x(u_{i-1})\,y(u_i) - x(u_i)\,y(u_{i-1})\},$$

where I_i *is* (u_{i-1}, u_i).

If $\int_a^b g(I)$ *exists and* $= A$, *then* A *is the area of the curve.*

Prof. Young points out that a sufficient condition for the existence of the area is that the curve should be rectifiable or, more generally, semi-rectifiable [*i.e.* $x(u)$, $y(u)$ are both continuous and one at least has bounded variation]. But this condition is not necessary (see Example 1 at the end

of this section) and there seems to be no necessary and sufficient condition which can be simply expressed in terms of $x(u)$, $y(u)$.

A sharp distinction here presents itself between one and two dimensions. The corresponding problem in one dimension of the directed length of the set of segments traced out by a point moving continuously along a straight line is always possible, the length being x_2-x_1, where x_1, x_2 are the initial and final positions of the point. The distinction arises from the fact that the boundary of a connected portion of a straight line is necessarily of a simple nature (consisting of two points), whereas the curve which bounds a portion of a plane may be of an erratic character.

It should be noticed that the definition that we adopt does not determine the area even of all simple curves (see Example 2); but it seems impossible to substitute an equally elementary definition which shall lead to an analytical formula for the area.

The area of the closed curve $x = x(u),\ y = y(u)$.

By definition the area is the limit of the sum

$$\tfrac{1}{2}\sum_i (x_{i-1}\,y_i - x_i\,y_{i-1}), \tag{I}$$

where x_i is written for $x(u_i)$ and u_i is a typical point of subdivision of (a, b). Previous writers have confined themselves to the case when the curve is rectifiable or semi-rectifiable; this is because the limit (I) has been at once converted into the difference of two Stieltjes integrals

$$\tfrac{1}{2}\int_a^b (x\,dy - y\,dx)$$

and we cannot assert that these integrals exist unless at least one of $x(u)$, $y(u)$ has bounded variation.

We have, in fact,

$$\Sigma(x_{i-1}\,y_i - x_i\,y_{i-1}) = \Sigma x_{i-1}(y_i - y_{i-1}) - \Sigma y_{i-1}(x_i - x_{i-1}),$$

and if $\int x\,dy$ exists, it is the limit of $\Sigma x_{i-1}(y_i - y_{i-1})$.

But it is to be observed that the Stieltjes integral $\int x\,dy$ is defined as the limit of a wider class of approximative sums than $\Sigma x_{i-1}(y_i - y_{i-1})$, namely the class $\Sigma \xi_{i-1}(y_i - y_{i-1})$ where ξ_{i-1} is the value of x at an arbitrary point of the closed interval (u_{i-1}, u_i).

Hence by transforming at once the sum (I) to Stieltjes integrals, we sacrifice the generality of the problem in two ways, namely

(*a*) the existence of the limit of $\Sigma(x_{i-1}y_i - x_i y_{i-1})$ does not imply the existence of the limits of $\Sigma x_{i-1}(y_i - y_{i-1})$ and $\Sigma y_{i-1}(x_i - x_{i-1})$.

(*b*) the existence of the limit of $\Sigma x_{i-1}(y_i - y_{i-1})$ does not imply the existence of the Stieltjes integral $\int x\,dy$.

The first case (*a*) is illustrated in Example 1.

We notice in passing a simple sufficient condition for the existence of the area, deduced directly from the expression (I).

The sum (I) is unaltered by a change of origin and we may therefore assume that

$$0 < B \leqslant x \leqslant C, \qquad 0 < H \leqslant y \leqslant K$$

for all points of the curve.

THEOREM 3.1.—*A sufficient condition that the curve has an area is that* $\psi(u) = y(u)/x(u)$ *has bounded variation, and the area is then* $\frac{1}{2}\int_a^b x^2\,d\psi$ *(in the Riemann-Stieltjes sense).*

Proof.—The sum (I) $= \frac{1}{2}\Sigma x_{i-1}x_i(\psi_i - \psi_{i-1}) \to \frac{1}{2}\int_a^b x^2\,d\psi$.

Corollary.—A sufficient condition that a curve has an area is that an origin exists for which the vectorial angle has bounded variation.

For, assuming as above that x, y lie between two positive bounds, we have

$$0 < F \leqslant \theta \leqslant G < \tfrac{1}{2}\pi.$$

Therefore
$$0 < K_1 \leqslant \frac{\tan\theta_1 - \tan\theta_2}{\theta_1 - \theta_2} \leqslant K_2.$$

Now $y/x = \tan\theta$; therefore y/x has bounded variation.

Our result then follows from the theorem.

We now make an application of a general theorem on functions of intervals. If we denote, as before, $\frac{1}{2}(x_{i-1}y_i - x_i y_{i-1})$ by $g(I_i)$, then we have, from F.I., Theorem 7.6:—

THEOREM 3.2.—*Let* $g(I)$ *be absolutely continuous in* (a, b).

If the curve has an area A, *then* $A = \int_a^b g'(u)\,du$; *and conversely, if* $\int_a^b g'(u)\,du$ *exists and is equal to* A, *then the curve has an area* A.

This solution of the problem of area for the curve $x = x(u),\ y = y(u)$ satisfies the conditions which we imposed in the introduction and is more general than any solution hitherto given. An example of the application of Theorem 3 . 2 will be found at the end of the section (Example 3).

The area of the curve $x = x(u, v),\ y = y(u, v)$, *where* (u, v) *is a point of the perimeter of a rectangle* R.

We suppose that $x(u, v)$, $y(u, v)$ are continuous functions of (u, v) in R. Write A for the area of the curve.

We shall say that x, y *satisfy condition* (α) *in* R if every curve in the (x, y)-plane which is the image of a sub-rectangle of R (with sides parallel to the sides of R) has a definite area.

The theory of functions of intervals then yields the following result.

THEOREM 3 . 3.—*If*

(1) x, y *satisfy condition* (α) *in* R,

(2) (x, y) *is absolutely continuous with respect to* (u, v) *in* R,

(3) $L(x, y\ ;\ u, v)$ *exists a.e. in* R,

then

$$A = \iint_R L\,du\,dv.$$

Proof.—Divide R into a set of meshes such as 1 2 3 4.

Then the sides of R are divided into segments such as $\alpha\beta$.

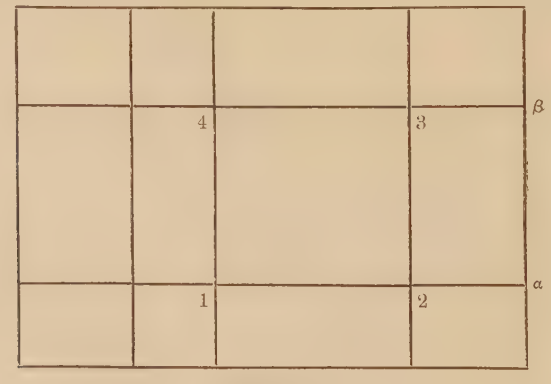

FIG. 2.

By definition,

$$A = \lim \tfrac{1}{2}\Sigma(x_\alpha y_\beta - x_\beta y_\alpha)$$

as $\max \alpha\beta \to 0$, the summation being round the perimeter of R.

Therefore

$$A = \lim \tfrac{1}{2}\Sigma(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_4 - x_4y_3 + x_4y_1 - x_1y_4)$$

as the norm of the mesh-system tends to zero, the Σ denoting summation over the meshes.

Hence $$A = \int_R \Delta_I(x, y).$$

By F.I., Theorem 7.5, $$A = \iint_R L\,du\,dv.$$ Q.E.D.

Attention is particularly drawn to the simplicity of this theorem, as no better justification could be needed of the Principle of Unseparated Variables, which led us to replace the Jacobian J by L.

Comparing Theorem 3.3 with results for the ordinary Jacobian, we can assert

(i) that *Theorem* 3.3 *includes Young's first result* (F.A., p. 344), in which it is assumed that the partial derivates of x and y with respect to u and v are bounded in R.

(ii) that *Theorem* 3.3 *is not included in any previously known result.* We construct in Example 4 a curve satisfying the hypotheses of Theorem 3.3, to which no previous set of conditions applies.

To justify the statement (i), we observe that condition (α) includes the hypotheses made by Young (F.A., p. 348), namely that the image of every stretch on a straight line parallel to a side of the (u, v) rectangle is a rectifiable curve.

Secondly, the partial derivates of x and y are bounded in R, so that each of x, y satisfies the Lipschitz condition with respect to each of u, v; therefore by Theorem 1.5, (x, y) satisfies the Lipschitz condition with respect to (u, v) and *a fortiori* is absolutely continuous with respect to (u, v).

Lastly, Theorem 2.8 shows that, at almost every point of R, L exists and equals J.

Therefore all the conditions of Theorem 3.3 are satisfied and

$$A = \iint_R J\,du\,dv.$$

We close this section with examples of curves illustrating the general theory.

Example 1.—*A simple closed curve possessing an area according to Young's definition, for which the Stieltjes integrals* $\int x\,dy$, $\int y\,dx$ *do not exist.**

Define $x(u)$, $y(u)$ in $0 \leqslant u \leqslant 2$.

For $u = 0$, $x = 1$.

For $u = \dfrac{n-1}{n}$ $(n = 2, 3, \ldots)$, $x = (-1)^{n-1}\dfrac{1}{\sqrt{n}}$.

Let $x(u)$ be linear in each of the intervals $(0, \frac{1}{2})$, $(\frac{1}{2}, \frac{2}{3})$, ...

For $1 \leqslant u \leqslant 2$, $x = u-1$.

Again, for $u = 0$, $y = 0$.

For $u = \dfrac{n-1}{n}$, $y = \dfrac{1}{\sqrt{n}}$ or 0 according as n is even or odd.

Let $y(u)$ be linear in each of the intervals $(0, \frac{1}{2})$, $(\frac{1}{2}, \frac{2}{3})$, ...

For $1 \leqslant u \leqslant \frac{3}{2}$, $y = -u+1$.

For $\frac{3}{2} \leqslant u \leqslant 2$, $y = u-2$.

Then $x(u)$, $y(u)$ are continuous and $x(0) = x(2)$, $y(0) = y(2)$, so that (x, y) describes a closed curve, which is easily seen to be simple.

Denoting the points $u = 0, \frac{1}{2}, \frac{2}{3}, \ldots$, by $u_1, u_2, u_3, \ldots$, and the corresponding values of x, y by $x_1, x_2, \ldots$, $y_1, y_2, \ldots$, we have

$$x_1(y_2-y_1) = 1 \cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}},$$

$$x_2(y_3-y_2) = \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{2},$$

$$x_3(y_4-y_3) = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} = \frac{1}{3}, \quad \text{etc.},$$

and Pollard's argument (*loc. cit.*, p. 82) shows that $\int x\,dy$ does not exist (and similarly $\int y\,dx$ does not exist).

* This example is adapted from Pollard, "The Stieltjes Integral and its Generalisations", *Quarterly Journal*, Vol. 49 (1921), p. 81.

But
$$x_1y_2-x_2y_1 = \frac{1}{\sqrt{2}},$$
$$x_2y_3-x_3y_2 = \frac{1}{\sqrt{2}\sqrt{3}},$$
$$x_3y_4-x_4y_3 = \frac{1}{\sqrt{3}\sqrt{4}}, \quad \text{etc.}$$

and it is plain from the convergence of $\Sigma(x_{i-1}y_i-x_iy_{i-1})$ that the curve possesses an area.

Example 2.—*A simple closed curve built up of linear stretches, which does not possess an area, according to Young's definition.*

On a pair of perpendicular axes take two sequences of points $P_1, P_2, \ldots$ and $Q_1, Q_2, \ldots$, as shown in the diagram, such that

$$OP_n = \frac{1}{\sqrt{(2n-1)}} \quad \text{and} \quad OQ_n = \frac{1}{\sqrt{(2n)}}.$$

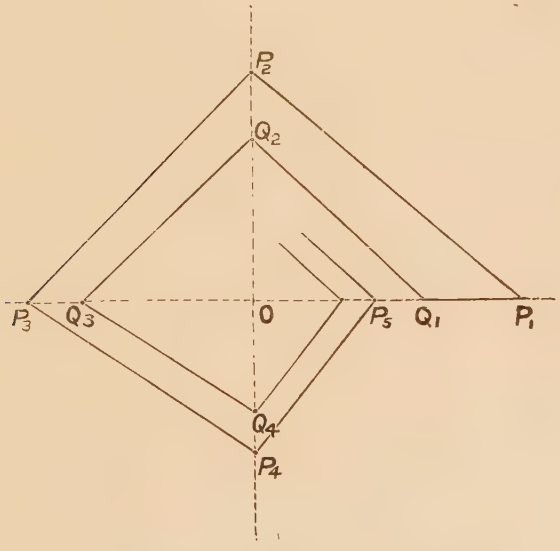

FIG. 3.

Join P_1P_2, P_2P_3, ..., P_1Q_1, Q_1Q_2, ..., by straight lines.
Then $P_1P_2P_3 \ldots Q_3Q_2Q_1P_1$ forms a simple closed curve.
The area of the triangle OP_nP_{n+1} is

$$\frac{1}{2\sqrt{\{(2n-1)(2n+1)\}}}$$

and the area of the triangle OQ_nQ_{n+1} is

$$-\frac{1}{2\sqrt{\{2n(2n+2)\}}}.$$

Each of the series

$$\Sigma\frac{1}{\sqrt{\{(2n-1)(2n+1)\}}}, \qquad \Sigma\frac{1}{\sqrt{\{2n(2n+2)\}}}$$

diverges.

Hence by choosing among our points of division $P_1, \ldots, P_m$ and $Q_1, \ldots, Q_n$ for suitable values of m, n, we can make the sum $\frac{1}{2}\Sigma(x_{i-1}y_i - x_iy_{i-1})$ tend to any arbitrary value.

Therefore the curve does not possess an area according to Young's definition.

Example 3.—*A curve* $x = x(u)$, $y = y(u)$ *whose area can be calculated by Theorem* 3.2, *but by no formula previously known.*

In the interval $0 \leqslant u \leqslant 1$, define

$$x(u) = W(u) \cos 2\pi u$$

$$y(u) = W(u) \sin 2\pi u,$$

where $W(u)$ is continuous, positive and satisfies $W(0) = W(1)$.

Then (x, y) describes a simple closed curve.

We prove that the $g(I)$ of Theorem 3.2 satisfies the Lipschitz condition and *a fortiori* is absolutely continuous.

For, if I is the interval (u, U),

then $$g(I) = \tfrac{1}{2}W(u)\,W(U) \sin 2\pi(U-u).$$

Therefore $$|g(I)| \leqslant \pi K^2(U-u),$$

where K is the upper bound of $W(u)$ in (0, 1).

Clearly, at any point of (0, 1),

$$g'(u) = \pi\{W(u)\}^2,$$

so that $g'(u)$ is continuous and therefore integrable.

Theorem 3.2 then shows that the curve has an area

$$\pi\int_0^1 \{W(u)\}^2 du.$$

$W(u)$ may be of such a form that $\int x\,dy$, $\int y\,dx$ do not exist; we may suppose for instance that $W(u)$ has not a derivative at any point of (0, 1).

Example 4.—*A curve* $x = x(u, v)$, $y = y(u, v)$ *whose area can be calculated by Theorem* 3.3, *but by no formula previously known.*

Let R be the rectangle $a \leqslant u \leqslant b$, $c \leqslant v \leqslant d$, where $a > 0$ and $d - c < 1$.

Define
$$x = uW(v) \cos 2\pi v,$$
$$y = uW(v) \sin 2\pi v,$$

where $W(v)$ satisfies the conditions of Example 3.

Then, if I is the rectangle whose opposite corners are (u, v), (U, V),

$$\Delta_I(x, y;\ u, v) = \tfrac{1}{2}(U^2 - u^2) W(v) W(V) \sin 2\pi(V - v),$$

Hence
$$|\Delta| < K(U - u)(V - v),$$

where K is a constant.

Thus (x, y) satisfies the Lipschitz condition and therefore is absolutely continuous in R.

Moreover, at any point of R,

$$L(x, y;\ u, v) = 2\pi u \{W(v)\}^2.$$

Theorem 3.3 then shows that

$$A = 2\pi \int_a^b u\,du \int_c^d \{W(v)\}^2 dv = \pi(b^2 - a^2) \int_c^d \{W(v)\}^2 dv.$$

If we suppose that $W(v)$ is a non-differentiable function, then $\frac{\partial x}{\partial v}$, $\frac{\partial y}{\partial v}$ do not exist and the formula $\iint_R J\,du\,dv$ does not apply.

4. *The length of a curve.*

The results stated in this section are wholly classical. The presentation, however, is novel, being in terms of functions of intervals, and the proofs are merely applications of the general theorems that we have established for functions of intervals.

Every statement is carefully worded so as to admit of immediate generalisation to the problem of the area of a surface, whenever such generalisation is possible.

We take for simplicity a two-dimensional curve. The theory may at once be extended to curves in any greater number of dimensions.

Definition.—A *curve* S is a set of points (x, y) defined by $x = x(u)$,

$y = y(u)$ for values of u in an interval R, $a \leqslant u \leqslant b$, where $x(u)$, $y(u)$ are continuous functions of u.

Notation.—We divide R into meshes $I_1, \ldots, I_n$, where I_i is (u_{i-1}, u_i).

Define $g_1(I_i) = \Delta_{I_i}(x) = x_i - x_{i-1}$ with the notation of Section 1, and $g_2(I_i) = \Delta_{I_i}(y)$.

Define $$G_1(I_i) = \int_{I_i} g_1(I) \quad \text{and} \quad G_2(I_i) = \int_{I_i} g_2(I).$$

We then have the following property which we shall find to be peculiar to the case of one independent variable :

Property (α).—For any interval I,

$$G_1(I) = g_1(I) \quad \text{and} \quad G_2(I) = g_2(I).$$

Definition. Length of curve.

Divide R into a system of meshes I_i. Let P_{i-1}, P_i be the points of S corresponding to the end-points of the mesh I_i. Let p_i be the length of the chord $P_{i-1}P_i$. Then the upper limit of $\Sigma_i p_i$ as the norm of the mesh-system tends to zero is called the length of the curve S.

Thus the length of a curve always exists, though it may be infinite.

To express this definition in analytical form, write

$$g(I) = \sqrt{\{G_1^2(I) + G_2^2(I)\}}.$$

Then $\overline{\int}_R g(I)$ is the length of S.

We shall say for brevity that a curve is absolutely continuous if the defining functions $x(u)$, $y(u)$ are absolutely continuous.

The following theorems contain the characteristic properties of the length of a curve.

THEOREM 4.1.—*The necessary and sufficient conditions that the length of S is finite are that $\overline{\int}_R |G_1(I)|$ and $\overline{\int}_R |G_2(I)|$ are finite, i.e. that $x(u)$ and $y(u)$ have bounded variation in R.*

Proof.—Each of

$$|G_1(I_i)|, \ |G_2(I_i)| \leqslant g(I_i) \leqslant |G_1(I_i)| + |G_2(I_i)|.$$

Hence each of

$$\overline{\int}_R |G_1(I), \ \overline{\int}_R |G_2(I)| \leqslant \overline{\int}_R g(I) \leqslant \overline{\int}_R |G_1(I)| + \overline{\int}_R |G_2(I)|,$$

giving the result.

THEOREM 4.21.—*$\sum_i p_i$ tends to a limit (which may be $+\infty$) as the norm of the mesh-system tends to zero; i.e. $\int_R g(I)$ exists.*

Proof.—The function $g(I)$ increases by subdivision,* for

$$\begin{aligned} g(I_1+I_2) &= \sqrt{[\{G_1(I_1+I_2)\}^2+\{G_2(I_1+I_2)\}^2]} \\ &= \sqrt{[\{G_1(I_1)+G_1(I_2)\}^2+\{G_2(I_1)+G_2(I_2)\}^2]} \\ &\leqslant \sqrt{\{G_1^2(I_1)+G_2^2(I_1)\}}+\sqrt{\{G_1^2(I_2)+G_2^2(I_2)\}} \\ &= g(I_1)+g(I_2). \end{aligned}$$

Also $g(I)$ is continuous.

Therefore, by F.I., Theorem 5.1 (for one dimension), $\int_R g(I)$ exists (in fact $\int_I g(I)$ exists over any subinterval I of R).

THEOREM 4.22.—*If S is absolutely continuous, its length is finite.*

Proof.—The result follows from F.I., Theorem 5.1.

THEOREM 4.3.—*If S is absolutely continuous, its length is*

$$\int_a^b \sqrt{[\{x'(u)\}^2+\{y'(u)\}^2]}\,du.$$

Proof.— $0 \leqslant g(I) \leqslant |G_1(I)| + |G_2(I)|$.

Therefore $g(I)$ is absolutely continuous.

We proceed to prove that $g(I)$ has *a.e.* in R a derivative

$$\sqrt{[\{x'(u)\}^2+\{y'(u)\}^2]}.$$

From F.I., Theorem 7.6, corollary, $G_1(I)$ has *a.e.* in R a derivative $x'(u)$.

Applying F.I., Theorem 6.3, to the function $\sqrt{(t_1^2+t_2^2)}$, we see that $g(I)$ has *a.e.* in R a derivative

$$\sqrt{[\{x'(u)\}^2+\{y'(u)\}^2]}.$$

Our result then follows from F.I., Theorem 7.6, corollary.

THEOREM 4.4.—*If S has finite length l and R is divided into meshes R_i ($i = 1, 2, \ldots, n$), then each corresponding part S_i of S has a finite length (l_i say) and $l = \sum_{i=1}^{n} l_i$.*

* See F.I., § 5.

Proof.—From the proof of Theorem 4.21, $\int_{R_i} g(I)$ exists for each i. The result then follows from F.I., Theorem 2.8 (4).

5. *The area of a curved surface.*

We now attack our original problem of setting up a theory of the area of a curved surface which shall satisfy as far as possible the conditions laid down in the introduction.

W. H. Young has remarked* that attempts to define the area of a surface as the limit of the areas of approximative polyhedra (usually assumed to have triangular faces) have generally been successful only if the surface has a continuously varying tangent plane. An exception is to be made in favour of the recent work of Rademacher,† who proves that a surface has an area when the functions $x(u, v)$, $y(u, v)$, $z(u, v)$ satisfy Lipschitz conditions; but his results will be found to be less general than Young's.

The definition of the area of a curved surface which Young has introduced has a much wider scope, and it is his definition that we adopt (with a trivial modification). The fundamental concept of Young's theory is that of the area of a closed skew curve.‡

Definition.—Area of a skew curve.

The method used in § 3 to define the area of a closed plane curve applies to define the area of a closed skew curve as a vector whose magnitude is the limit of the magnitude of a certain couple and whose direction is the limit of its direction. We shall here be concerned only with the magnitude of the area.

If C is a skew curve and C_1, C_2, C_3 are its projections on the coordinate planes, we see by resolving the forces that

If C_1, C_2, C_3 have areas A_1, A_2, A_3, then C has area $\sqrt{(A_1^2+A_2^2+A_3^2)}$.

We now develop the theory of the area of a curved surface, preserving where possible the analogy with the simpler theory of the length of a curve; moreover, where no such analogy presents itself, we show that its

* Young, "On the Area of Surfaces", *Proc. Roy. Soc. A*, Vol. 96 (1920), p. 72.

† H. Rademacher, "Über partielle und totale Differenzierbarkeit von Funktionen mehrerer Variabeln und über die Transformation der Doppelintegrale", *Math. Annalen*, Band 79 (1919), p. 340, and Band 81 (1920), p. 52.

‡ Young, "On the Area of Surfaces," p. 77.

absence lies, not in any defect of our method, but in the nature of things, in the essential distinctions between the properties of functions of intervals in one and two dimensions.

Definition.—A *surface* S is a set of points (x, y, z) defined by $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ for values of (u, v) in a rectangle R, $a \leqslant u \leqslant b$, $c \leqslant v \leqslant d$, where $x(u, v)$, $y(u, v)$, $z(u, v)$ are continuous functions of (u, v).

Notation.—We divide R into a system of meshes $I_1, \ldots, I_n$.

Define $$g_1(I_i) = \Delta_{I_i}(y, z), \quad \text{with the notation of § 1,}$$

$$g_2(I_i) = \Delta_{I_i}(z, x),$$

$$g_3(I_i) = \Delta_{I_i}(x, y).$$

Define $G_1(I_i) = \int_{I_i} g_1(I)$ and similarly for $G_2(I_i)$, $G_3(I_i)$.

Then what we called in § 4 Property (a), namely that $G_1(I)$ exists and equals $g_1(I)$, is not in general true for two independent variables. To replace Property (a) we have to introduce a restriction on the class of surfaces that we consider :—

Condition (a).—We say that a surface S satisfies Condition (a) if, for every sub-rectangle I of R, $G_1(I)$, $G_2(I)$, $G_3(I)$ exist and are finite, *i.e.* if the projection on any coordinate plane of the curve on S which is the image of the perimeter of I has a definite finite area.

We shall in future be concerned only with surfaces satisfying Condition (a).

We may remark in passing that our Condition (a) is wider than the conditions which previous writers have been compelled to impose at the outset of their work.

Lebesgue, for instance, confines his attention to "rectifiable surfaces," namely surfaces S such that the image on S of any rectifiable curve in R is rectifiable.*

If, then, S is rectifiable, the image on S of any sub-rectangle I of R is rectifiable ; therefore the projections of this image on the coordinate planes are also rectifiable ; therefore each of these projections bounds a definite area, and Condition (a) is satisfied.

* H. Lebesgue, "Intégrale, longueur, aire", *Annali di Mat.*, Ser. 3a, Tome VII (1902), p. 315.

Definition. Area of surface.

Divide R into a system of meshes I_i. Let C_i be the closed curve on S corresponding to the perimeter of the mesh I_i. Let p_i be the area of the skew curve C_i. Then the upper limit of $\sum_i p_i$ as the norm of the mesh-system tends to zero is called the area of the surface S.

Thus the area of a surface always exists, though it may be infinite.

To express this definition in analytical form, write

$$g(I) = \sqrt{\{G_1^2(I)+G_2^2(I)+G_3^2(I)\}}.$$

Then $\overline{\int}_R g(I)$ is the area of S.

We differ from Young in defining the area of S as the upper limit of $\sum_i p_i$ instead of the limit of $\sum_i p_i$. Theorem 5.2, below, gives a condition that our definition reduces to his.

We shall say for brevity that a surface is absolutely continuous if each of the three pairs of functions (y, z), (z, x), (x, y) is absolutely continuous with respect to (u, v) in the sense of § 1.

The following theorems contain the characteristic properties of the area of a surface.

THEOREM 5.1.—*The necessary and sufficient conditions that the area of S is finite are that* $\overline{\int}_R |G_1(I)|$, $\overline{\int}_R |G_2(I)|$, $\overline{\int}_R |G_3(I)|$ *are finite.*

Proof.—Each of

$$|G_1(I_i)|,\ |G_2(I_i)|,\ |G_3(I_i)| \leqslant g(I_i) \leqslant |G_1(I_i)|+|G_2(I_i)|+|G_3(I_i)|.$$

Hence each of

$$\overline{\int}_R |G_1(I)|,\ \overline{\int}_R |G_2(I)|,\ \overline{\int}_R |G_3(I)| \leqslant \overline{\int}_R g(I) \leqslant \overline{\int}_R |G_1(I)|+\overline{\int}_R |G_2(I)|+\overline{\int}_R |G_3(I)|,$$

giving the result.

THEOREM 5.2.—*If S is absolutely continuous, then $\sum_i p_i$ tends to a finite limit as the norm of the mesh-system tends to zero; i.e. $\int_R g(I)$ exists and is finite.*

Proof.—We prove as in Theorem 4.21 that $g(I)$ increases by subdivision.

Also $g_1(I)$, $g_2(I)$, $g_3(I)$ are absolutely continuous.

Therefore by F.I., Theorem 4.1, $G_1(I)$, $G_2(I)$, $G_3(I)$ are absolutely continuous.

Now $$0 \leqslant g(I) \leqslant |G_1(I)| + |G_2(I)| + |G_3(I)|.$$

Hence $g(I)$ is absolutely continuous.

Therefore by F.I., Theorem 5.1, $\int_R g(I)$ exists and is finite (in fact $\int_I g(I)$ exists over any sub-rectangle I of R).

THEOREM 5.3.—*If*

(1) *S is absolutely continuous,*

(2) *$L(y, z)$, $L(z, x)$, $L(x, y)$ exist a.e. in R,*

then the area of S is

$$\iint_R \sqrt{[\{L(y, z)\}^2 + \{L(z, x)\}^2 + \{L(x, y)\}^2]}\, du\, dv.$$

Proof.—We have shown in Theorem 5.2 that $g(I)$ is absolutely continuous in R and that $G(I)$ exists for every $I \subset R$.

We proceed to prove that $g(I)$ has *a.e.* in R a derivative

$$\sqrt{[\{L(y, z)\}^2 + \{L(z, x)\}^2 + \{L(x, y)\}^2]}.$$

From F.I., Theorem 7.5, $G_1(I)$ has *a.e.* in R a derivative $L(y, z)$.

Applying F.I., Theorem 6.3, to the function $\sqrt{(t_1^2 + t_2^2 + t_3^2)}$, we see that $g(I)$ has *a.e.* in R a derivative

$$\sqrt{[\{L(y, z)\}^2 + \{L(z, x)\}^2 + \{L(x, y)\}^2]}$$

Our result then follows from F.I., Theorem 7.5.

THEOREM 5.4.—*If S is absolutely continuous and has area A, and if R is divided into meshes R_i ($i = 1, 2, \ldots, n$), then each corresponding part S_i of S has a finite area (A_i say) and $A = \sum_{i=1}^{n} A_i$.*

Proof.—From the proof of Theorem 5.2, $\int_{R_i} g(I)$ exists for each i. The result then follows from F.I., Theorem 2.8 (4).

We may add a few words as to the *generality* of our main result—that of Theorem 5.3. It is the direct extension to curved surfaces of Theorem 3.3 for plane surfaces, and the remarks which we made on the scope of the latter theorem apply in the present instance :—

(i) The conditions of Theorem 5.3 are satisfied if the partial derivates of x, y, z with respect to u, v are bounded. This case has been stated by Young.*

(ii) There are surfaces (cf. Example 4 of § 3) whose area can be calculated by Theorem 5.3, but by no formula previously known.

Let us now examine the differences which present themselves between the theorems of §§ 4 and 5.

In Theorems 5.2, 5.4 we have to assume that the surface is absolutely continuous, while no such hypothesis is necessary in Theorems 4.21, 4.4. We see that this is a consequence of the sharper form which F.I., Theorem 5.1, takes in one dimension.

The other difference is that in Theorem 5.3 we have to *assume* the existence (*a.e.*) of the modified Jacobians $L(y, z), \ldots$, while in Theorem 4.3 the existence (*a.e.*) of the derivatives $x'(u) \ldots$ is established in the course of the argument. We trace this back to the fact that in one dimension the original and extended definitions of the integral of a function of intervals coincide, so that in one dimension we are able to replace F.I., Theorem 7.5, by F.I., Theorem 7.6 (corollary).

* "On the Area of Surfaces," p. 77.

SOME ELECTROSTATIC DISTRIBUTIONS IN TWO DIMENSIONS.

By A. E. H. LOVE.

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SUMMARY.

1. The subject of this paper is the application of the method of conformal transformation to determine the potential due to two charged conductors. The conductors are cylinders with parallel generators and of infinite length. If the cylinders become thin plates with straight parallel edges, and the planes of the plates are parallel, and if the sum of the charges per unit of length is zero, the system is a parallel plate condenser. For a condenser with plates of equal breadth, so situated that the ends of the rectilinear segments in which they are cut by a transverse plane are at the corners of a rectangle, the solution was obtained by Michell,* and afterwards in a different way by Bromwich.† In the present paper the solution is obtained in a slightly more direct fashion, leading almost immediately to the important approximate formula given by Bromwich for the capacity per unit of length. It is then shown how it may be modified so as to apply to asymmetric positions and unequal breadths, the planes of the plates remaining parallel. In the case of plates of equal breadth, displaced from the symmetric position, various approximate formulae, applicable to different circumstances, are given for the capacity per unit of length. For plates of unequal breadths limiting cases arise when one shrinks to a line parallel to the edges of the other, and the resulting problem of a plate under the influence of a line charge is discussed independently. There are two symmetrical cases, one of which leads by inversion to the solution of the problem of free distribution of charge on a thin sheet, in the form of a portion of a circular cylinder with edges along two generators. The solution for the general case is then obtained by a second inversion. The problem of the potential due to charges, whose sum is not zero, is then taken up for any forms of cylinders for which the condenser problem can be solved. The solution

* J. H. Michell, *Messenger of Math.*, Vol. 23 (1894), p. 72.

† T. J. I'A. Bromwich, *Messenger of Math.*, Vol. 31 (1902), p. 184.

of the condenser problem renders possible a conformal transformation of the region outside the curves, in which the cylinders are cut by a transverse plane, into the ring between two concentric circles, the point at ∞ in the first region and the corresponding point in the ring being excluded. The problem of given charges on the conductors is thus reduced to that of two thin conducting circular cylindrical sheets with a common axis, carrying given charges, and under the influence of a charged line, which carries the sum of the given charges with changed sign, and is placed in a given position between the cylindrical sheets. This problem is solved by transforming the concentric circles into two equal rectilinear segments in the same straight line. It is thus proved that the problem of given charges on given cylinders can always be solved when the condenser problem can be solved for the same two cylinders. The case of two circular cylinders is worked out as an illustration.

Parallel Plate Condenser.

2. *Equal breadths. Symmetrical position.*—The two thin conducting plates cut a plane at right angles to their edges in two straight segments of the same length, directly opposite to each other, as DB, HF in Fig. 1. Let the lengths of these segments be $2b$ and their distance apart $2h$.

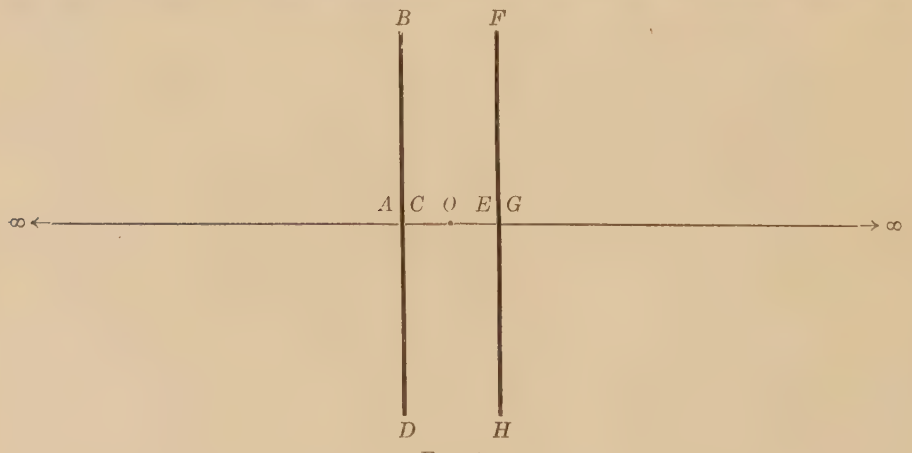

Fig. 1.

Then $2b$ is the breadth of either conducting plate, and $2h$ is the distance apart of the plates. Let the charge on the plate passing through HF be e_0 per unit of length of an edge of the plate, and that of the other $-e_0$.

Let the plane of the figure be the plane of a complex variable z, or $x+iy$, let the axis of x bisect the two segments at right angles, and let the origin be midway between the segments.

Let ϕ be the potential due to the charged plates. It is the real part of a function χ of z. Let the conjugate function be denoted by ψ, so that $\chi = \phi + i\psi$. The function χ is indeterminate to the extent of an additive complex constant.

The axis of x divides the z figure symmetrically, and the parts of it, as divided by the segments, are lines of force. We can therefore draw the figure in the χ plane that corresponds to the upper half of the figure in the z plane. The line of force CE becomes a line $\psi = \text{const.}$, starting at the value of ϕ on DB, and going to the greater value of ϕ on HF. This is continued by a line $\phi = \text{const.}$, rising from the value of ψ at E to the value of ψ at G. The height through which it rises is $2\pi e_0$. This is continued by a line $\psi = \text{const.}$, along which the potential ϕ diminishes from its value at HF to its value at the infinitely distant point of the plane of z, as x increases indefinitely. Since the system is a condensing system, there is no discontinuity in ψ at ∞; the line of force running

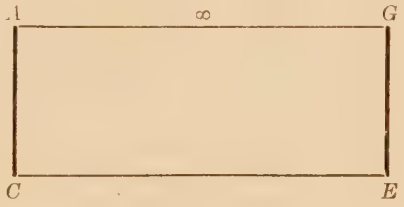

FIG. 2.

from $x = -\infty$ to A is a continuation of the same line $\psi = \text{const.}$, and it terminates in the χ plane at the value of ϕ on DB. This is continued by a line $\phi = \text{const.}$, falling to the starting point. The figure in the χ plane is therefore a rectangle (Fig. 2). Its height is $2\pi e_0$, and its length the difference of potential of the two plates. This may be denoted by $2\phi_1$.

The solution of the condenser problem is expressed by that relation between the complex variables χ and z by which the upper half of the z plane, bounded by the axis of x and the two half-segments, is transformed conformally into the rectangle in the χ plane. By the theory of Schwarz* both regions can be transformed into the upper half-plane of an auxiliary variable t, and the points 0, E, ∞ in the z plane can be transformed to the points 0, 1, ∞ in the t plane. Then to the points F, G

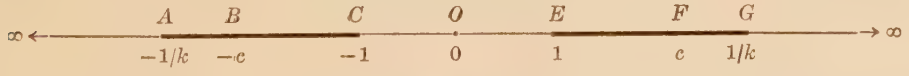

FIG. 3.

* The classical reference is H. A. Schwarz, *Ges. math. Abhandlungen*, Bd. 2, Berlin 1890, p. 65. Accounts of the theory are given in many books.

will correspond some real values of t, which can be written c and $1/k$. From the symmetry it appears that the values of t corresponding to the points A, B, C must be $-1/k$, $-c$, -1. The correspondence is illustrated in Fig. 3.

The equations, by which the transformations are effected, are of the forms

$$\frac{dz}{dt} = A \frac{t^2-c^2}{\left\{\left(t+\frac{1}{k}\right)(t+1)(t-1)\left(t-\frac{1}{k}\right)\right\}^{\frac{1}{2}}},$$

$$\frac{d\chi}{dt} = B \frac{1}{\left\{\left(t+\frac{1}{k}\right)(t+1)(t-1)\left(t-\frac{1}{k}\right)\right\}^{\frac{1}{2}}},$$

where A and B are real.

The integrations are performed by putting

$$t = \operatorname{sn} w,$$

where the modulus of the elliptic function is k, and w may be taken to vanish with t. Then χ is simply proportional to w if the arbitrary additive constant in χ is adjusted suitably. The values of w at various points of the z figure are shown in the Table :

A	C	E	G	∞
$-K+iK'$	$-K$	K	$K+iK'$	iK'

Since ψ increases by $2\pi e_0$ as w goes from K to $K+iK'$, we have

$$\chi = \frac{2\pi e_0}{K'} w ;$$

and, since ϕ increases by $2\phi_1$ as w goes from $-K$ to K, we have

$$2\phi_1 = \frac{4\pi e_0}{K'} K.$$

Hence the capacity of the condenser per unit of length, which is $e_0/(2\phi_1)$, is $K'/4\pi K$.

The value c of t at F is not arbitrary, but must be such that z has the same value at G as at E. Now

$$\frac{dz}{dw} = -Ak(\operatorname{sn}^2 w - c^2)$$

$$= \frac{A}{k}(1-k^2\operatorname{sn}^2 w - 1 + k^2c^2),$$

giving $$z = \frac{A}{k}\left\{Z(w)+w\left(\frac{E}{K}-1+k^2c^2\right)\right\},$$

where Z is the symbol of Jacobi's Zeta-function. We must therefore have

$$Z(K+iK')+(K+iK')\left(\frac{E}{K}-1+k^2c^2\right) = Z(K)+K\left(\frac{E}{K}-1+k^2c^2\right),$$

and since

$$Z(K) = 0, \qquad Z(K+iK')= -\frac{i\pi}{2K}, \qquad EK'+E'K-KK' = \frac{\pi}{2},$$

this gives $$k^2c^2 = \frac{E'}{K'},$$

and then the formula* for z becomes

$$z = \frac{A}{k}\left\{Z(w)+\frac{\pi w}{2KK'}\right\}.$$

It is convenient to put $$c = \operatorname{sn}(K+iv_0),$$

where $K' > v_0 > 0$, and then

$$K'k^2\operatorname{sn}^2(K+iv_0) = E'.$$

To make $z = h$ at $w = K$ we must have

$$h = \frac{A}{k}\frac{\pi}{2K'},$$

and then to make $z = h+ib$ at $w = K+iv_0$ we must have

$$\frac{b}{h} = \frac{2K'}{\pi}\left\{-i\,Z(K+iv_0)+\frac{\pi v_0}{2KK'}\right\}.$$

Since v_0 is known when k is known, this is an equation to determine k.

3. *Calculation of capacity.*—The elementary theory gives for the capacity of the condenser per unit of length the value $b/(4\pi h)$ when h/b is

* The formula may be identified with Michell's by noting that

$$Z(w, k)+\frac{\pi w}{2KK'} = -\frac{\pi}{2K'}+iZ\{i(w+K-iK'), k'\}.$$

In the *Messenger of Math.*, Vol. 20 (1891), p. 144, W. Burnside records the result that the real part of z, as given by the formula, is the stream-function for fluid streaming, according to the electrical law of flow, in the plane of t past the two linear obstacles defined by $1/k > t > 1$ and $-1 > t > -1/k$, the general direction of the stream being transverse to the line of the obstacles. For this reference I am indebted to Prof. L. N. G. Filon.

small. When this is so, K'/K must be of the same order as b/h, and then K must be nearly equal to $\frac{1}{2}\pi$, and k must be very small of the order $e^{-K'}$. Tables of elliptic integrals are not then of any use, and the simplest way to proceed seems to be by use of the q-series. These may be written

$$Z(w) = \frac{\pi}{K} \sum_{n=1}^{\infty} \frac{\sin(n\pi w/K)}{\sinh(n\pi K'/K)},$$

$$k^2 \operatorname{sn}^2 w = 1 - \frac{E}{K} - \frac{\pi^2}{K^2} \sum_{n=1}^{\infty} \frac{n \cos(n\pi w/K)}{\sinh(n\pi K'/K)},$$

by substituting for q its value $e^{-\pi K'/K}$. If we put

$$w = u + iv,$$

the series hold for all values of u, and for all values of v such that $K' > v > -K'$.

The equation $\operatorname{sn}^2(K + iv_0) = E'/(K'k^2)$, in which K' is large, E' nearly unity, and k^2K' very small, shows that v_0 must be large compared with K. In the serial expressions we may put with sufficient approximation $E = K = \frac{1}{2}\pi$. Then we have

$$k^2 \operatorname{sn}^2(K + iv_0) = 4 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n \cosh 2nv_0}{\sinh 2nK'}$$

$$= 4 \sum_{n=1}^{\infty} (-1)^{n+1} n e^{-2n(K'-v_0)}$$

approximately.

Reducing the series to its first term, we find that the equation giving v_0 becomes approximately $e^{2(K'-v_0)} = 4K'$.

To the same order of approximation, we have

$$Z(K + iv_0) = -2ie^{-2(K'-v_0)} = -i/(2K'),$$

and hence

$$\frac{b}{h} = \frac{K'}{K} \left\{1 - \frac{1}{2K'}(1 + \log 4K')\right\},$$

or, to the same order,

$$\frac{K'}{K} = \frac{b}{h} \left\{1 + \frac{h}{\pi b}\left(1 + \log \frac{2\pi b}{h}\right)\right\}.$$

This is the approximate formula given by Bromwich, and shown by him to be deducible from a result of Kirchhoff's, dating back to 1877. It appears that the value for the capacity per unit of length, that is given by elementary theory, should be increased by a fraction of itself, which

is approximately $(h/\pi b)\{1+\log(2\pi b/h)\}$. With $b/h = 100$ this is about 0·0237.

Closer approximations can be obtained if desired, but, in view of the errors involved in neglecting the thickness of the plates, it is doubtful whether they would have much physical meaning.

4. *Distribution of charge.*—The surface density σ at any point of the positively charged plate is given by the equation

$$\sigma = \frac{1}{4\pi}\left|\frac{d\chi}{dz}\right| = \frac{\pi e_0}{4k^2hK'^2|t^2-c^2|},$$

in which $1/k > t > 1$. When y is near to b the equation giving dz/dt shows that $b-y$ is nearly proportional to $(t-c)^2$, and therefore σ becomes infinite at the edge in the order $(b-y)^{-\frac{1}{2}}$, as we should expect. But there does not seem to be any simple formula expressing σ as a function of y with $(b^2-y^2)^{-\frac{1}{2}}$ as a factor.

The increment of ψ as w goes from K to $K+iv_0$ being $2\pi e_0 v_0/K'$, the fraction v_0/K' of the charge on either plate is on the face nearer to the other, and the fraction $(K'-v_0)/K'$ is on the more remote face. When b/h is large the latter fraction is approximately

$$\{h/(\pi b)\}\log(2\pi b/h).$$

If $b/h = 100$ this is about 0·0205.

5. *Further discussion of the solution.*—The solution has been obtained by considering one half of the symmetrical figure formed by the conducting segments and the line that bisects them both. Exactly similar work would apply to the other half of the figure, and the same relations would connect z, t, w, χ in the corresponding regions. For example, the lower half of the t plane is transformed by the relation $t = \mathrm{sn}\, w$ into a rectangle in the w plane, whose corners can be taken to be at the points $-K$, $-K-iK'$, $K-iK'$, K. They will be taken so here, but they might be at any points whose affixes differ from these by multiples of $2iK'$. The values of z and t, in the two half-planes of these variables, fit along the lines that answer to the common side $v = 0$ of the two rectangles in the w plane, and therefore the artificial barriers corresponding to this side may be removed; and then the whole plane of z, with certain barriers, is transformed into the whole plane of t, with certain barriers, and both these regions are transformed into the rectangle in the w plane bounded by $u = \pm K$, $v = \pm K'$, which is half of a period rectangle for $\mathrm{sn}\, w$. The barriers in the z plane are the conducting segments DB, HF, the line

along the real axis from G to ∞ on the right, and the line along the real axis from A to ∞ on the left. The barriers in the t plane are the segments $-1 > t > -1/k$ and $1/k > t > 1$, the line along the real axis from $1/k$ to ∞ on the right, and the line along the real axis from $-1/k$ to ∞ on the left.

Since t and z, as functions of w, have the period $2iK'$, the values of t and z in the two half-planes of these variables also fit along the artificial barriers which correspond to the edges $v = \pm K'$ of the w rectangle. By the further transformation

$$\zeta = Ae^{\pi w/K'},$$

where A is a real positive constant, the sides $u = \pm K$ of this rectangle are transformed into two concentric circles, and the sides $v = \pm K'$ are brought to coincidence along the line joining $\zeta = -Ae^{-\pi K/K'}$ and $\zeta = -Ae^{\pi K/K'}$. This line and the corresponding artificial barriers in the z and t planes may then be omitted, and then the whole plane of z, or of t, with barriers along the conducting segments only, is transformed into the ring between two concentric circles in the ζ plane.* The solution of any plane condenser problem (for cylinders each outside the other) is effectively the same thing as transforming the region of a plane that is bounded internally by two closed curves, external to each other, into such a circular ring. It is also the same thing as transforming this region into one in which the two curves become two equal straight segments in line, as in the t plane.

In the solution already obtained the relation between dz/dt and t contains an unknown constant c, whose value is virtually determined by the condition that z must be a uniform function of t.

It is noteworthy that, in this solution, the ends of the segments in one plane do not correspond to the ends of the segments in the other.

6. *Equal breadths. Asymmetric position.*—If we add to the expression for dz/dt in Art. 2 a term of the form iC, where C is a positive constant, we shall add to z a term of the form iCt. Then every point on the segment HF will be displaced in the positive direction of the axis of y through a distance proportional to the t of the point, and every point of the segment DB will be displaced in the opposite direction.

* The relation between the two planes ceases to be conformal at the point at ∞ in the z, or t, plane and the corresponding point in the ζ plane, because $d\zeta/dz$, or $d\zeta/dt$, vanishes there.

The resulting expression may be written

$$\frac{dz}{dt} = Ak\left\{\frac{c^2-t^2}{\sqrt{\{(1-t^2)(1-k^2t^2)\}}} + i\frac{B}{k}\right\}$$

by introducing a new positive constant B in place of C. If, as before, $t = \operatorname{sn} w$, dz/dt has no zeroes within the w rectangle bounded by $u = \pm K$ and $v = \pm K'$, and it has four zeroes on the upright sides of the rectangle. The values of w at these zeroes are of the form

$$K+iv_1, \quad K-iv_2, \quad -K+iv_2, \quad -K-iv_1.$$

The proof is easy and may be omitted here. These values correspond to the ends of the segments.

To make z a uniform function of t we must have, as before,

$$k^2K'c^2 = E',$$

and, if $c = \operatorname{sn}(K+iv_0)$, as before, it is not difficult to prove that

$$K' > v_1 > v_0 \quad \text{and} \quad v_0 > v_2 > 0.$$

With this value of c the formula for z becomes

$$z = \frac{A}{k}\left\{Z(w) + \frac{\pi w}{2KK'} + iBk\operatorname{sn} w\right\}.$$

The right-hand member is an uneven function of w, and thence it follows that the segments are equal and are situated symmetrically with respect

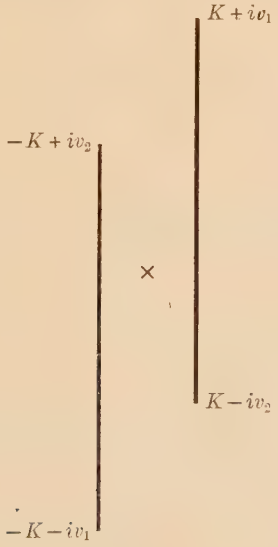

FIG. 4.

to the origin of z, as shown in Fig. 4, where the values attached to the ends of the segments are values of w.

Since z and t, as functions of w, have the period $2iK'$, the relations connecting z, t, w transform the whole plane of z outside the two parallel conducting segments into the whole plane of t outside two equal conducting segments in line; and the condenser problem for two parallel plates of equal breadths, but not directly opposite to each other, is solved by combining with the above relation between z and w the relation $\chi = 2\pi e_0 w/K'$. The capacity per unit of length is $K'/(4\pi K)$, as before.

Let the segments in the z plane be of length $2b$ and at a distance $2h$ apart; and let one project a distance d beyond the other, so that the length of the part of either which is directly opposite to the other is $2b-d$. Then we must have

$$h = \frac{A}{k}\frac{\pi}{2K'},$$

$$2ib = \frac{A}{k}\Big[Z(K+iv_1)-Z(K-iv_2)+\frac{i\pi(v_1+v_2)}{2KK'} + iBk\{\mathrm{sn}(K+iv_1)-\mathrm{sn}(K-iv_2)\}\Big],$$

$$id = \frac{A}{k}\Big[Z(K+iv_1)-Z(-K+iv_2)+\frac{i\pi(v_1-v_2)}{2KK'} + iBk\{\mathrm{sn}(K+iv_1)-\mathrm{sn}(-K+iv_2)\}\Big],$$

and v_1, v_2, B must be connected by the equations

$$\frac{\mathrm{sn}^2(K+iv_1)-\mathrm{sn}^2(K+iv_0)}{\mathrm{cn}(K+iv_1)\,\mathrm{dn}(K+iv_1)} = \frac{\mathrm{sn}^2(K-iv_2)-\mathrm{sn}^2(K+iv_0)}{\mathrm{cn}(K-iv_2)\,\mathrm{dn}(K-iv_2)} = i\frac{B}{k},$$

in which v_0 is known when k is known. These are sufficient equations to determine A, B, v_1, v_2, k.

With a view to approximate calculation we shall require, in addition to the q-series for $k^2\mathrm{sn}^2 w$ and $Z(w)$ already cited, the q-series for $k^2\,\mathrm{sn}w\,\mathrm{cn}w\,\mathrm{dn}w$. It may be written

$$k^2\,\mathrm{sn}w\,\mathrm{cn}w\,\mathrm{dn}w = \frac{\pi^3}{2K^3}\sum_{n=1}^{\infty}\frac{n^2\sin(n\pi w/K)}{\sinh(n\pi K'/K)},$$

and is valid for the same values of w as the others. When $w = K+iv$, the series become

$$k^2\mathrm{sn}^2(K+iv) = 1-\frac{E}{K}-\frac{\pi^2}{K^2}\sum_{n=1}^{\infty}\frac{(-1)^n n\cosh(n\pi v/K)}{\sinh(n\pi K'/K)},$$

$$k^2\,\mathrm{sn}(K+iv)\,\mathrm{cn}(K+iv)\,\mathrm{dn}(K+iv) = \frac{i\pi^3}{2K^3}\sum_{n=1}^{\infty}\frac{(-1)^n n^2\sinh(n\pi v/K)}{\sinh(n\pi K'/K)},$$

$$Z(K+iv) = \frac{i\pi}{K}\sum_{n=1}^{\infty}\frac{(-1)^n\sinh(n\pi v/K)}{\sinh(n\pi K'/K)}.$$

7. *Calculation of capacity. Small displacement.*—Whenever $2b-d$ is large compared with h, the elementary theory gives for the capacity per unit of length the value $(b-\frac{1}{2}d)/(4\pi h)$.

When d is of the same order as h, v_1 and v_2 do not differ much from v_0, and the method of Art. 3 gives the approximate formulae

$$k\,\mathrm{sn}(K+iv_1) = 2e^{-(K'-v_1)}, \qquad k\,\mathrm{cn}(K+iv_1)\,\mathrm{dn}(K+iv_1) = -2ie^{-(K'-v_1)},$$

$$Z(K+iv_1) = -2ie^{-2(K'-v_1)},$$

$$k\,\mathrm{sn}(K-iv_2) = 2e^{-(K'-v_2)}, \qquad k\,\mathrm{cn}(K-iv_2)\,\mathrm{dn}(K-iv_2) = 2ie^{-(K'-v_2)},$$

$$Z(K-iv_2) = 2ie^{-2(K'-v_2)},$$

as well as $$k^2\mathrm{sn}^2(K+iv_0) = 4e^{-2(K'-v_0)} = \frac{1}{K'}.$$

To the same order the equations connecting v_1, v_2, B become

$$2\{e^{-(K'-v_1)}-e^{-(K'-2v_0+v_1)}\} = -2\{e^{-(K'-v_2)}-e^{-(K'-2v_0+v_2)}\} = B,$$

from which $$\sinh(v_1-v_0) = \sinh(v_0-v_2),$$

or $$v_1+v_2 = 2v_0,$$

and $$B = 2\{e^{-(K'-v_1)}-e^{-(K'-v_2)}\}.$$

Hence we have, to the same order,

$$\frac{b}{h} = \frac{K'}{K}\left[-e^{-2(K'-v_1)}-e^{-2(K'-v_2)}+\frac{v_0}{K'}+2\{e^{-(K'-v_1)}-e^{-(K'-v_2)}\}^2\right]$$

$$= \frac{K'}{K}\left\{1-\frac{\log 4K'}{2K'}+e^{-2(K'-v_1)}+e^{-2(K'-v_2)}-4e^{-2(K'-v_0)}\right\}$$

$$= \frac{K'}{K}\left[1-\frac{\log 4K'}{2K'}+\frac{1}{4K'}\{e^{2(v_1-v_0)}+e^{-2(v_1-v_0)}-4\}\right],$$

giving approximately

$$\frac{K'}{K} = \frac{b}{h}\left\{1+\frac{h}{\pi b}\left(2-\cosh\gamma+\log\frac{2\pi b}{h}\right)\right\},$$

where γ is written for $2(v_1-v_0)$. Also we have, to the same order,

$$\frac{d}{h} = \frac{K'}{K}\left[-2e^{-2(K'-v_1)}+2e^{-2(K'-v_2)}+\frac{v_1-v_2}{K'}+4\left\{e^{-2(K'-v_1)}-e^{-2(K'-v_2)}\right\}\right]$$

$$= \frac{K'}{K}\left[\frac{2(v_1-v_0)}{K'}+\frac{1}{2K'}\left\{e^{2(v_1-v_0)}-e^{-2(v_1-v_0)}\right\}\right],$$

so that γ is given by the equation

$$\gamma+\sinh\gamma = \frac{\pi d}{2h}.$$

For any value of d/h in the neighbourhood of unity the value of K'/K is but little affected by displacement. For example, if $b/h = 100$, the change in K'/K as d increases from 0 to h is a decrease from 102·37 to 102·27.

8. *Displacement comparable with half-breadth.*—When $b-\frac{1}{2}d$ and d are both large compared with h this approximation fails, because v_1 is so near to K' that $e^{-(K'-v_1)}$ cannot be treated as small.

The series for $k^2\,\mathrm{sn}^2(K+iv_1), \ldots$ can be summed approximately, yielding

$$k\,\mathrm{sn}(K+iv_1) = 2e^{-(K'-v_1)}\left\{1+e^{-2(K'-v_1)}\right\}^{-1},$$

$$k\,\mathrm{cn}(K+iv_1)\,\mathrm{dn}(K+iv_1) = -2ie^{-(K'-v_1)}\left\{1-e^{-2(K'-v_1)}\right\}\left\{1+e^{-2(K'-v_1)}\right\}^{-2},$$

$$Z(K+iv_1) = -2ie^{-2(K'-v_1)}\left\{1+e^{-2(K'-v_1)}\right\}^{-1}.$$

To simplify the Algebra we shall write

$$e^{K'-v_1} = 2\beta_1\sqrt{K'}, \quad e^{K'-v_2} = 2\beta_2\sqrt{K'}, \quad e^{K'-v_0} = 2\sqrt{K'}.$$

The equations connecting v_1, v_2, B become

$$\frac{4\beta_1^2K'+1}{4\beta_1^2K'-1}\left(\frac{4\beta_1K'}{4\beta_1^2K'+1}-\frac{4\beta_1^2K'+1}{4\beta_1K'}\right) = \beta_2-\frac{1}{\beta_2} = B\sqrt{K'}.$$

The equations connecting b, d, h with v_1, v_2 give

$$\frac{2b+d}{h} = \frac{2K'}{\pi}\left\{\frac{2}{i}Z(K+iv_1)+\frac{\pi v_1}{KK'}+2Bk\,\mathrm{sn}(K+iv_1)\right\},$$

and this is approximately

$$\frac{b+\frac{1}{2}d}{h} = \frac{K'}{K}\left\{-\frac{2e^{-2(K'-v_1)}}{1+e^{-2(K'-v_1)}}+1-\frac{K'-v_1}{K'}+2B\frac{e^{-(K'-v_1)}}{1+e^{-2(K'-v_1)}}\right\}$$

$$= \frac{K'}{K}\left\{1-\frac{1}{K'}\log(2\beta_1\sqrt{K'})-\frac{2}{4\beta_1^2K'+1}+B\sqrt{K'}\,\frac{4\beta_1}{4\beta_1^2K'+1}\right\}$$

$$= \frac{K'}{K}\left[\frac{4\beta_1^2K'-1}{4\beta_1^2K'+1}+\frac{16\beta_1^2K'}{(4\beta_1^2K'-1)(4\beta_1^2K'+1)}\right.$$

$$\left.-\frac{1}{K'}\left\{\frac{4\beta_1^2K'+1}{4\beta_1^2K'-1}+\log(2\beta_1\sqrt{K'})\right\}\right]$$

$$= \frac{K'}{K}\left\{\frac{4\beta_1^2K'+1}{4\beta_1^2K'-1}\left(1-\frac{1}{K'}\right)-\frac{1}{K'}\log(2\beta_1\sqrt{K'})\right\},$$

or, still more roughly,

$$\frac{b+\frac{1}{2}d}{h} = \frac{K'}{K}\,\frac{4\beta_1^2K'+1}{4\beta_1^2K'-1}.$$

Again, the equations connecting b, d, h with v_1, v_2 give

$$\frac{2b-d}{h} = \frac{2K'}{\pi}\left\{-\frac{2}{i}Z(K-iv_2)+\frac{\pi v_2}{KK'}-2Bk\,\mathrm{sn}\,(K-iv_2)\right\},$$

and the approximation used before for v_2 gives

$$\frac{b-\frac{1}{2}d}{h} = \frac{K'}{K}\left\{-2e^{-2(K'-v_2)}+1-\frac{K'-v_2}{K'}-2Be^{-(K'-v_2)}\right\}$$

$$= \frac{K'}{K}\left\{1-\frac{1}{K'}\log(2\beta_2\sqrt{K'})-\frac{1}{2\beta_2^2K'}-\frac{1}{K'}\left(1-\frac{1}{\beta_2^2}\right)\right\}$$

$$= \frac{K'}{K}\left[1-\frac{1}{K'}\left\{1+\log(2\beta_2\sqrt{K'})-\frac{1}{2\beta_2^2}\right\}\right].$$

To determine β_1 it is a sufficient approximation to take $(b-\frac{1}{2}d)/h$ to be the same as K'/K. Then β_1 is given approximately by the equation

$$\frac{4\beta_1^2K'+1}{4\beta_1^2K'-1} = \frac{b+\frac{1}{2}d}{b-\frac{1}{2}d};$$

when β_1 is found from this equation, β_2 is given approximately by the equation

$$\beta_2 = \frac{4\beta_1K'}{4\beta_1^2K'-1},$$

and then K'/K is given approximately by the equation

$$\frac{K'}{K} = \frac{b-\frac{1}{2}d}{h}\left[1+\frac{2h}{\pi(b-\frac{1}{2}d)}\left\{1+\log(2\beta_2\sqrt{K'})\right\}\right],$$

in which, on the right, K' may be given the value $\pi(b-\frac{1}{2}d)/(2h)$.

The following short Table gives some corresponding values of d/b and $K'h/\{K(b-\frac{1}{2}d)\}$, the factor by which the capacity per unit of length (given by the elementary theory) ought to be multiplied. The value of b/h has been taken to be 100, and the approximations made above have been assumed to be sufficient.

d/b	0·2	0·3	0·4	0·5	0·6	0·7	0·8
$K'h/\{K(b-\frac{1}{2}d)\}$	1·053	1·062	1·074	1·090	1·114	1·154	1·233

9. *Large displacement and relatively small capacity.*— When d is large compared with h, but $2b-d$ is not, v_2 will be small compared with K', and it will no longer be sufficient to replace $\cosh 2v_2$ and $\sinh 2v_2$ by $\frac{1}{2}e^{2v_2}$. For quite moderate values of K'/K, however, $e^{\pi K'/K}$ will be large, and $\frac{1}{2}\pi$ will be a sufficient approximation to K, so that the previous approximation will be adequate for v_1. Thus we have

$$\frac{b+\frac{1}{2}d}{h} = \frac{K'}{K}\left[\frac{4\beta_1^2K'+1}{4\beta_1^2K'-1} - \frac{1}{K'}\left\{\frac{4\beta_1^2K'+1}{4\beta_1^2K'-1}+\log(2\beta_1\sqrt{K'})\right\}\right].$$

The serial expressions for the functions give, with sufficient approximation,

$$k^2\,\mathrm{sn}^2(K-iv_2) = 8e^{-2K}\cosh 2v_2,$$

$$k\,\mathrm{cn}(K-iv_2)\,\mathrm{dn}(K-iv_2) = 2\sqrt{2}\,.\,ie^{-K'}\frac{\sinh 2v_2}{\sqrt{(\cosh 2v_2)}},$$

$$Z(K-iv_2) = 4ie^{-2K'}\sinh 2v_2,$$

and thence we have

$$\frac{b-\frac{1}{2}d}{h} = \frac{K'}{K}\left\{-4e^{-2K'}\sinh 2v_2+\frac{v_2}{K'}-B\,2\sqrt{2}\,.\,e^{-K'}\sqrt{(\cosh 2v_2)}\right\}.$$

Also the equation

$$\frac{\mathrm{sn}^2(K-iv_2)-\mathrm{sn}^2(K+iv_0)}{\mathrm{cn}(K-iv_2)\,\mathrm{dn}(K-iv_2)}=i\frac{B}{k}$$

gives $$2\sqrt{2}\,.\,B=\frac{\sqrt{(\cosh 2v_2)}}{\sinh 2v_2}\left(\frac{e^{K'}}{K'}-8\cosh 2v_2\,.\,e^{-K'}\right),$$

so that, if $K'e^{-2K'}$ is neglected,

$$\frac{b-\frac{1}{2}d}{h}=\frac{2}{\pi}(v_2-\coth 2v_2).$$

When v_2 is found from this equation, B is given with sufficient approximation by the formula

$$B=\frac{\sqrt{(\cosh 2v_2)}\,e^{K'}}{2\sqrt{2}\,.\,K'\sinh 2v_2},$$

and the equation

$$B=\frac{4\beta_1\sqrt{K'}}{4\beta_1^2K'-1}-\frac{1}{K'}\frac{(4\beta_1^2K'+1)^2}{(4\beta_1^2K'-1)4\beta_1\sqrt{K'}}$$

becomes an equation giving β_1 in terms of K'. In this we put

$$2\beta_1\sqrt{K'}=1+\beta,$$

where β is small, and get approximately

$$\frac{1}{\beta}\left(1-\frac{1}{K'}\right)=\frac{\sqrt{(\cosh 2v_2)}\,e^{K'}}{2\sqrt{2}\,.\,K'\sinh 2v_2}.$$

Then the equation expressing $(b+\frac{1}{2}d)/h$ in terms of K, K', β_1 becomes approximately

$$\frac{b+\frac{1}{2}d}{h}=\frac{K'}{K}\,\frac{1}{\beta}\left(1-\frac{1}{K'}\right),$$

or $$e^{K'}=\frac{\pi\sqrt{2}\,.\sinh 2v_2}{\sqrt{(\cosh 2v_2)}}\,\frac{b+\frac{1}{2}d}{h},$$

an equation giving an approximate value for K'

For example, if $d=2b$, so that no part of one conducting plate is directly opposite to any part of the other, but the plane of the two nearest edges is at right angles to the planes of the plates, the equation for v_2 is $\coth 2v_2=v_2$, the root is 1·03267 ... ; and, if $b/h=100$, the value of K' is 7·4516, and the corresponding value of K'/K is about 4·74.

It may be noted that even this moderate value of K'/K is outside the limits of the range for which the elliptic integrals were tabulated by Legendre, the corresponding value of k being about $\sin(0^\circ 9')$.

10. *Plates of unequal breadths in the same plane.* — The condenser problem for two thin plates can also be solved in the case where the plates cut a plane at right angles to their edges in two segments of straight lines of unequal lengths. We begin with the case where the segments are in the same straight line, which we take to be the axis of x in the plane of a complex variable z.

Let the ends of the segments in order from left to right be at x_1, x_2, x_3, x_4, let the lengths of the segments x_2-x_1 and x_4-x_3 be b_1 and b_2, and let the length of the gap x_3-x_2 between them be $2a$. We shall suppose $b_2 > b_1$. The region in the plane of z, bounded internally by the two segments, can be transformed into the region in the plane of an auxiliary complex variable t, bounded internally by two equal segments in line, by an equation of the form

$$z = z_0 + \frac{\lambda}{t+\nu},$$

where $-\nu$ is the (unknown) value of t that answers to the infinitely distant point in the z plane. The constants λ, z_0, and ν are real, and can be adjusted so that the ends (x_1, x_2, x_3, x_4) of the segments in the z plane may correspond to the points $1/k$, 1, -1, $-1/k$ in order in the t plane, k being at present undetermined. The equations

$$b_1 = x_2-x_1 = \lambda\frac{1-k}{(\nu+1)(k\nu+1)}, \qquad 2a = x_3-x_2 = \frac{2\lambda}{\nu^2-1},$$

$$b_2 = x_4-x_3 = \lambda\frac{1-k}{(\nu-1)(k\nu-1)}$$

determine λ, ν, k, and then the equation

$$z_0 = x_1 - \frac{\lambda k}{k\nu+1}$$

determines z_0. It appears that k satisfies the equation

$$k^2-2k\left\{1+\frac{b_1 b_2}{a(2a+b_1+b_2)}\right\}+1 = 0$$

which has real positive roots, one of them being less than unity. This value being taken for k, ν is given by the equation

$$\nu = \frac{b_2-b_1}{(2a+b_1+b_2)k-2a},$$

and this ν is positive and greater than $1/k$, b_2 being greater than b_1. Then, ν being known, λ is $a(\nu^2-1)$.

11. *Parallel plates of unequal breadths.*—When the segments in which the plates cut a plane at right angles to their edges are parallel, there is a symmetrical position, shown in Fig. 5. The line bisecting both

FIG. 5.

segments at right angles is made up of lines of force. As in the previous case of symmetry, we may consider half the figure at a time. By a transformation of the form

$$\frac{dz}{dz'} = A' \frac{(z'-c_1')(z'-c_2')}{\sqrt{\{(z'-x_1')(z'-x_2')(z'-x_3')(z'-x_4')\}}}$$

we may transform half the figure in the plane of the parallel segments (the plane of z) into the region bounded by the line of two segments in line in the plane of z'. These segments will be unequal, and a further transformation of the form

$$z' = z_0 + \frac{\lambda}{t+\nu}$$

is required to transform them into equal segments in line. The result is that the plane of z, outside the two unequal parallel segments, is transformed into the plane of t, outside two equal segments in line, by an equation of the form

$$\frac{dz}{dt} = A \frac{(t-c_1)(t-c_2)}{(t+\nu)^2\sqrt{\{(1-t^2)(1-k^2t^2)\}}},$$

provided the constants satisfy the conditions which secure that z is a uniform function of t.

It requires only straightforward work on the elliptic integrals to show that these conditions are

$$c_1 c_2 = -\frac{\nu^2 E' - K'}{k^2\nu^2 K' - E'}, \qquad (\nu + c_1)(\nu + c_2) = \frac{(\nu^2-1)(k^2\nu^2-1)K'}{k^2\nu^2 K' - E'},$$

and to show that, when these conditions are satisfied, z has the form

$$z = \text{const.} + B\left\{Z(w) + \frac{\pi w}{2KK'} + \frac{\operatorname{cn} w \operatorname{dn} w}{\nu + \operatorname{sn} w}\right\},$$

in which B is a real constant and $t = \operatorname{sn} w$, as before. The constants B, ν, k are then to be determined by the conditions that the segments have given lengths and are at a given distance apart. The solution of the condenser problem can be completed as usual.

For an asymmetric position, the parallelism of the plates being maintained, a term of the form iC must be added to the above expression for dz/dt.

Plate and Line Charge.

12. *Charged line in plane of plate.* — If in the preceding theory the breadth of the narrower plate is diminished indefinitely, there arises the limiting case of a thin plate with straight parallel edges under the influence of a uniformly charged line parallel to the edges. Such cases can be investigated independently. The simplest is that where the plate and line meet a transverse plane in a straight segment and a point in line with the segment. Let the plane be the plane of a complex variable z, and the line of the segment the axis of x, and let the charged point and the ends of the segment be at $z = -\nu, -1, 1$. Let the charges on the plate and line be e_0 and $-e_0$ per unit of length.

Exactly as in Art. 2 we may draw the χ figure answering to the upper half of the z plane. It is as shown in Fig. 6, where the values attached

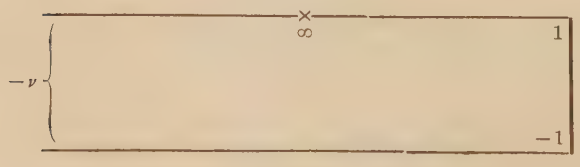

Fig. 6.

to points are values of z. The breadth of the half-strip is $2\pi e_0$. The relation between χ and z is then given by an equation of the form

$$\frac{d\chi}{dz} = \frac{A}{(z+\nu)\sqrt{(z^2-1)}},$$

where A is real. To integrate it we put

$$z = \cosh w,$$

and we may take $w = u+iv$, $u \geqslant 0$, $\pi \geqslant v \geqslant 0$. Then the upper half of the z plane is transformed into a half-strip of breadth π in the

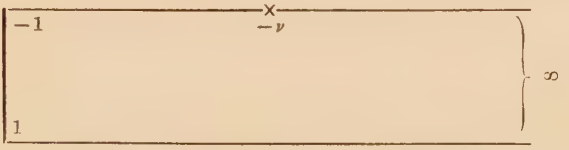

FIG. 7.

w plane, as shown in Fig. 7, where the values attached to points are values of z. We shall put

$$\nu = \cosh \beta$$

and suppose β to be real and positive. To the value $-\nu$ of z there answers the value $\beta+i\pi$ of w. Then we have

$$\frac{d\chi}{dw} = \frac{A}{\cosh w + \cosh \beta},$$

and this equation can be integrated in the form

$$\chi = \frac{A}{\sinh \beta} \log \frac{\cosh \frac{1}{2}(w+\beta)}{\cosh \frac{1}{2}(w-\beta)},$$

if the arbitrary additive constant in χ is chosen so that $\phi = 0$ on the segment, where $u = 0$.

In the lower half of the z plane the same relations would connect z, χ, w, if w were taken to lie in an adjacent half-strip, for example, that given by $u \geqslant 0$, $2\pi \geqslant v \geqslant \pi$.

The constant A may be shown to be $-2e_0 \sinh \beta$. Then

$$\chi = 2e_0 \log \frac{\cosh \frac{1}{2}(w-\beta)}{\cosh \frac{1}{2}(w+\beta)},$$

and the value of ϕ at the infinitely distant point in the z plane is $-2e_0\beta$.

The surface density σ at any point of the plate is given by the formula

$$\sigma = \frac{1}{4\pi} \left| \frac{d\chi}{dz} \right|_{v=0} = \frac{e_0\sqrt{(\nu^2-1)}}{2\pi(\nu+x)\sqrt{(x^2-1)}};$$

and, if O, A, B are the charged point and the ends of the segment, and P is any point of the segment, this may be written

$$\sigma = \frac{e_0}{2\pi} \frac{\sqrt{(OA \,.\, OB)}}{OP\sqrt{(AP \,.\, BP)}}.$$

13. *Charged line and plate in full view.*—A second simple case of the problem of the plate and charged line arises when the point, in which the line cuts a plane transverse to the edges of the plate, is equidistant from the ends of the segment, in which the plate cuts the same plane. In this case the perpendicular from the point to the segment bisects the segment. We shall take the plane of the point and segment to be the plane of a complex variable z_1, and this line of symmetry as the axis of x_1: we shall also take the origin at the middle point of the segment, the charged point at $z_1 = -d$, the length of the segment as $2b$, and the charges on the plate and line as e_0 and $-e_0$ per unit of length.

We can transform the upper half of the figure bounded by the axis of x_1 and the segment into the upper half of the z plane in the previous problem, by a relation of the form

$$\frac{dz_1}{dz} = A \frac{z-a}{\sqrt{(z^2-1)}},$$

where a is the value of z that answers to the end $z_1 = ib$ of the segment. The constant a is not arbitrary but must be determined by the condition that z_1 has the same value zero at $z = 1$ as at $z = -1$. This condition requires that $a = 0$. We have then

$$z_1 = A\sqrt{(z^2-1)}.$$

To make $z_1 = ib$ when $z = 0$ we must have $A = b$. The present problem will then be transformed into the previous problem if $z_1 = -d$ when $z = -\nu$. The solution is then expressed by the equations

$$z_1 = b \sinh w, \qquad \chi = 2e_0 \log \frac{\cosh \frac{1}{2}(w-\beta)}{\cosh \frac{1}{2}(w+\beta)}, \qquad d = b \sinh \beta.$$

If we put

$$\cosh \beta = \sec \alpha, \qquad \sinh \beta = \tan \alpha,$$

d is $b \tan \alpha$ and $\pi - 2\alpha$ is the angle subtended by the segment at the charged point.

The solution holds in the whole plane of z_1 if w is subject to the

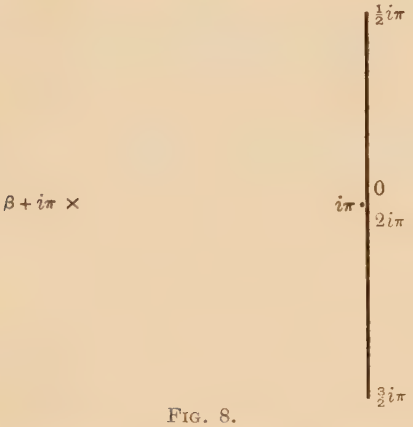

FIG. 8.

restrictions expressed by $u \geqslant 0$, $2\pi \geqslant v \geqslant 0$. The variation of w on the segment is illustrated in Fig. 8, where the values attached to points are values of w.

The surface density σ is given by the formula

$$\sigma = \frac{1}{4\pi}\left|\frac{d\chi}{dz_1}\right|_{u=0} = \frac{1}{4\pi}\,\frac{2e_0\sinh\beta}{|\sin v|(\cosh\beta+\cos v)}\,\frac{1}{b}\,\frac{|\sin v|}{|\cos v|}$$

$$= \frac{e_0\tan\alpha}{2\pi b}\,\frac{\sec\alpha-\cos v}{|\cos v|(\sec^2\alpha-\cos^2 v)} = \frac{e_0\tan\alpha}{2\pi b}\,\frac{\sec\alpha-\cos v}{(\tan^2\alpha+\sin^2 v)|\cos v|}.$$

Now $\tan\alpha = d/b$, and, on the segment, $\sin v = y_1/b$, so this is

$$\sigma = \frac{e_0}{2\pi}\,\frac{d}{d^2+y_1^2}\,\frac{\sqrt{(d^2+b^2)}\mp\sqrt{(b^2-y_1^2)}}{\sqrt{(b^2-y_1^2)}}$$

where the upper sign applies to the side of the segment which is remote

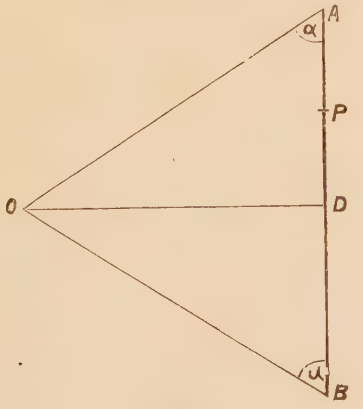

FIG. 9.

from the charged point. With the lettering in Fig. 9 the result may be written

$$\sigma = \frac{e_0}{2\pi}\,\frac{OD}{OP^2}\,\frac{OA \mp \sqrt{(AP.BP)}}{\sqrt{(AP.BP)}}.$$

The fractions of the total charge which are on the two faces of the plate can be evaluated by finding the increment of χ as w goes from 0 to $\frac{1}{2}i\pi$. This is found to be $-4e_0 i \tan^{-1}(\tanh \frac{1}{2}\beta)$, or $-2e_0 i\alpha$. The increment of χ as w goes round the segment from 0 to $2i\pi$ is $-4e_0 i\pi$. It follows that the charge on the remote side is $e_0\alpha/\pi$, and that on the near side is $e_0(\pi-\alpha)/\pi$.

14. *Free charge on cylindrical sheet with parallel edges.*—If we transform the figure in the z_1 plane (Fig. 8) by a complex inversion with respect to the point $z_1 = -d$, the segment becomes an arc of a circle of diameter d, and the charged point recedes to an infinite distance. We thus have the solution of the problem of free distribution of charge on a cylindrical sheet bounded by two generators.

We shall take the plane of the arc to be the plane of a complex variable z_2, and transform the results of the last problem by putting

$$z_2(z_1+d) = d^2.$$

Then, in general,
$$z_2 = d\,\frac{\tan\alpha}{\sinh w + \tan\alpha},$$

where $\pi-2\alpha$ is the angle subtended by the arc at any point, which is on the circumference of the circle and outside the arc; and, at points on the arc,

$$x_2 = d\,\frac{\tan^2\alpha}{\tan^2\alpha+\sin^2 v}, \qquad y_2 = -d\,\frac{\tan\alpha\sin v}{\tan^2\alpha+\sin^2 v},$$

so that the circle is
$$x_2^2+y_2^2 = dx_2.$$

The arrangement of the values of w on the arc is shown in Fig. 10, and

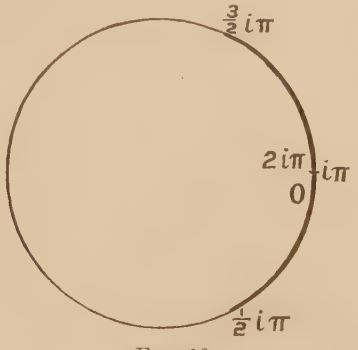

Fig. 10.

the charges on the convex and concave sides of the arc are $e_0(\pi-\alpha)/\pi$ and $e_0\alpha/\pi$, as found in the previous problem. The surface density at any point on the cylindrical sheet is given by the formula

$$\sigma = \frac{1}{4\pi}\left|\frac{d\chi}{dz_2}\right|_{u=0} = \frac{e_0\tan^2\alpha}{2\pi d}\,\frac{\sec\alpha-\cos v}{(\tan^2\alpha+\sin^2 v)|\cos v|}\,\frac{d^2}{|z_2^2|_{u=0}}$$

$$= \frac{e_0}{2\pi d}\,\frac{\sec\alpha-\cos v}{|\cos v|}.$$

In this

$$\sin v = -(y_2/x_2)\tan\alpha = -\tan\theta\tan\alpha,$$

where θ is the vectorial angle of a point on the arc measured from the point on the circumference opposite to the middle point of the arc, as shown in Fig. 11.

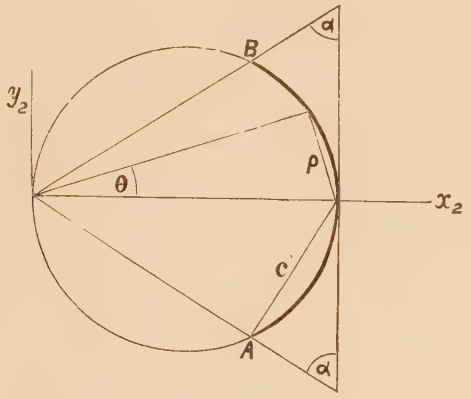

FIG. 11.

Let ρ be the distance of any point on the arc from the middle point of the arc, c the greatest value of ρ, or the distance of either end of the arc from its middle point, Then, from Fig. 11,

$$\rho^2 = d^2\sin^2\theta, \qquad c = d\cos\alpha,$$

and $$\sin^2 v = \frac{\rho^2}{d^2-\rho^2}\,\frac{d^2-c^2}{c^2}, \qquad \cos^2 v = \frac{d^2(c^2-\rho^2)}{c^2(d^2-\rho^2)}.$$

The formula for σ now becomes

$$\sigma = \frac{e_0}{2\pi d}\left\{\sqrt{\left(\frac{d^2-\rho^2}{c^2-\rho^2}\right)} \mp 1\right\},$$

where the upper sign applies to the concave face of the sheet.

15. *Plate and line charge. General position.*—By a complex inversion of the figure in the z_2 plane from a point O on the unoccupied arc the circular arc will be transformed into a segment of a straight line, and we shall have the solution for this segment under the influence of a charged point at O, the segment and point being the intersections of a transverse plane by a charged conducting plate (with parallel edges) and a charged line parallel to its edges. The charges are equal and opposite, as before. Let the plane of z_3 be the plane of the point and segment, and let $\frac{1}{2}\pi-\gamma$ be the vectorial angle of O measured from the origin of z_2 in the plane of the arc (see Fig. 12). Also let D be the point opposite to O on the

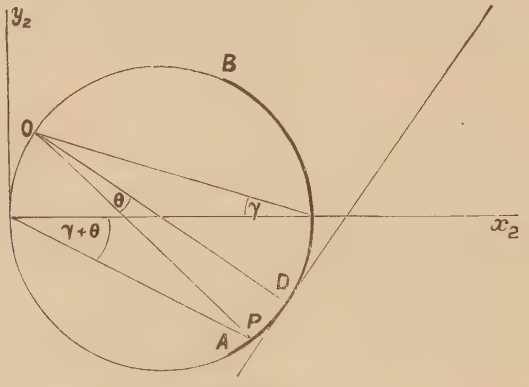

Fig. 12.

circumference, and let the transformation be such that D transforms into itself and OD becomes the axis of x_3. These conditions give

$$z_3\{z_2-d\sin\gamma\, e^{i(\frac{1}{2}\pi-\gamma)}\} = d^2 e^{-2i\gamma}.$$

The arc AB in the plane of z_2 is then transformed into a segment of the tangent at D, in such a way that if the line joining O to any point P on the arc makes with OD an angle θ on one side, the line joining O to the corresponding point on the segment makes with OD an equal angle on the other side. The ends of the segment corresponding to the ends A, B of the arc are

$$z_3 = d\{1+i\cot(\alpha+\gamma)\}, \qquad z_3 = d\{1-i\cot(\alpha-\gamma)\}.$$

The angle a is such that the segment subtends at O an angle $\pi-2a$, and the angle γ is the angle between the internal bisector OE of this angle and the perpendicular OD to the segment (see Fig. 13). The

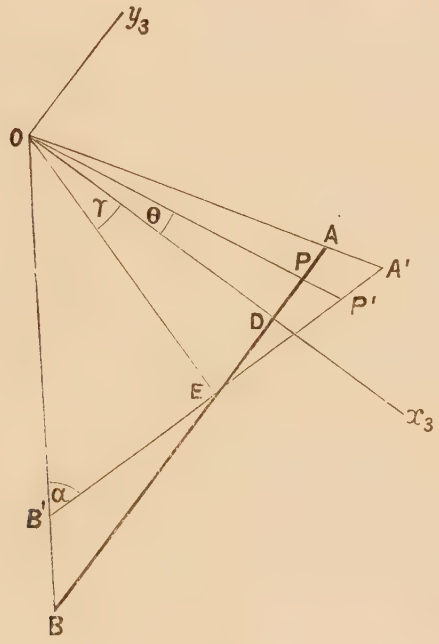

FIG. 13.

length of this perpendicular is d. As before the ratio of charges on the near and remote sides of the segment is $(\pi-a)/a$.

The surface density σ is given by the formula

$$\sigma = \frac{1}{4\pi}\left|\frac{d\chi}{dz_3}\right|_{u=0} = \frac{e_0}{2\pi d}\,\frac{\sec a-\cos v}{|\cos v|}\,\frac{d^2}{|z_3^2|_{u=0}}.$$

Here the value of v at any point P on the segment is given by the equation

$$\sin v = -\tan a\,\frac{y_2}{x_2},$$

where (x_2, y_2) is the corresponding point on the arc. Now if OP makes with OD an angle θ, as shown in Fig. 12, y_2/x_2 is $-\tan(\gamma+\theta)$. Thus

$$\sin v = \tan a \tan(\gamma+\theta),$$

a formula expressing v at a point of the segment. We have now

$$\cos^2 v = 1-\tan^2 a\{\sec^2(\gamma+\theta)-1\} = \sec^2 a\{1-\sin^2 a \sec^2(\gamma+\theta)\}.$$

In Fig. 13 OA, OB, OP meet a line through E at right angles to OE in A', B', P', and the figure shows that

$$\sin\alpha : \cos(\gamma+\theta) = \sin A' : \sin P' = OP' : OA',$$

and therefore

$$\cos v = \pm\sec\alpha \sqrt{\frac{OA'^2-OP'^2}{OA'^2}} = \pm\sec\alpha \frac{\sqrt{(A'P'.B'P')}}{OA'}$$

where the upper sign refers to the side of the segment remote from O. We have then

$$\sigma = \frac{e_0}{2\pi}\frac{OD}{OP^2}\frac{OA' \mp \sqrt{(A'P'.B'P')}}{\sqrt{(A'P'.B'P')}}.$$

The problem solved here includes the case of a semi-infinite plate with a straight edge under the influence of a parallel line charge, for which the solution is known, and it can be verified that the known distribution of charge is the limit of that found here.

Cylinders Charged Freely.

16. *General theory.*—We consider the general problem of determining the distribution of given charges on two given cylindrical conductors, with parallel generators and of infinite length, and the special case where the sum of the charges per unit of length is zero, so that the system forms a condenser. The charges per unit of length will be denoted in general by e_1 and e_2, and the curves in which the corresponding cylinders cut a transverse plane will be denoted by c_1 and c_2. It will be assumed that they are external to each other. Let the plane of c_1 and c_2 be the plane of a complex variable z. The condenser problem ($e_2 = -e_1$) is solved by a particular relation between z and the complex function χ, whose real part is the potential. We shall suppose this relation to be known, and, to fix ideas, shall take e_1 to be positive. We may suppose the line of force $\psi = \text{const.}$, issuing from c_1, going away to ∞, returning, and falling on c_2, to be drawn. Let z travel along this line from c_2 to c_1, then round c_1 in either the positive or the negative sense, then back along the original line of force to c_2, then round c_2 in the same sense (positive or negative) to the starting point. The corresponding point in the χ plane describes a rectangle, whose sides are parallel to the axes of ϕ and ψ, and the relation between z and χ effects a conformal transformation of the region of the z plane that is outside c_1 and c_2 (rendered simply-connected by a barrier along the line of force), into the region of the χ plane that is bounded by the rectangle. Just as in Art. 6, the rectangle may be transformed into the ring between two concentric circles in the plane of

an auxiliary variable ζ, and then the barrier may be omitted, and the region of the z plane outside c_1 and c_2 will be transformed into the ring in the ζ plane. In the condenser problem the point ζ_0 in the ζ ring, which corresponds to the point $z = \infty$, is an ordinary point for χ. But, when $e_1+e_2 \neq 0$, χ must become infinite at ζ_0, as if this point were the intersection by a transverse plane of a uniformly charged line, carrying charge $-(e_1+e_2)$ per unit of length; and ϕ must be constant on the concentric circles, as if they were the traces on a transverse plane of two thin cylindrical conducting sheets, carrying charges e_1 and e_2 per unit of length. If this problem of two concentric circles and a charged point can be solved for any ratio of charges other than -1, it can be solved for any charges. For let χ_1 be the χ function for the particular ratio, and χ_0 that for the condenser problem. Then real constants A and B can be adjusted so that the real part of $A\chi_0+B\chi_1$ is the potential due to any given charges on the concentric circles and their sum (with changed sign) at the special point. On transforming back to the plane of the two curves we shall have the solution for given charges on the conducting cylinders, the special point receding to an infinite distance.

Now this problem of two concentric circles and a special point can be solved; and therefore the problem of the two conducting cylinders with given charges can be solved whenever the condenser problem can be solved for the same two cylinders.

17. *Charged point between concentric circles.*—Let the radii of the outer and inner conducting circles be r_1 and r_2, and the charges on them e_1 and e_2, and let there be a point with charge $-(e_1+e_2)$ between them. We shall determine the potential under two limitations. The first limitation is that the charged point is within the concentric circle of radius $\sqrt{(r_1 r_2)}$. This limitation is not really a restriction of generality, because a preliminary inversion with respect to this circle will transform either of the two concentric circles (r_1 and r_2) into the other, and a point between them, and outside this critical circle, into a point between them and within this circle. The second limitation is that the ratio of the charges e_1, e_2 is not assigned beforehand, but is chosen in the course of the solution in a way that simplifies the analysis. This limitation also is not a restriction of generality, because, as we have seen, if the solution is known for one other ratio of charges in addition to the ratio -1, the problem can be solved for any charges.

Let ζ be a complex variable fixing position in the plane of the concentric circles, and put $\zeta = \sqrt{(r_1 r_2)}e^{(\rho+i\theta)}$. If the doubly-connected region between the two circles is rendered simply-connected by an arti-

ficial barrier along the radius $\theta = 0$, this region is at once transformed into a rectangle in the plane of the variable $\rho + i\theta$, the boundaries being $\rho = \pm \log(r_1/r_2)$, $\theta = 0$, and $\theta = 2\pi$. The ratio $\pi \div \log(r_1/r_2)$ of the sides of this rectangle can be the ratio $K' : K$ associated with a set of Jacobian elliptic functions. If we now put

$$\rho + i\theta = \pi w/K', \qquad w = u + iv,$$

the region between the two concentric circles (with the barrier) is transformed into the w rectangle bounded by $u = \pm K$, $v = 0$, $v = 2K'$. Then by the further transformation $t = \operatorname{sn} w$, this region (without the barrier) is tranformed into the region of the t plane outside the two segments of the real axis defined by $-1 > t > -1/k$ and $1/k > t > 1$.

Let the charged point be at $\xi = \sqrt{(r_1 r_2)}\, e^{(\rho_0 + i\pi)}$. Then, by supposition, ρ_0 is negative, and the corresponding point in the w plane is at $-a + iK'$, where $K > a > 0$. The corresponding point in the t plane is at $t = -\nu$, where $\nu = 1/(k \operatorname{sn} a)$. The problem is now transformed into that of two equal conducting segments in the t plane and a charged point in that plane. The segments and the point are in one straight line, the real axis, and the point is to the left of the left-hand segment. The charges are e_1 on the right-hand segment, e_2 on the left-hand segment, and $-(e_1 + e_2)$ at the point. There is a point of equilibrium on the real axis, and the analysis is simplified a little if the ratio $e_1 : e_2$ is adjusted so that this point may be at $t = 0$.

The real axis is a line of symmetry of the t figure, and it is made up of parts which are all either conductors or lines of force. The boundary in the χ plane of the region which, by the relation between χ and t, is transformed into the upper half of the t plane can be drawn. It is shown in Fig. 14, where the values attached to points are values of t. The figure

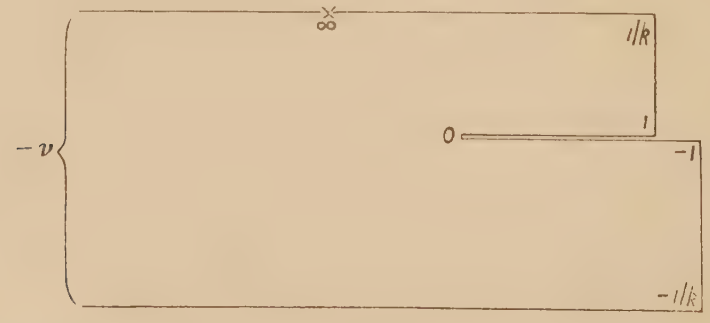

Fig. 14.

is drawn as if $e_2 > e_1$, as is likely, but this does not affect the form of the relation between χ and t. The transformation is effected by an equation

of the form

$$\frac{d\chi}{dt} = A\,\frac{t}{(t+\nu)\{(t+1/k)(t+1)(t-1)(t-1/k)\}^{\frac{1}{2}}}$$

$$= \frac{-Akt}{(t+\nu)\sqrt{\{(1-t^2)(1-k^2t^2)\}}},$$

where A is a real constant. Since ψ must diminish by $2\pi(e_1+e_2)$ as t passes over $-\nu$ from left to right, we have

$$A = -2(e_1+e_2)\,\frac{\operatorname{cn} a \operatorname{dn} a}{k \operatorname{sn} a},$$

and

$$\frac{d\chi}{dt} = 2(e_1+e_2)\,\frac{\operatorname{cn} a \operatorname{dn} a}{\operatorname{sn} a}\,\frac{t}{(t+\nu)\sqrt{\{(1-t^2)(1-k^2t^2)\}}}$$

$$= 2(e_1+e_2)\,\frac{\operatorname{cn} a \operatorname{dn} a}{\operatorname{sn} a}\,\frac{t^2-t\nu}{(t^2-\nu^2)\sqrt{\{(1-t^2)(1-k^2t^2)\}}}.$$

Hence

$$\frac{d\chi}{dw} = -2(e_1+e_2)\,\frac{k^2 \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a \operatorname{sn}^2 w}{1-k^2 \operatorname{sn}^2 a \operatorname{sn}^2 w} + 2(e_1+e_2)\,\frac{k \operatorname{cn} a \operatorname{dn} a \operatorname{sn} w}{1-k^2 \operatorname{sn}^2 a \operatorname{sn}^2 w}.$$

The first term leads to an elliptic integral of the third kind, and the second to an elementary integral, and the result can be written

$$\chi = -2(e_1+e_2)\,\Pi(w, a) + 2(e_1+e_2)\{\tfrac{1}{2}\log(1-k^2 \operatorname{sn}^2 a \operatorname{sn}^2 w)$$

$$-\log(\operatorname{dn} a \operatorname{dn} w + k \operatorname{cn} a \operatorname{cn} w)\} + \text{const.}$$

Here $\Pi(w, a)$ is the Jacobian expression for the elliptic integral of the third kind in the normal form, which may also be written in terms of the Jacobian Theta-function, thus:

$$\Pi(w, a) = \tfrac{1}{2}\log\frac{\Theta(w-a)}{\Theta(w+a)} + w\,Z(a).$$

The function χ is many-valued; but, when the constant of integration is chosen, t is confined to the upper half-plane, and w is confined to the corresponding half of the rectangle, the logarithms in the expression for χ

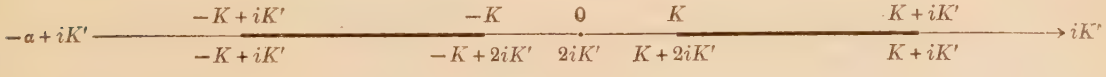

FIG. 15.

are single-valued. The restriction to half the figure can be removed, and then the whole plane of t, with a barrier extending from $-\nu$ to ∞ on the

right, is transformed into a certain region of the plane of w, and into a certain region of the plane of χ. The boundary in the t plane in shown

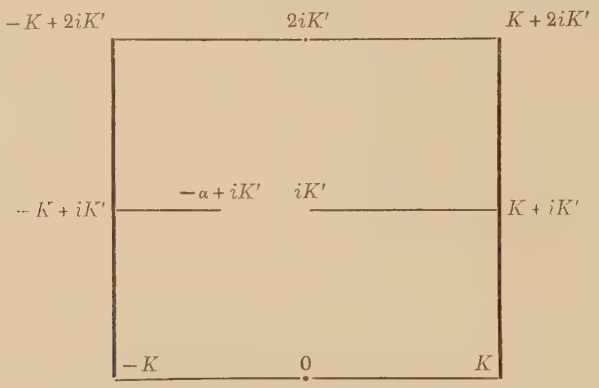

Fig. 16.

in Fig. 15, where the values attached to points are values of w. The region of the w plane is bounded by the rectangle whose sides are $u=\pm K$, $v=0$, $v=2K'$, and two artificial barriers, one extending from $-K+iK'$ to $-\alpha+iK'$, and the other extending from iK' to $K+iK'$. This boundary is shown in Fig. 16. The boundary of the corresponding region of the χ plane is shown in Fig. 17, where the values attached to points are values of w.

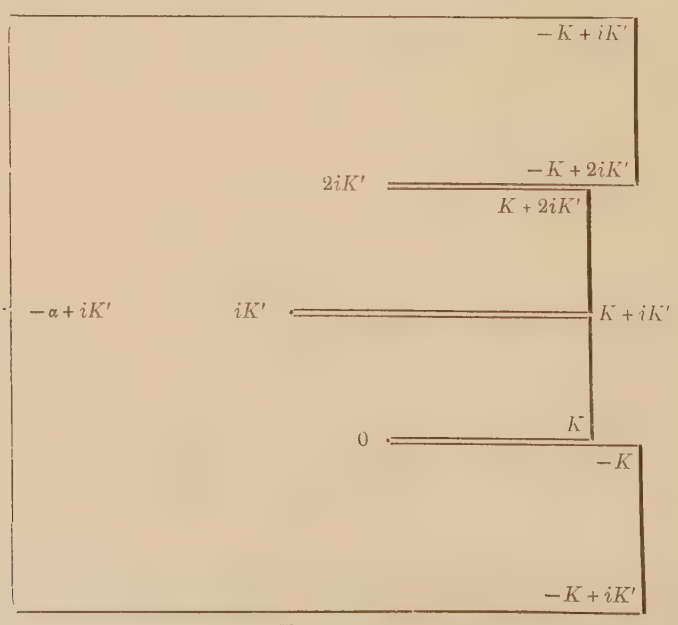

Fig. 17.

To express the charges, we observe that ψ increases by $2\pi e_1$ as w goes along $u = K$ from K to $K+iK'$. It is easy to prove that the second term of χ increases by $2(e_1+e_2)\frac{1}{2}i\pi$. The increment of $wZ(a)$ is, of course, $iK'Z(a)$. There remains the term in $\log\{\Theta(w-a)/\Theta(w+a)\}$. Now we have the formula

$$\frac{\Theta(u+iK')}{\Theta(u)} = i\sqrt{k}\, e^{\pi(K'-2iu)/(4K)} \operatorname{sn} u,$$

from which

$$\frac{\Theta(K-a+iK')}{\Theta(K-a)} \frac{\Theta(K+a)}{\Theta(K+a+iK')} = e^{i\pi a/K}.$$

Hence
$$2\pi i e_1 = -2(e_1+e_2)\left\{\frac{i\pi a}{2K} + iK'Z(a)\right\} + (e_1+e_2)i\pi,$$

or
$$e_1 = \tfrac{1}{2}(e_1+e_2)\left\{1 - \frac{a}{K} - \frac{2K'}{\pi} Z(a)\right\}.$$

In like manner we should find

$$e_2 = \tfrac{1}{2}(e_1+e_2)\left\{1 + \frac{a}{K} + \frac{2K'}{\pi} Z(a)\right\}.$$

It may be noted that e_1 and e_2 have the same sign, which has been taken positive, and their ratio is determined in terms of a by the equation

$$\frac{e_2-e_1}{e_2+e_1} = \frac{a}{K} + \frac{2K'}{\pi} Z(a).$$

In the particular case where $a = 0$, the charged point in the t plane recedes to ∞, and the charges on the segments are equal. In this case

$$\chi = -4e_1 \log(\operatorname{dn} w + k \operatorname{cn} w).$$

In the ζ plane the charged point is then on the circle $\rho = 0$ or $|\zeta| = \sqrt{(r_1 r_2)}$.

18. *Two circles external to each other.*—As an example of this theory we consider the case where the two cylindrical conductors are circular, and a section of them by a transverse plane is a pair of circles, forming two members of a coaxal system with real limiting points. Let A, B be the centres, a, b the radii, I, J the limiting points, distant $2c$ apart, O the

middle point of IJ, see Fig. 18. Let the plane of the figure be the plane of a complex variable z, O the origin of z, BA the real axis. The region

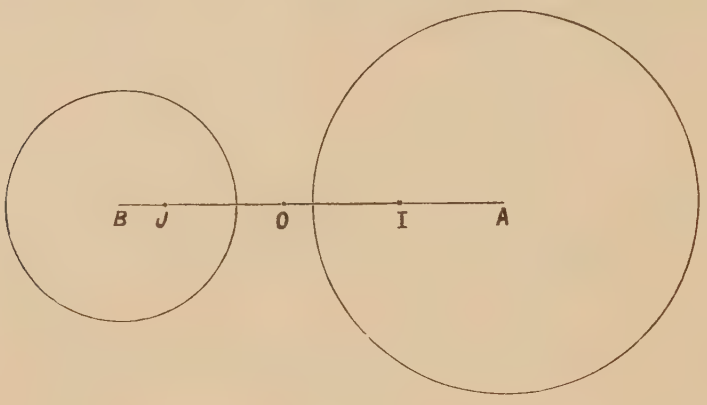

FIG. 18.

of z outside the two circles is transformed into a ring in the plane of ζ, between two concentric circles, by the complex inversion formula

$$(\zeta+2c)(z+c) = 4c^2,$$

or

$$\zeta = 2c\frac{c-z}{c+z}.$$

The circle of radius b in the z plane is transformed into the outer circle in the ζ plane, and its radius is given by

$$|\zeta| = r_1 = 2c\frac{\sqrt{(b^2+c^2)}+c}{b},$$

and the circle of radius a in the z plane is transformed into the inner circle in the ζ plane, and its radius is given by

$$|\zeta| = r_2 = 2c\frac{\sqrt{(a^2+c^2)}-c}{a}.$$

The solution of the condenser problem for the circles a and b is given by

$$\chi = \chi_0 = 2e_0 \log\left\{\frac{2c}{\sqrt{(r_1 r_2)}}\ \frac{c-z}{c+z}\right\},$$

in which r_1, r_2 have the values given in terms of a, b, c.

The infinitely distant point in the z plane is transformed to the point $\zeta = -2c$, and the imposed condition that this point lies within the

circle $|\xi| = \sqrt{(r_1, r_2)}$ is $a > b$. Thus it amounts to choosing as centre of inversion that limiting point which is within the smaller circle (in the z plane). To proceed with the problem of charges e_1, e_2 on the circles b, a, we must introduce Jacobian elliptic functions with such K and K' that

$$e^{2\pi K/K'} = \frac{\sqrt{(b^2+c^2)}+c}{b} \; \frac{\sqrt{(a^2+c^2)}+c}{a},$$

for this is the ratio $r_1 : r_2$, and we must also introduce an elliptic argument w such that

$$\frac{c-z}{c+z} = \left\{\frac{\sqrt{(b^2+c^2)}+c}{b} \; \frac{\sqrt{(a+c^2)}-c}{a}\right\}^{\frac{1}{2}} e^{\pi w/K'}.$$

Then $iK'-a$ is the value of w at the infinitely distant point in the z plane, and therefore a is given by the equation

$$e^{2\pi a/K'} = \frac{\sqrt{(b^2+c^2)}+c}{b} \; \frac{\sqrt{(a^2+c^2)}-c}{a}.$$

Then the ratio of charges for which the problem is solved is given by

$$\frac{e_2-e_1}{e_2+e_1} = \frac{a}{K} + \frac{2K'}{\pi} Z(a),$$

and the function χ_1, whose real part is the potential due to charges e_1, e_2 (in this ratio) on the circles b, a, is

$$\chi_1 = -2(e_1+e_2)\left\{\tfrac{1}{2}\log\frac{\Theta(w-a)}{\Theta(w+a)} + w\,Z(a)\right\}$$

$$+2(e_1+e_2)\left\{\tfrac{1}{2}\log(1-k^2\,\mathrm{sn}^2\,a\,\mathrm{sn}^2\,w) - \log(\mathrm{dn}\,a\,\mathrm{dn}\,w + k\,\mathrm{cn}\,a\,\mathrm{cn}\,w)\right\}.$$

The functions χ_0 and χ_1 being thus determined, the problem can be solved for any charges.

If the circles are equal, a vanishes, the solution corresponds to equal charges of the same sign, K and K' are such that

$$e^{\pi K/K'} = \frac{\sqrt{(a^2+c^2)}+c}{a},$$

the elliptic argument w is

$$\frac{K'}{\pi}\log\frac{c-z}{c+z},$$

and the function χ_1 is

$$-4e_1\log(\mathrm{dn}\,w + k\,\mathrm{cn}\,w).$$

THE PRODUCT OF TWO GENERALISED HYPERGEOMETRIC FUNCTIONS

By C. T. PREECE.

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AMONG the mathematical notes left by Ramanujan occurs, without proof, the theorem expressed by the equation

$$ {}_1F_1(\alpha\,;\ \rho\,;\ x) \times {}_1F_1(\alpha\,;\ \rho\,;\ -x) = {}_2F_3(\alpha, \rho-\alpha\,;\ \rho, \tfrac{1}{2}\rho, \tfrac{1}{2}(\rho+1)\,;\ \tfrac{1}{4}x^2). $$

The object of the present paper is to prove this theorem, and to investigate what similar theorems concerning these products exist. The conditions of validity, depending upon convergence, are so easily determined that they will not again be mentioned.

Long ago Clausen* determined the conditions (1) that the square of a hypergeometric function should be a generalised function of type ${}_3F_2$, (2) that the product of two different hypergeometric functions should be a generalised function of type ${}_3F_2$. He determined the differential equation of order three satisfied by the square of a hypergeometric function, and, by a proper selection of constants, made it identical with the equation satisfied by a function ${}_3F_2$. As regards the product of two different series he considered only the case in which the product satisfied a differential equation of order three and proceeded as he had proceeded for the square of a single function.

* *Über die Fälle wenn die Reihe von der Form*

$$ y = 1 + \frac{\alpha.\beta}{1.\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma.(\gamma+1)}x^2 + \text{etc.} $$

ein Quadrat von der Form, etc. Crelle, Bd. 3 (1828), S. 89–95.

Orr* made a more substantial use of the same method to prove a theorem on hypergeometric functions stated without proof by Cayley† and to add further results of the same nature. He employed the normal forms of the equations satisfied by two independent hypergeometric functions, obtained the linear differential equation satisfied by the product of the solutions of these, and investigated in what cases it could be identical with the equation satisfied by the product of $(x-1)^{-k}x^{-h}$ and a function of the type indicated on the right of the equation given by Cayley.

In the present paper the normal form of the differential equation of order two is not employed. The differential equation of order four (in special cases of order three) satisfied by the product of two functions which separately satisfy any linear equations of order two is determined. Application is made to the product ${}_1F_1(\alpha\,;\,\rho\,;\,x)\,{}_1F_1(\beta\,;\,\sigma\,;\,cx)$; application to the product of hypergeometric functions requires only slightly greater algebraical labour. The equation obtained is compared with and, when possible, made identical with the equations satisfied by functions of the types ${}_1F_2$, $e^x\,{}_1F_2$, ${}_2F_3$, $e^x\,{}_2F_3$. The function ${}_3F_2$, with its singularity at $x=1$, is inadmissible, and powers of $1-x$, suitable in the case of the hypergeometric function, are not here, unlike e^x, possible factors for functions ${}_1F_2$, ${}_2F_3$. When identity of differential equations is secured it is easy to determine those solutions of the differential equations which are equal to the products considered.

Differential Equation satisfied by the product of two functions which are solutions of linear equations of order two.

Let y and z be two functions of x satisfying the equations

$$y'' = (L+M)y' + (P+Q)y,$$

$$z'' = (L-M)z' + (P-Q)z,$$

in which primes denote differential coefficients.

Let $\qquad w = yz, \qquad u = yz' - z'y, \qquad v = y'z'.$

* Orr, *Trans. Camb. Phil. Soc.*, Vol. 17 (1899), pp. 1–15.

† Cayley, *Phil. Mag.* (4), Vol. 16 (1858), pp. 356–357.

By differentiation we obtain

$$w'' = Lw' - Mu + 2Pw + 2v,$$

$$v' = 2Lv + Qu + Pw',$$

$$u' = Lu - Mw' - 2Qw.$$

Elimination of v, v', and u' yields

$$u(M' - LM - 2Q) = (2MQ - 4LP + 2P')w + (L' - 2L^2 + M^2 + 4P)w' + 3Lw'' - w'''.$$

If $M' - LM - 2Q = 0$ then w satisfies the equation of order *three*

$$w''' - 3Lw'' - (L' - 2L^2 + M^2 + 4P)w' - (2MQ - 4LP + 2P')w = 0 \quad (1)$$

That $M' - LM - 2Q$ should be equal to zero is easily seen to be the condition that the two differential equations considered should have identical invariants, so that, removing a trivial factor, both equations are reducible to the same canonical form $\theta'' + I\theta = 0$. In this case, therefore, the problem of the product yz will degenerate into the problem of finding a linear equation of order three satisfied by θ^2.

If $M' - LM - 2Q \neq 0$, we obtain the equation of order *four*

$$\left(\frac{d}{dx} - L\right)\left[\frac{w''' - 3Lw'' - (L' - 2L^2 + M^2 + 4P)w' - (2MQ - 4LP + 2P')w}{2Q + LM - M'}\right] + Mw' + 2Qw = 0 \quad (2)$$

We shall write these equations in the form

$$w''' - 3Lw'' - Aw' - Bw = 0, \quad (3)$$

$$\left(\frac{d}{dx} - L\right)\left[\frac{w''' - 3Lw'' - Aw' - Bw}{C}\right] + Mw' + 2Qw = 0. \quad (4)$$

Application to the functions $y = {}_1F_1(\alpha\,;\,\rho\,;\,x)$, $z = {}_1F_1(\beta\,;\,\sigma\,;\,cx)$.

The differential equations satisfied by these functions are

$$x\frac{d^2y}{dx^2} = (x - \rho)\frac{dy}{dx} + \alpha y,$$

$$x\frac{d^2z}{dx^2} = (cx - \sigma)\frac{dz}{dx} + c\beta z.$$

For these equations

$$L = \frac{1+c}{2} - \frac{\rho+\sigma}{2x}, \qquad P = \frac{\alpha+c\beta}{2x},$$

$$M = \frac{1-c}{2} - \frac{\rho-\sigma}{2x}, \qquad Q = \frac{\alpha-c\beta}{2x},$$

$$A = \frac{-(1+6c+c^2)}{4} + \frac{4(\alpha+c\beta)+\rho(1+3c)+\sigma(3+c)}{2x} + \frac{2(\rho+\sigma)-\rho^2-6\rho\sigma-\sigma^2}{4x^2},$$

$$B = -\frac{\alpha(1+3c)+c\beta(3+c)}{2x} + \frac{\alpha(\rho+3\sigma-2)+c\beta(3\rho+\sigma-2)}{2x^2},$$

$$C = \frac{1-c^2}{4} + \frac{2(\alpha-c\beta)-(\rho-c\sigma)}{2x} + \frac{(\rho-\sigma)(\rho+\sigma-2)}{4x^2}.$$

Differential Equations satisfied by certain functions of type ${}_pF_q(a_1, \ldots, a_p;\ \rho_1, \ldots, \rho_q;\ \kappa x^\mu)$ *and related functions.*

The equations given are those with which equations (3) and (4) are to be compared.

For functions of the form ${}_1F_2$ the equation is

$$[x^3D^3 + S_1x^2D^2 + (S_2 - \mu^2\kappa x^\mu)xD - \mu^3\kappa a_1 x^\mu]\, y = 0;$$

for functions of the form $e^{px}\,{}_1F_2$ the equation is

$$[x^3D^3 + (S_1 - 3px)x^2D^2 + (S_2 - 2pxS_1 + 3p^2x^2 - \mu^2\kappa x^\mu)xD - pxS_2 + p^2x^2S_1 - p^3x^3 - \mu^2\kappa x^\mu(\mu a_1 - px)]\, y = 0;$$

and for functions of the form $x^p\,{}_1F_2$ the equation is

$$[x^3D^3 + (S_1 - 3p)x^2D^2 + (S_2 - 2pS_1 + 3p_2 - \mu^2\kappa x^\mu)xD - p_3 + p_2S_1 - pS_2 + p\mu^2\kappa x^\mu - \mu^3\kappa a_1 x^\mu]\, y = 0,$$

where

$$S_1 = \mu(\rho_1 + \rho_2 - 2) + 3,$$

$$S_2 = \{\mu(\rho_1 - 1) + 1\}\{\mu(\rho_2 - 1) + 1\},$$

$$p_2 = p(p+1), \qquad p_3 = p(p+1)(p+2).$$

The functions ${}_2F_3$ and $e^{px}\,{}_2F_3$ satisfy respectively the equations

$$[x^4D^4 + T_1x^3D^3 + (T_2 - \mu^2\kappa x^\mu)x^2D^2 + \{T_3 - \mu^2\kappa x^\mu(\mu a_1 + \mu a_2 + 1)\}xD - \mu^4\kappa a_1 a_2 x^\mu]\, y = 0,$$

and $[x^4(D^4-4pD^3+6p^2D^2-4p^3D+p^4)+T_1x^3(D^3-3pD^2+3p^2D-p^3)$

$+(T_2-\mu^2\kappa x^\mu)x^2(D^2-2pD+p^2)$

$+\{T_3-\mu^2\kappa x^\mu(\mu a_1+\mu a_2+1)\}x(D-p)-\mu^4\kappa a_1 a_2 x^\mu]y=0,$

where

$$T_1=\mu\Sigma(\rho-1)+6,$$

$$T_2=\mu^2\Sigma(\rho_2-1)(\rho_3-1)+3\mu\Sigma(\rho-1)+7,$$

$$T_3=\mu^3(\rho_1-1)(\rho_2-1)(\rho_3-1)+\mu^2\Sigma(\rho_2-1)(\rho_3-1)+\mu\Sigma(\rho-1)+1.$$

It will be found impossible, by any choice of constants, to make the equation satisfied by the product ${}_1F_1(\alpha\,;\ \rho\,;\ x)\times{}_1F_1(\beta\,;\ \sigma\,;\ cx)$ identical with the equation satisfied by $x^\rho{}_2F_3$ or with the equation satisfied by $x^\rho{}_1F_2$. Proceeding now to investigate in what ways it is possible by a suitable choice of constants to identify equations (3) and (4) with any of the differential equations given above, we note that when C is equal to zero the simpler equation (3) of order three is the one satisfied by the product, and we therefore consider in turn the cases C equal to zero, C constant, C variable.

Case I (1) $\quad C=0,\quad c=1,\quad \rho=\sigma,\quad \beta=\alpha.$

$$L=1-\rho/x,$$

$$A=-2+\frac{4(\alpha+\rho)}{x}+\frac{\rho-2\rho^2}{x^2},$$

$$B=-\frac{4\alpha}{x}+\frac{\alpha(4\rho-2)}{x^2}.$$

The equation for w is

$$w'''-3\left(1-\frac{\rho}{x}\right)w''+\left\{2-\frac{4(\alpha+\rho)}{x}-\frac{\rho-2\rho^2}{x^2}\right\}w'+\left\{\frac{4\alpha}{x}-\frac{\alpha(4\rho-2)}{x^2}\right\}w=0.$$

The term $-3w''$ makes it impossible to reconcile this equation with an equation for ${}_1F_2$. Taking the equation satisfied by $e^{\mu\nu}{}_1F_2$, we have to satisfy

$$-3+\frac{3\rho}{x}=\frac{S_1}{x}-3p,$$

$$2-\frac{4(\alpha+\rho)}{x}-\frac{\rho-2\rho^2}{x^2}=\frac{S_2}{x^2}-\frac{2pS_1}{x}+3p^2-\frac{\mu^2\kappa x^\mu}{x^2},$$

$$\frac{4\alpha}{x}-\frac{\alpha(4\rho-2)}{x^2}=-\frac{pS_2}{x^2}+\frac{p^2S_1}{x}-p^3-\frac{\mu^3\kappa a_1 x^\mu}{x^3}+\frac{\mu^2\kappa p x^\mu}{x^2}$$

Hence we must take $p=1,\ S_1=3\rho.$

Since $3p^2 \neq 2$, it follows from the second of the three equations that $\mu = 2$; and then $\kappa = \frac{1}{4}$, $S_1 = 2(\alpha+\rho)$, $S_2 = 2\rho^2-\rho$. From the expressions for S_1, $\rho = 2\alpha$; from the third equation, $\alpha_1 = \alpha$; and, from the values of S_1 and S_2, it follows that $\rho_1 = 2\alpha$, $\rho_2 = \alpha+\frac{1}{2}$. Hence the product

$$ {}_1F_1(\alpha;\ 2\alpha;\ x)\,{}_1F_1(\alpha;\ 2\alpha;\ x) $$

satisfies the same differential equation as $e^x\,{}_1F_2(\alpha;\ 2\alpha,\ \alpha+\frac{1}{2};\ \frac{1}{4}x^2)$. The common differential equation implies a common recurrence formula, and verification as far as the second term of each expansion is sufficient to establish the equality

$$ \{{}_1F_1(\alpha;\ 2\alpha;\ x)\}^2 = e^x\,{}_1F_2(\alpha;\ 2\alpha,\ \alpha+\tfrac{1}{2};\ \tfrac{1}{4}x^2) \qquad (5) $$

Two other solutions of the differential equation independent of each other and of $e^x\,{}_1F_2(\alpha;\ 2\alpha,\ \alpha+\frac{1}{2};\ \frac{1}{4}x^2)$ are $x^{2-4\alpha}e^x\,{}_1F_2(1-\alpha;\ 2-2\alpha,\ \frac{3}{2}-\alpha;\ \frac{1}{4}x^2)$ $x^{1-2\alpha}e^x\,{}_1F_2(\frac{1}{2};\ \alpha+\frac{1}{2},\ \frac{3}{2}-\alpha;\ \frac{1}{4}x^2)$. A second product solution of the common differential equation is $\{x^{1-2\alpha}\,{}_1F_1(1-\alpha;\ 2-2\alpha,\ x)\}^2$; and this is equal to $x^{2-4\alpha}e^x\,{}_1F_2(1-\alpha;\ 2-2\alpha,\ \frac{3}{2}-\alpha;\ \frac{1}{4}x^2)$, a result which is not, in effect, different from equation (5). A third independent product solution,

$$ x^{1-2\alpha}\,{}_1F_1(\alpha;\ 2\alpha;\ x)\,{}_1F_1(1-\alpha;\ 2-2\alpha;\ x), $$

must be a linear function of the three solutions involving functions ${}_1F_2$: and it is easy to verify that

$$ {}_1F_1(\alpha;\ 2\alpha;\ x)\,{}_1F_1(1-\alpha;\ 2-2\alpha;\ x) = e^x\,{}_1F_2(\tfrac{1}{2};\ \alpha+\tfrac{1}{2},\ \tfrac{3}{2}-\alpha;\ \tfrac{1}{4}x^2) \qquad (6) $$

Case I (2) $\quad C = 0, \quad c = 1, \quad \rho+\sigma = 2, \quad \beta = \alpha-\rho+1.$

This can only give the same two results as Case I (1); equation (6) would be the first one obtained.

Case I (3). $\quad C = 0, \quad c = -1, \quad \rho = \sigma, \quad \beta = \rho-\alpha.$

The equation for w is

$$ w''' + \frac{3\rho}{x}w'' - \left\{1 + \frac{2(2\alpha-\rho)}{x} + \frac{\rho-2\rho^2}{x^2}\right\}w' - \left\{\frac{\rho}{x} + \frac{(2\alpha-\rho)(2\rho-1)}{x^2}\right\}w = 0. $$

Identification with the equation satisfied by ${}_1F_2$ is possible, and we find that

$$ {}_1F_1(\alpha;\ 2\alpha;\ x)\,{}_1F_1(\alpha;\ 2\alpha;\ -x) = {}_1F_2(\alpha;\ 2\alpha,\ \alpha+\tfrac{1}{2};\ \tfrac{1}{4}x^2), \qquad (7) $$

$$ {}_1F_1(\alpha;\ 2\alpha;\ x)\,{}_1F_1(1-\alpha;\ 2-2\alpha;\ -x) = {}_1F_2(\tfrac{1}{2};\ \alpha+\tfrac{1}{2},\ \tfrac{3}{2}-\alpha;\ \tfrac{1}{4}x^2) \qquad (8) $$

From equations (6) and (8) follows a result involving generalised functions of type ${}_1F_1$ only :*

$$ {}_1F_1(1-a;\ 2-2a;\ -x) = e^{-x}\,{}_1F_1(1-a;\ 2-2a;\ x), $$

or, in simpler notation,

$$ {}_1F_1(a;\ 2a;\ -x) = e^{-x}\,{}_1F_1(a;\ 2a;\ x) \qquad (9) $$

Case I (4). $C=0, \quad c=-1, \quad \rho+\sigma=2, \quad \beta=1-a.$

The results are the same as in Case I (3), equation (8) being obtained first.

Case II, C constant.—If we try to identify the equation satisfied by the product with the equation for a function of type $e^{px}\,{}_2F_3$ we are forced to make p equal to zero or 1, which leads to $c=1$ or $c=-1$, so that zero is he only constant value for C which permits the identity of the two differential equations.

Case III.—If we attempt to identify the product

$$ {}_1F_1(a;\ \rho;\ x)\,{}_1F_1(\beta;\ \sigma;\ cx) $$

with a function ${}_2F_3$ we find that $c^2=1$, and if $c=1$ then $C=0$, a case already dealt with.

We take $c=-1$.

Then
$$ L = -\frac{\rho+\sigma}{2x}, \quad M = 1-\frac{\rho-\sigma}{2x}, \quad Q = \frac{a+\beta}{2x}, $$

$$ A = 1+\frac{2(a-\beta)-\rho+\sigma}{2x}+\frac{2(\rho+\sigma)-\rho^2-6\rho\sigma-\sigma^2}{4x^2}, $$

$$ B = \frac{a+\beta}{x}+\frac{a(\rho+3\sigma-2)-\beta(3\rho+\sigma-2)}{4x^2}, $$

$$ C = \frac{2(a+\beta)-(\rho+\sigma)}{2x}+\frac{(\rho-\sigma)(\rho+\sigma-2)}{4x^2} $$

$$ = \frac{U}{2x}+\frac{V}{4x^2}, \text{ say.} $$

We have to reconcile the equations

$$ \left\{w''''-w'''\left(\frac{4LC+C'}{C}\right)+w''\left(\frac{3L^2C-AC-3CL'+3C'L}{C}\right)\right. $$

$$ \left.+w'\left(\frac{LAC+MC^2-BC-CA'+C'A}{C}\right)+w\left(\frac{2QC+LBC-CB'+C'B}{C}\right)\right\}=0 $$

* For an account of such formulæ involving functions ${}_1F_1$ and ${}_0F_1$, which are due to Kummer, see Watson, *Theory of Bessel Functions* (1922), pp. 100–105.

and $[x^4D^4+T_1x^3D^3+(T_2-\mu^2\kappa x^\mu)x^2D^2+\{T_3-\mu^2\kappa x^\mu(\mu a_1+\mu a_2+1)\}xD$
$-\mu^4\kappa a_1a_2x^\mu]y=0,$

which is satisfied by a function ${}_2F_3$.

Equating coefficients, we obtain easily the following:

$$U=0 \quad \text{or} \quad T_1=2(\rho+\sigma)+1,$$
$$V=0 \quad \text{or} \quad T_1=2(\rho+\sigma)+2,$$
$$\mu=2 \quad \text{and} \quad \kappa=\tfrac{1}{4},$$
$$\rho-\sigma=2(\alpha-\beta).$$

It is therefore evident that either $U=0$ or $V=0$. To obtain a new case we suppose that if one of these vanishes the other does not. If $U=0$, then $\rho+\sigma=2(\alpha+\beta)$; $\rho=2\alpha$, $\sigma=2\beta$. When we deal with w' we find that either $\alpha=\beta$ or $\alpha+\beta=1$, and therefore $V=0$. We therefore turn to the supposition that V vanishes while U does not.

Let $$V=0, \quad \rho=\sigma.$$

Then $$L=-\frac{\rho}{x}, \quad M=1. \quad Q=\frac{\alpha+\beta}{2x},$$

$$A=1+\frac{2(\alpha-\beta)}{x}+\frac{\rho-2\rho^2}{x^2},$$

$$B=\frac{\alpha+\beta}{x}+\frac{(\alpha-\beta)(2\rho-1)}{x^2},$$

$$C=\frac{\alpha+\beta-\rho}{x}.$$

Equating coefficients in the two differential equations, we find

$$\alpha=\beta, \quad a_1+a_2=\rho, \qquad a_1a_2=\alpha(\rho-\alpha),$$

$$T_1=4\rho+1, \quad T_2=5\rho^2-\rho, \quad T_3=\rho(\rho-1)(2\rho-1).$$

Hence $a_1=\alpha$, $a_2=\rho-\alpha$, and, after substitution for T_1, T_2, T_3,

$$\rho_1=\rho, \quad \rho_2=\tfrac{1}{2}\rho, \quad \rho_3=\tfrac{1}{2}\rho+\tfrac{1}{2}.$$

We have therefore proved that the product ${}_1F_1(\alpha;\ \rho;\ x)\,{}_1F_1(\alpha;\ \rho;\ -x)$ and the function ${}_2F_3(\alpha,\ \rho-\alpha;\ \rho,\ \tfrac{1}{2}\rho,\ \tfrac{1}{2}\rho+\tfrac{1}{2};\ \tfrac{1}{4}x^2)$ satisfy the same differ-

ential equation. Four independent solutions of the differential equation are

$$
\begin{aligned}
&{}_2F_3(\alpha,\ \rho-\alpha;\ \rho,\ \tfrac{1}{2}\rho,\ \tfrac{1}{2}\rho+\tfrac{1}{2};\ \tfrac{1}{4}x^2),\\
&x^{2-2\rho}\,{}_2F_3(\alpha-\rho+1,\ 1-\alpha;\ 2-\rho,\ 1-\tfrac{1}{2}\rho,\ \tfrac{3}{2}-\tfrac{1}{2}\rho;\ \tfrac{1}{4}x^2),\\
&x^{2-\rho}\,{}_2F_3(\alpha-\tfrac{1}{2}\rho+1,\ \tfrac{1}{2}\rho-\alpha+1;\ \tfrac{1}{2}\rho+1,\ 2-\tfrac{1}{2}\rho,\ \tfrac{3}{2};\ \tfrac{1}{4}x^2),\\
&x^{1-\rho}\,{}_2F_3(\alpha-\tfrac{1}{2}\rho+\tfrac{1}{2},\ \tfrac{1}{2}\rho-\alpha+\tfrac{1}{2};\ \tfrac{1}{2}\rho+\tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{3}{2}-\tfrac{1}{2}\rho;\ \tfrac{1}{4}x^2),
\end{aligned}
$$

and in connection with them are to be considered the four products having for factors one of ${}_1F_1(\alpha;\ \rho;\ x)$, $x^{1-\rho}\,{}_1F_1(\alpha-\rho+1;\ 2-\rho;\ x)$ and one of ${}_1F_1(\alpha;\ \rho;\ -x)$, $x^{1-\rho}\,{}_1F_1(\alpha-\rho+1;\ 2-\rho;\ -x)$. It is verified immediately that

$$
{}_1F_1(\alpha;\ \rho;\ x)\,{}_1F_1(\alpha;\ \rho;\ -x) = {}_2F_3(\alpha,\ \rho-\alpha;\ \rho,\ \tfrac{1}{2}\rho,\ \tfrac{1}{2}\rho+\tfrac{1}{2};\ \tfrac{1}{4}x^2), \tag{10}
$$

and this is Ramanujan's result.

The second formula to be obtained

$$
\begin{aligned}
&{}_1F_1(\alpha-\rho+1;\ 2-\rho;\ x)\,{}_1F_1(\alpha-\rho+1,\ 2-\rho;\ -x)\\
&\qquad = {}_2F_3(\alpha-\rho+1,\ 1-\alpha;\ 2-\rho,\ 1-\tfrac{1}{2}\rho,\ \tfrac{3}{2}-\tfrac{1}{2}\rho;\ \tfrac{1}{4}x^2)
\end{aligned}
$$

is merely Ramanujan's Formula with a more complicated notation.

For a third formula consider the product

$$
{}_1F_1(\alpha;\ \rho;\ x)x^{1-\rho}\,{}_1F_1(\alpha-\rho+1;\ 2-\rho;\ -x).
$$

It is indicated that the odd terms of this product form the series

$$
x^{1-\rho}\,{}_2F_3(\alpha-\tfrac{1}{2}\rho+\tfrac{1}{2},\ \tfrac{1}{2}\rho-\alpha+\tfrac{1}{2};\ \tfrac{1}{2}\rho+\tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{3}{2}-\tfrac{1}{2}\rho;\ \tfrac{1}{4}x^2),
$$

while the even terms form a multiple of

$$
x^{2-\rho}\,{}_2F_3(\alpha-\tfrac{1}{2}\rho+1,\ \tfrac{1}{2}\rho-\alpha+1;\ \tfrac{1}{2}\rho+1,\ 2-\tfrac{1}{2}\rho,\ \tfrac{3}{2};\ \tfrac{1}{4}x^2);
$$

and this we find to be correct, the formula being

$$
\begin{aligned}
&{}_1F_1(\alpha;\ \rho;\ x)\,{}_1F_1(\alpha-\rho+1;\ 2-\rho;\ -x)\\
&\quad = {}_2F_3(\alpha-\tfrac{1}{2}\rho+\tfrac{1}{2},\ \tfrac{1}{2}\rho-\alpha+\tfrac{1}{2};\ \tfrac{1}{2}\rho+\tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{3}{2}-\tfrac{1}{2}\rho;\ \tfrac{1}{4}x^2)\\
&\qquad + \frac{(2\alpha-\rho)(1-\rho)}{\rho(2-\rho)}\,x\,{}_2F_3(\alpha-\tfrac{1}{2}\rho+1,\ \tfrac{1}{2}\rho-\alpha+1;\ \tfrac{1}{2}\rho+1,\ 2-\tfrac{1}{2}\rho,\ \tfrac{3}{2};\ \tfrac{1}{4}x^2)
\end{aligned} \tag{11}
$$

The fourth formula is merely the third with $-x$ written in place of x.

In this investigation we made V vanish by making $\sigma = \rho$. Nothing new is to be expected, and nothing new is obtained, if we make V vanish by making $\rho+\sigma$ equal to 2. In equation (11) we have already $\rho+\sigma = 2$,

and it is this equation that would be obtained if we pursued our investigation with $\rho+\sigma=2$.

There still seems the possibility that a product of two functions ${}_1F_1$ may be equal to a function of type $e^{px}{}_2F_3$. However, if we equate coefficients in the differential equations and avoid as long as possible the hypothesis $p=0$, we are led to the equations $p=\frac{1}{2}+\frac{1}{2}c$, $c^2=1$, which necessitates $c=1$, $p=1$, and the result will be the same as in Case I (1).

Each of the identities established is of course a collection of identities obtained by equating coefficients of equal powers of x.

Consider first equation (5)

$$\{{}_1F_1(\alpha;\ 2\alpha;\ x)\}^2 = e^x\,{}_1F_2(\alpha;\ 2\alpha,\ \alpha+\tfrac{1}{2};\ \tfrac{1}{4}x^2).$$

The coefficient of x^n on the left, multiplied by $n!$, is equal to

$$n!\sum_{r=0}^{n}\frac{(\alpha)_r}{r!\,(2\alpha)_r}\,\frac{(\alpha)_{n-r}}{(n-r)!\,(2\alpha)_{n-r}}$$

$$=\sum_{r=0}^{n}\frac{(-)^r(-n)_r(\alpha)_r(\alpha)_n(1-2\alpha-n)_r}{r!\,(2\alpha)_r(1-\alpha-n)_r(2\alpha)_n}$$

$$=\frac{\Gamma(2\alpha)\,\Gamma(\alpha+n)}{\Gamma(\alpha)\,\Gamma(2\alpha+n)}\,{}_3F_2(\alpha,\ 1-2\alpha-n,\ -n;\ 1-\alpha-n,\ 2\alpha;\ -1),$$

since
$$(\alpha)_{n-r}=\frac{(-)^r(\alpha)_n}{(1-\alpha-n)_r}.$$

The coefficient of x^n on the right, multiplied by $n!$, is

$$\sum_{r=0}^{\frac{1}{2}n}\frac{n!\ (\alpha)_r}{r!\,(2\alpha)_r(\alpha+\frac{1}{2})_r 4^r(n-2r)!}=\Sigma\frac{(-n)_{2r}\,(\alpha)_r}{r!\,(2\alpha)_r(\alpha+\frac{1}{2})_r 4^r}$$

$$=\Sigma\frac{\{-(n/2)\}_r\{(1-n)/2\}_r(\alpha)_r}{r!\,(2\alpha)_r(\alpha+\frac{1}{2})_r}$$

$$={}_3F_2(\alpha,\ -\tfrac{1}{2}n,\ \tfrac{1}{2}-\tfrac{1}{2}n;\ 2\alpha,\ \alpha+\tfrac{1}{2};\ 1).$$

Hence, when n is a positive integer,

$${}_3F_2(\alpha,\ 1-2\alpha-n,\ -n;\ 1-\alpha-n,\ 2\alpha;\ -1)$$

$$=\frac{\Gamma(\alpha)\,\Gamma(2\alpha+n)}{\Gamma(2\alpha)\,\Gamma(\alpha+n)}\,{}_3F_2(\alpha,\ -\tfrac{1}{2}n,\ \tfrac{1}{2}-\tfrac{1}{2}n;\ 2\alpha,\ \alpha+\tfrac{1}{2};\ 1) \qquad (12)$$

Dealing in the same way with equation (6), we obtain

$$ {}_3F_2(a,\ 2a-n-1,\ -n;\ 2a,\ a-n;\ -1) $$

$$ = \frac{\Gamma(1-a)\,\Gamma(2-2a+n)}{\Gamma(2-2a)\,\Gamma(1-a+n)}\ {}_3F_2(\tfrac{1}{2},\ -\tfrac{1}{2}n,\ \tfrac{1}{2}-\tfrac{1}{2}n;\ a+\tfrac{1}{2},\ \tfrac{3}{2}-a;\ 1) \qquad (13) $$

From equation (7), by equating coefficients of x^{2n}, we can obtain the value of the function ${}_3F_2(a,\ 1-2a-n,\ -n;\ 2a,\ 1-a-n;\ 1)$ in Gamma-Functions, but this is only a particular case of a theorem given by Dixon* concerning the value of the function ${}_3F_2(\alpha_1,\ \alpha_2,\ \alpha_3;\ \rho_1,\ \rho_2;\ 1)$ in which $\alpha_1+\rho_1 = \alpha_2+\rho_2 = \alpha_3+1$. The result of equating the coefficient of x^{2n+1} to zero may be noted:

$$ {}_3F_2(a,\ -2a-2n,\ -2n-1;\ 2a,\ -a-2n;\ 1) = 0, $$

due to the presence of the element $-2n-1$, which introduces the factor $\Gamma(-n)$ into Dixon's expression for the value of the function.

* A. C. Dixon, *Proc. London Math. Soc.*, Ser. 1, Vol. 35, pp. 284–289.

A CERTAIN TYPE OF FOURIER-BESSEL SERIES

By T. A. LUMSDEN.

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1. The problem of the transverse vibrations of a circular membrane leads to a solution of the form

$$z = \sum_{n=1}^{\infty} a_n J_0(j_n r) \cos p_n t,$$

on the assumption of symmetry about the z-axis, where j_n is the nth positive zero of $J_0(x)$; a_n is determined on consideration of initial conditions which give

$$f(r) = \sum_{n=1}^{\infty} a_n J_0(j_n r), \tag{1.1}$$

by using the result

$$\tfrac{1}{2} J_1^2(j_n)\, a_n = \int_0^1 t f(t) J_0(j_n t)\, dt.$$

From the point of view of mathematical analysis interest centres in the sufficient conditions to be placed on an arbitrary function $f(r)$, to make expansions of the type (1.1) valid. This problem has been discussed at length by many writers.* The object of this paper is (1) to determine sufficient conditions, with reference to $f(r)$, for the convergence of Σa_n, and (2) to find the sum of this series.

With respect to (1) it will be noticed that the results require more stringent conditions to be enforced on $f(r)$, to secure the convergence of Σa_n, than those which make the expansion $\Sigma a_n J_0(j_n r)$ $(r > 0)$ valid. This is discussed in §§ 5–6.

* A general reference may be made at this point to Watson, *Bessel Functions* (1922), and in particular to Ch. 18 of this book, for bibliography and analytical methods not specifically mentioned in this note.

With respect to (2) we shall establish the convergence of $\sum_{n=1}^{\infty} a_n$ to the value* $f(+0)$, subject to the restriction that $f(r)$ has limited total fluctuation in the interval $0 \leqslant r \leqslant 1$.

2. Let $$T_n(t) = \sum_{m=1}^{n} 2J_0(j_m t)/J_1^2(j_m);$$

and hence $$\sum_{m=1}^{n} a_m = \int_0^1 t\, T_n(t)\, f(t)\, dt.$$

It follows that, for $0 < a \leqslant 1$,

$$\int_0^a t\, T_n(t)\, dt = \sum_{m=1}^{n} \frac{2}{J_1^2(j_m)} \int_0^a t\, J_0(j_m t)\, dt \tag{2.1}$$

$$= \sum_{m=1}^{n} \frac{2}{J_1^2(j_m)} \left[\frac{a}{j_m} J_1(j_m t)\right]_0^a$$

$$= \sum_{m=1}^{n} 2a\, J_1(aj_m)/\{j_m J_1^2(j_m)\}.$$

Consider now $\int_R \frac{J_0(aw)}{w J_0^2(w)}\, dw$, where R is a rectangle in the w-plane with corners at $\pm Bi$, $A_n \pm Bi$, where B will be made to tend to infinity, and A_n is a number between j_n and j_{n+1} and may be given the value† $(n+\frac{1}{4})\pi$. A semi-circular indentation, to exclude $w = 0$, is made.

The poles of the integrand are at the points $w = j_m$ and are of order 2. By expanding in a Taylor's series valid near $w = j_m$, after simplification, we find the residue at j_m to be

$$\frac{-a\, J_1(aj_m)}{j_m J_1^2(j_m)}.$$

From (2.1) it follows that

$$\int_0^a t\, T_n(t)\, dt = -\frac{1}{\pi i} \int_R \frac{J_0(aw)}{w J_0^2(w)}\, dw. \tag{2.2}$$

* This sum has been stated, inaccurately, by Gray, Mathews, and MacRobert, *Bessel Functions* (1922), p. 96, to be $\frac{1}{2}f(+0)$. The absence of discontinuity in the physical problem suggests the incorrectness of this result.

† When n is large, the only case discussed here,

$$j_n = (n-\tfrac{1}{4}) \quad + O(1/n).$$

The integrand is an odd function, so that the integrals along the upper and lower parts of the imaginary axis cancel. Also

$$\left|\int_{\pm Bi}^{A_n \pm Bi} \frac{J_0(aw)}{wJ_0^2(w)}\,dw\right| = \left|\int_0^{A_n} \frac{J_0\{a(u\pm Bi)\}\,du}{(u\pm Bi)J_0^2(u\pm Bi)}\right|$$

$$< K\int_0^{A_n} \frac{\exp\{-B(2-a)\}\,du}{(u^2+B^2)^{\frac{1}{2}}}$$

which tends to zero as $B\to\infty$.

The two inequalities valid in the range of integration, and derived from the asymptotic expansion, used in the above reduction, are

$$(2.3)\qquad |J_0(w)| < \frac{c_1 \exp|I(w)|}{\sqrt{|w|}}, \quad |J_0(w)| > \frac{c_2 \exp|I(w)|}{\sqrt{|w|}}.$$

Since $J_0(aw)/J_0^2(w)\to 1$ uniformly as $w\to 0$ the integral round the indentation contributes $-\pi i$.

Summing up, we have

$$\int_0^a t\,T_n(t)\,dt = 1-\frac{1}{\pi i}\int_{A_n-Bi}^{A_n+Bi} \frac{J_0(aw)}{wJ_0^2(w)}\,dw.$$

On using (2.3) we deduce

$$\left|\int_{A_n-Bi}^{A_n+Bi} \frac{J_0(aw)}{wJ_0^2(w)}\,dw\right| = O\left(\frac{1}{A_n^{\frac{1}{2}}}\right) = O\left(\frac{1}{n^{\frac{1}{2}}}\right),$$

and, proceeding to the limit, we find the two results

$$(2.4)\qquad \lim_{n\to\infty}\int_0^a t\,T_n(t)\,dt = 1 \quad (0<a\leqslant 1),$$

$$(2.5)\qquad \lim_{n\to\infty}\int_a^{a'} t\,T_n(t)\,dt = 0 \quad (0<a\leqslant a'\leqslant 1).$$

3. We are now in a position to establish the theorem that, *if $f(r)$ have limited total fluctuation in the interval $0\leqslant r\leqslant 1$ the series $\sum_{n=1}^{\infty} a_n$ converges to $f(+0)$.*

First of all, write (3.1)

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty}\int_0^1 t f(t)\,T_n(t)\,dt = \lim_{n\to\infty}\int_0^\delta t f(t)\,T_n(t)\,dt + \lim_{n\to\infty}\int_\delta^1 t f(t)\,T_n(t)\,dt,$$

where δ, defined definitely later, satisfies $0<\delta\leqslant 1$.

In the second integral put

$$f(t) = \psi_1(t) - \psi_2(t),$$

the two functions on the right being taken to be bounded positive increasing functions. The second mean value theorem gives

$$\lim_{n\to\infty} \int_\delta^1 t\,\psi_1(t)\,T_n(t)\,dt = \psi_1(1) \lim_{n\to\infty} \int_{\delta_1}^1 t\,T_n(t)\,dt \quad (\delta \leqslant \delta_1 \leqslant 1)$$

$$= 0, \quad \text{by (2.5)}.$$

A similar result follows for $\psi_2(t)$.

Equation (3.1) now becomes

$$\text{(3.2)} \qquad \sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \int_0^\delta t f(t)\,T_n(t)\,dt$$

$$= f(+0) \lim_{n\to\infty} \int_0^\delta t\,T_n(t)\,dt + \lim_{n\to\infty} \int_0^\delta t\{f(t) - f(+0)\}\,T_n(t)\,dt.$$

In the second integral put

$$f(t) - f(+0) = \phi_1(t) - \phi_2(t),$$

where $\phi_1(t)$ and $\phi_2(t)$ are bounded positive increasing functions, since $\{f(t) - f(+0)\}$ has limited total fluctuation, and

$$\phi_1(+0) = \phi_2(+0) = 0.$$

For an arbitrary positive ϵ, define δ to be such that

$$\phi_1(t) < \epsilon, \quad \phi_2(t) < \epsilon \quad \text{for} \quad 0 < t \leqslant \delta,$$

which is clearly possible while retaining $\delta > 0$.

Another use of the second mean value theorem gives

$$\lim_{n\to\infty} \int_0^\delta t\,\phi_1(t)\,T^n(t)\,dt = \phi_1(\delta) \lim_{n\to\infty} \int_{\delta_2}^\delta t\,T_n(t)\,dt \quad (0 \leqslant \delta_2 \leqslant \delta).$$

$$< 2\epsilon.$$

With the same result for $\phi_2(t)$ we find that the second integral in (3.2) can be made arbitrarily small, consistent with δ remaining greater than zero.

The result,

$$\sum_{n=1}^{\infty} a_n = f(+0) \lim_{n\to\infty} \int_0^\delta t\,T_n(t)\,dt = f(+0),$$

is therefore established.

4. The preceding result is easily confirmed in the case of (i) $f(r) = 1$, (ii) $f(r) = 1-r^2$.

(i) $f(r) = 1, \quad f(+0) = 1.$

The formula for the coefficients gives

$$\begin{aligned} a_m &= \frac{2}{J_1^2(j_m)} \int_0^1 t\, J_0(j_m t)\, dt \\ &= 2\{j_m J_1^2(j_m)\}^{-1} [t\, J_1(j_m t)]_0^1 \\ &= 2\{j_m J_1(j_m)\}^{-1}. \end{aligned}$$

The series thus found converges very slowly and hence the sum of a small number of terms suggests the correctness of, but in no way verifies the preceding theory.

We may, however, avail ourselves of the Eulerian method* if we can show this series to consist of terms of alternating sign which decrease numerically to zero as $m \to \infty$.

From the known fact that the zeros of $J_0(x)$ and $J_1(x)$ interlace, the alternation of sign is evident.

Also $$\int_{j_m}^{j_{m+1}} J_0^2(t)\, t\, dt = \tfrac{1}{2} [t^2 \{J_0^2(t) + J_1^2(t)\}]_{j_m}^{j_{m+1}}$$

$$= \tfrac{1}{2} \{j_{m+1}^2 J_1^2(j_{m+1}) - j_m^2 J_1^2(j_m)\}.$$

Since the integral is positive

$$j_{m+1} |J_1(j_{m+1})| > j_m |J_1(j_m)|,$$

and hence the terms are, numerically, monotonically decreasing.

Finally, since

$$\frac{1}{j_m J_1(j_m)} \sim \frac{1}{(m-\frac{1}{4})\pi} \cdot \frac{\pi (m-\frac{1}{4})^{\frac{1}{2}}}{2^{\frac{1}{2}} \cos{(m-1)\pi}} = O(m^{-\frac{1}{2}}),$$

the nature of this series allows it to be summed by the suggested method.

The sum of the first ten terms of the above series gives the value 0·77937872.

On using the Eulerian method† it is found that $\sum_{n=1}^{\infty} a_n$ lies between 1·00000542 and 0·99999492.

* Bromwich, *Infinite Series*, pp. 55–59.

† The major values are produced by summing the first three terms of the series together, while the minor values differ only in having the first four terms summed.

(ii) $f(r) = 1-r^2$, $f(+0) = 1$.—Here

$$a_m = 2\int_0^1 \frac{t(1-t^2)\,J_0(j_m t)}{J_1^2(j_m)}\,dt = 8[j_m^3 J_1(j_m)]^{-1}.$$

The sum of the first ten terms of this series gives the value 0·99915480.

The use of the Eulerian method, justified by example (1), shows that $\sum_{n=1}^{\infty} a_n$ lies between 1·00000184 and 0·99999364.

The result is therefore confirmed in both cases.

5. The stringent restrictions enforced on $f(r)$, though wide enough for all cases in the physical problem, call for comment. To that end the function $T_n(t, r)$ defined as

$$T_n(t, r) = \sum_{m=1}^{n} \frac{2J_0(j_m t)\,J_0(j_m r)}{J_1^2(j_m)}, \tag{5.1}$$

so that

$$T_n(t, 0) = T_n(t),$$

will be introduced.

It is shown, on similar lines to above, by a contour integral, that *near* $r = 0$,

$$T_n(t, r) = O(r^{-\frac{1}{2}}) \quad (n \text{ fixed}).$$

Accordingly for the interval $(\delta, 1)$, where $\delta > 0$, $|T_n(t, r)|$ is *bounded*.

This important fact immediately makes the application of an analogue of the Riemann-Lebesgue lemma for Fourier series possible. We find therefore that, for $r > 0$, if $f(r)$ satisfy Dirichlet conditions in the interval (0, 1), while it *only necessarily* has limited total fluctuation in an arbitrary interval, within (0, 1), which contains r as an internal point, the expansion converges to the value $\frac{1}{2}\{f(r+0)+f(r-0)\}$.

We shall now investigate the boundedness of $T_n(t)$.

$$T_n(t) = \sum_{m=1}^{n} \frac{2J_0(j_m t)}{J_1^2(j_m)} = \frac{1}{\pi i}\int_R \frac{J_1(t w)}{t\,J_0^2(w)}\,dw$$

$$= \frac{1}{\pi i}\int_{An-Bi}^{An+Bi} \frac{J_1(t w)}{t\,J_0^2(w)}\,dw = \frac{1}{\pi i t} I,$$

where R is the rectangle considered previously, and a similar reduction takes place. Now use the inequalities,

$$|J_1(t w)| \leqslant \frac{c_3 \exp|I(t w)|}{\sqrt{|t w|}}, \qquad |J_0(w)| \geqslant \frac{c_4 \exp|I(w)|}{\sqrt{|w|}},$$

the second of which holds as w evades any zero of $J_0(w)$ with

$$A_n = (n+\tfrac{1}{4})\pi.$$

We find, thus, making $B \to \infty$,

$$|I| < \frac{2c_3}{c_4^2} \int_0^\infty \frac{\exp\{-(2-t)v\}(A_n^2+v^2)^{\frac{1}{4}}}{t^{\frac{1}{2}}}\, dv$$

$$= \frac{2c_3}{c_4^2 t^{\frac{1}{2}}} \cdot \frac{A_n^{\frac{1}{2}}}{2-t} + O(A_n^{-\frac{3}{2}}).$$

Since the extension of the conditions imposed on $f(r)$ concerns only values of t greater than 0, it follows that $t^{-\frac{3}{2}}$ is bounded and that all we can prove is

$$|T_n(t)| = O(A_n^{\frac{1}{2}}) = O(n^{\frac{1}{2}}) \quad (t > 0). \tag{5.2}$$

This result is in striking contrast to the result deduced for $|T_n(t, r)|$, though the cause of the difference is evident. Without attempting a closer approximation to $|T_n(t)|$ we shall consider a particular case.

Let $$f(r) = (1-r^2)^{-\frac{1}{2}}.$$

This function evidently satisfies the conditions for expansion in a Fourier-Bessel series when r lies in the interval (δ_1, δ_2), where $0 < \delta_1 < \delta_2 < 1$. Now

$$a_m = \frac{2}{J_1^2(j_m)} \int_0^1 \frac{t\, J_0(j_m t)}{\sqrt{(1-t^2)}}\, dt$$

$$= \frac{2}{J_1^2(j_m)} \int_0^{\frac{1}{2}\pi} \sin\theta\, J_0(j_m \sin\theta)\, d\theta$$

$$= \frac{2}{J_1^2(j_m)} \frac{\sin j_m}{j_m},$$

a result which is found on expanding $J_0(j_m \sin\theta)$ in powers of $\sin\theta$ and integrating term by term.

Hence we find, on using asymptotic approximations,

$$a_m \sim \frac{\pi \sin(m-\frac{1}{4})\pi}{\cos^2(m-1)\pi},$$

that is, $$a_m \neq o(1).$$

The series $\sum_{m=1}^{\infty} a_m$ evidently does not converge, but oscillates finitely.

Thus it is concluded that $|T_n(t)|$ is certainly not a bounded function of n, for the closed interval $(\delta, 1)$, where $\delta > 0$, otherwise the Riemann-

Lebesgue lemma* could be applied and the function considered above would be expansible in a Fourier-Bessel series the coefficients of whose terms would give a convergent series.

6. In the previous example the function considered evades the restriction of limited total fluctuation by not being bounded. The consideration of bounded functions and unlimited total fluctuation is of interest.

C. N. Moore has discussed a function which is substantially

$$f(r) = \sum_{n=1}^{\infty} \{\sqrt{r}/n^2\} \cos(2^n - \tfrac{1}{4})\pi r,$$

and shown it to be non-summable (Ck) for $0 \leqslant k \leqslant \frac{1}{2}$ at the point $r = 0$.

On the other hand it is easily shown on using methods similar to those in this paper that the functions $\sin \log(1-r^2)$, $\cos \log(1-r^2)$, give convergent series at the point $r = 0$. These functions are evidently bounded for the interval (0, 1) but have not limited total fluctuation therein.

Accordingly the condition of limited total fluctuation throughout the interval (0, 1) is sufficient, but not necessary, for the convergence of the series of coefficients in the Fourier-Bessel expansion of a function $f(r)$ to the value $f(+0)$.

* Watson, *Bessel Functions*, p. 589.

ON THE ELECTROSTATIC FIELD OF A PLANE OR CIRCULAR GRATING FORMED OF THICK ROUNDED BARS.

By H. W. RICHMOND.

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1.1. Mathematical investigations of the electrical or hydrodynamical properties of gratings are due to a very large extent to English writers, Clerk Maxwell,* H. Lamb,† and J. J. Thomson.‡ The potential of a plane grating formed of parallel wires of infinitesimal thickness (or of small thickness compared with their distance apart), its effect upon induction, the hydrodynamical effects of a plane interrupted by parallel and equal gaps, the influence of a grating of thin wires in reflecting electric or other waves of long wave length, etc., etc., have all been discussed.

A paper§ contributed in 1911 to this Society was the outcome of an attempt by Mr. Page in 1907 to bring about closer numerical agreement between the mathematical treatment of the reflection and transmission of Hertzian waves by gratings and certain experiments ‖ in which the bars were cylinders or tubes of finite diameter: it was possible that the discrepancy could be accounted for by the fact that the actual bars were not infinitesimal. Mr. Page investigated the problem mathematically when the bars were rectangular, by the Schwarz-Christoffel method; he also considered means of rounding off the angles of the polygons to which that method is applicable.

Advanced physical problems such as this are beyond the range of the present writer. In this paper a simple means of rounding off rectangular

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* *Electricity and Magnetism*, 2nd edition, pp. 286–291.

† *Proc. London Math. Soc.* (1898); and *Hydrodynamics*.

‡ *Recent Researches*, pp. 406, 425.

§ W. M. Page, "Some two-dimensional problems in Electrostatics and Hydrodynamics", *Proc. London Math. Soc.*, Ser. 2, Vol. 11, pp. 313–328.

‖ See Schaefer and Laugwitz, *Ann. der Phys.*, Vols. 21 and 23; G. H. Thomson, *Ann. der Phys.*, Vol. 22; H. Lamb and G. Cook, *Phil. Mag.*, Vol. 20, 6th Series, p. 303.

bars into practically perfect circles is explained, and the electrostatic field, produced when a grating with thick round bars is charged or is placed in a uniform field of force, is obtained; the functions involved are elementary ones, whose values are given in many books of mathematical tables. Apart from any intrinsic interest these problems possess, they are also definite steps in the solution of the more difficult one that Mr. Page attempted, and it is permissible to hope that the method may help in the solution of more important problems.

2.1. *A plane grating with rectangular bars.*—Fig. 2.1 shows part of the section of such a grating by a plane xOy perpendicular to the length of the bars. The origin O is on the axis of a bar, and Ox, Oy are normal to its faces. The axes of the other bars meet the plane xOy in points lying at equal intervals along Oy. C is midway between O and O', the axis of the next bar, so that the figure is symmetrical about Ox, Oy, CX.

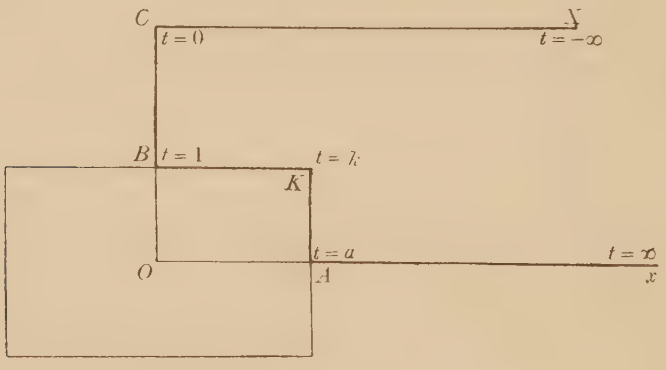

Fig. 2.1.

If now $z(\equiv x+iy)$ is defined as a function of t by the equation

$$E\frac{dz}{dt} = \frac{(t-k)^{\frac{1}{2}}}{(t-a)^{\frac{1}{2}}(t-1)^{\frac{1}{2}}t^{\frac{1}{2}}} \qquad (2.1)$$

where E is positive and $a > k > 1$, then as t diminishes through real values from $+\infty$ to $-\infty$, the point z describes a rectilinear path like $xAKBCX$, it being understood that each root is positive initially when t is large, and becomes a positive quantity multiplied by i as t passes through the value which makes the root vanish. The value of t at each critical point of the path is noted in the figure, and the values of E a, k

can be adapted to suit the dimensions of any such path. If k approaches a or 1, the rectangles flatten into straight lines, either all parallel to Ox or all lying on Oy; the bars of the grating approximate to strips of finite breadth, zero thickness, and unlimited length. In these extreme cases the relation (2 . 1) can be integrated in terms of simple functions.

2 . 2. *Three electrostatic problems.*—This mapping of the "z" contour $xAKBCX$ upon a real "t" axis solves three electrostatic problems concerning the grating. Let a second complex variable w ($\equiv u+iv$) be introduced such that the curves $u =$ constant are equipotential, and $v =$ constant lines of force. [Occasionally it is convenient to make $u =$ constant the lines of force and $v =$ constant the equipotentials.]

I. *Let the bars of the grating carry equal charges.*—Then AKB is part of an equipotential (say $u = 0$) and Ax, BCX are lines of force ($v = 0$ and $v = \frac{1}{2}\pi$ respectively). The "w" contour which corresponds to $xAKBCX$ is an infinite rectangle formed by the lines $v = 0$, $u = 0$, $v = \frac{1}{2}\pi$ and is mapped upon the "t" axis by the equation

$$E_1 \frac{dw}{dt} = \frac{1}{(t-a)^{\frac{1}{2}}(t-1)^{\frac{1}{2}}} \tag{2 . 21}$$

with the conditions $w = 0$ when $t = a$, $w = \frac{1}{2}i\pi$ when $t = 1$. Hence

$$t = a\cosh^2 w - \sinh^2 w \tag{2 . 22}$$

II. *Let the grating (uncharged) be placed in a uniform field of force parallel to Ox.*—Here $AKBC$ is equipotential (say $u = 0$), and Ax, CX are lines of force ($v = 0$ and $v = \frac{1}{2}\pi$).

$$E_2 \frac{dw}{dt} = \frac{1}{(t-a)^{\frac{1}{2}} t^{\frac{1}{2}}} \tag{2 . 23}$$

where $w = 0$ if $t = a$, $w = \frac{1}{2}i\pi$ if $t = 0$. Hence

$$t = a\cosh^2 w. \tag{2 . 24}$$

III. *Let the grating (uncharged) be placed in a uniform field of force parallel to Oy.*—This is an occasion when it is convenient to interchange the roles of u and v. Here $BKAx$ and CX are (by symmetry) equipotentials ($v = 0$ and $v = \frac{1}{2}\pi$ respectively) and BC is a line of force ($u = 0$).

$$E_3 \frac{dw}{dt} = \frac{1}{(t-1)^{\frac{1}{2}} t^{\frac{1}{2}}}, \tag{2 . 25}$$

and $w = 0$ if $t = 1$, $w = \frac{1}{2}i\pi$ when $t = 0$. Hence

$$t = \cosh^2 w. \tag{2.26}$$

In each problem elimination of t from (2.1) gives the relation between z and w; but results for rectangular bars need not detain us.

2.3. *Oval or rounded bars.*—The relation (2.1) between z and the real variable t can be modified so that the rectangles $AKB\ldots$ are replaced by ovals, while the results of § 2.2 remain valid. Among the possible methods one is noteworthy inasmuch as it leads to results expressible in terms of elementary functions instead of the elliptic functions required by (2.1).

Replace $(t-k)^{\frac{1}{2}}$ in (2.1) by $P(t-a)^{\frac{1}{2}}+Q(t-1)^{\frac{1}{2}}$, where P and Q are positive constants. Since E may be absorbed into P and Q we now have

$$\frac{dz}{dt} = \frac{P(t-a)^{\frac{1}{2}}+Q(t-1)^{\frac{1}{2}}}{(t-a)^{\frac{1}{2}}(t-1)^{\frac{1}{2}}t^{\frac{1}{2}}} \tag{2.31}$$

$$= \frac{P}{(t-1)^{\frac{1}{2}}t^{\frac{1}{2}}} + \frac{Q}{(t-a)^{\frac{1}{2}}t^{\frac{1}{2}}} \tag{2.32}$$

where $a > 1$. As t diminishes from ∞ to $-\infty$, the direction of motion of the point z is not affected except if t lies between a and 1. So long as $t > a$, dz/dt is real and positive, as in § 2.1; x constantly diminishes with t, from an infinite to a finite value, and y does not vary: with the origin placed on the axis of a bar, we have $y = 0$. Again, as t diminishes from 1 to 0, dz/dt has a pure imaginary value, x is always 0 and y increases. When t is negative, dz/dt is real and negative as in § 2.1; y remains constant and x increases from 0 to ∞.

But when $a > t > 1$, the first fraction of (2.32) is real and the second is purely imaginary; dx/dt is positive, dy/dt negative, *i.e.* x steadily diminishes and y steadily increases: also

$$\frac{dy}{dx} = -\frac{Q}{P}\frac{(t-1)^{\frac{1}{2}}}{(a-t)^{\frac{1}{2}}},$$

i.e. is negative, and diminishes in absolute value from ∞ to 0. The point z describes a quadrant of an oval which cuts Ox and Oy at right angles. By the symmetry about Ox and Oy we see that the rectangular bars have been replaced by rounded oval bars, of a shape at present unknown but defined by the values of P, Q, c. The results of § 2.2 remain valid without change.

It will be useful to write down here in a real form the values of x and

y obtained by integrating equation (2 . 32) for all real values of t, *i.e.* for all parts of the contour $xABCX$.

(i) $t > a$ $\quad x = 2P\cosh^{-1}(t^{\frac{1}{2}}) + 2Q\cosh^{-1}(t^{\frac{1}{2}}/a^{\frac{1}{2}})$;
$\quad y = 0.$

(ii) $t = a$ $\quad x = 2P\cosh^{-1}(a^{\frac{1}{2}})$; $\quad y = 0.$

(iii) $a > t > 1$ $\quad x = 2P\cosh^{-1}(t^{\frac{1}{2}})$;
$\quad y = 2Q\cos^{-1}(t^{\frac{1}{2}}/a^{\frac{1}{2}}).$

(iv) $t = 1$ $\quad x = 0$; $\quad y = 2Q\cos^{-1}(1/a^{\frac{1}{2}}).$

(v) $1 > t > 0$ $\quad x = 0$;
$\quad y = 2P\cos^{-1}(t^{\frac{1}{2}}) + 2Q\cos^{-1}(t^{\frac{1}{2}}/a^{\frac{1}{2}}).$

(vi) $t = 0$ $\quad x = 0$; $\quad y = \pi(P+Q).$

(vii) $0 > t$ $\quad x = 2P\sinh^{-1}(\tau^{\frac{1}{2}}) + 2Q\sinh^{-1}(\tau^{\frac{1}{2}}/a^{\frac{1}{2}})$;
$\quad y = \pi(P+Q)$ where $\tau = -t.$

The origin is understood to be at the centre of the oval.

2 . 4. *The shape of the ovals.*—It is important for us to know how closely these ovals resemble circles; we will consider (iii) further. When $a > t > 1$ we separate the real and imaginary parts on either side of (2 . 32): thus

$$\frac{dx}{dt} = \frac{P}{(t-1)^{\frac{1}{2}}\,t^{\frac{1}{2}}}; \qquad \frac{dy}{dt} = \frac{-Q}{(a-t)^{\frac{1}{2}}\,t^{\frac{1}{2}}}.$$

With the origin at the centre of the oval, $y = 0$ when $t = a$ and $x = 0$ when $t = 1$. Hence

$$x = \int_1^t \frac{P\,dt}{(t-1)^{\frac{1}{2}}\,t^{\frac{1}{2}}} = 2P\cosh^{-1}(t^{\frac{1}{2}}); \qquad (2 . 41)$$

$$y = \int_t^a \frac{Q\,dt}{(a-t)^{\frac{1}{2}}\,t^{\frac{1}{2}}} = 2Q\cos^{-1}(t^{\frac{1}{2}}/a^{\frac{1}{2}}) \qquad (2 . 42)$$

as above. We infer that (for an assigned value of a) if the shape of the oval is known for certain values of P and Q, its shape for any other values can be deduced by homogeneous strain; and that if (as will be shown) for a certain value of P/Q the oval is to all intents a circle, it will for any other value bear an equally close resemblance to an ellipse.

2 . 5.—*Nearly circular bars. Dimensions.*—The points A and B where the oval cuts the coordinate axes correspond to the values a and 1

of t. If

$$t = a, \quad y = 0, \quad \text{and} \quad x = OA = 2Pb;$$

if

$$t = 1, \quad x = 0, \quad \text{and} \quad y = OB = 2Q\beta;$$

where

$$\cosh b = \sec \beta = a^{\frac{1}{2}} \tag{2.51}$$

To ensure that the bars are approximately circular we can choose P and Q so that $OA = OB = R$.

$$2P = R/b; \quad 2Q = R/\beta, \tag{2.52}$$

β being in radians. Thus on the oval

$$x/R = (1/b)\cosh^{-1}(t^{\frac{1}{2}}); \quad y/R = (1/\beta)\cos^{-1}(t^{\frac{1}{2}}/a^{\frac{1}{2}}), \tag{2.53}$$

the values of t ranging from a down to 1. Tentative calculations of the shapes of the ovals now make it clear that *the approximation to a circular form is extremely close* for all values of β from 0° up to the immediate neighbourhood of 90°. But if the bars can be considered as circular cylinders of radius R, the shape of the plane grating is completely specified if the distance ($2D$) between the axes of the bars is known. It is the ratio of R to D, and not a or b or β, that is of geometrical and physical importance, and must be the basis of numerical calculations. If O' is where the axis of the next bar meets Oy, since $2D = OO' = 2OC$ (Fig. 2.5), we have, by (vi) at the end of § 2.3,

$$D = \pi(P+Q), \tag{2.54}$$

$$\frac{D}{R} = \frac{\pi}{2}\left(\frac{1}{b} + \frac{1}{\beta}\right), \tag{2.55}$$

$$\frac{R}{D} = \frac{2b\beta}{\pi(b+\beta)}. \tag{2.56}$$

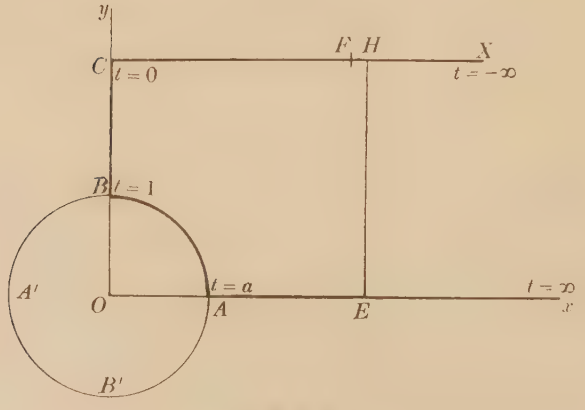

Fig. 2.5.

3.1. *Numerical results. Shape of the bars. Dimensions.*—In Dale's "Mathematical Tables", No. 23, values of b are given which correspond to clear values of β at intervals of half a degree from 0° to 90°. It is easy to tabulate (see Table 3.11) the values of b, P/D, Q/D, and R/D, which correspond to a series of values of β, and to derive Table 3.12 in which are shown the values of β, b, P/D, Q/D which correspond to a series of clear values of R/D. The value of a (2.51) can be found at once; those of P/D and Q/D are given for future use.

TABLE 3.11.

β degrees	β radians	b	P/D	Q/D	R/D
0	0	0	·159155	·159155	0
10	·17453	·17543	·15876	·15955	·05570
20	·34907	·35638	·15750	·16081	·11227
30	·52360	·54931	·15534	·16297	·17066
40	·68913	·76291	·15210	·16621	·23207
50	·87266	1·01068	·14749	·17082	·29814
60	1·04720	1·31696	·14099	·17732	·37138
70	1·22173	1·73542	·13151	·18680	·45644
80	1·39626	2·43625	·11579	·20234	·56504
85	1·48353	3·13130	·10233	·21598	·64086
89	1·55334	4·74135	·07855	·23976	·74486
90	1·57080	Infinite	0	·31831	1

TABLE 3.12

R D	β degrees	β radians	b	P/D	Q/D
$\frac{1}{10}$	17° 51′	·31154	·31671	·15785	·16046
$\frac{1}{5}$	34° 51′	·60825	·64964	·15392	·16439
$\frac{1}{3}$	54° 58′	·95935	1·15322	·14454	·17377
$\frac{1}{2}$	74° 24′	1·29852	1·98790	·12577	·19254
$\frac{2}{3}$	86° 18′ 36″	1·50640	3·4354	·0970	·2213
$\frac{3}{4}$	89° 6′ 36″	1·55526	4·858	·0772	·2411

With these data the shape of the ovals can be studied systematically. Whatever value is given to β, if points P on the oval are plotted by equations (2.53) it is found that as we recede from Ox (Fig. 2.5) the radius vector OP increases from its initial value R; it attains a maximum when the vectorial angle POA is somewhere near 45° and then diminishes to the minimum R as Oy is reached. The increase and decrease are quite regular, and, if the maximum value of OP/R is known, its excess above unity indicates the amount of divergence from a true circular form. The maximum value of OP/R for various values of R/D is shown in Table 3.14.

Details of the shape of an oval, typical of them all, are given in Table 3.13; this particular curve corresponds to the case when R/D is $\frac{1}{2}$, and $\beta = 74^\circ 24'$.

TABLE 3.13.

Showing the shape of the bars when $R/D = \frac{1}{2}$.
A series of values of x/R, y/R, OP/R, for points at roughly equal distances along the arc.

x/R	0	·250	·500	·707	·866	·966	1
y/R	1	·971	·882	·732	·525	·275	0
OP/R	1	1·005	1·014	1·018	1·013	1·004	1

TABLE 3.14.

Maximum value of OP/R for various values of R/D.

R/D	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$
Maximum of OP/R	1·00003	1·00043	1·004	1·018	1·063	1.11

Bars such as these (with the possible exception of the last case) would be regarded as circular anywhere, except in the abstractions of Pure Mathematics. They are possibly as nearly circular as the actual bars used in an experiment. If the free space between two bars exceeds in breadth the diameter of a bar (*i.e.* if $D > 2R$) no diameter of these theoretical bars differs from the mean diameter by 1 per cent. If the free space is half the diameter of a bar, the greatest difference is a trifle more than 3 per cent. In an actual bar, or in a drawing of the curve of section, so slight a discrepancy would hardly be detected. These theoretical bars may fairly be accepted as circular except in extreme cases when they are almost in contact.*

4.1. *The three electrostatic problems of* § 2.2 *for a plane grating with rounded bars of a thickness that is not negligible* are solved by combining the results of (2.22), (2.24), (2.26) with (2.32) so as to eliminate

* As the bars approach contact, the oval approximates to and finally assumes the form of a square contained by the lines $x = \pm R$, $y = \pm R$.

It may be shown that the oval defined by equations (2.41) and (2.42) has an intrinsic equation suggestive of that of an ellipse, viz.

$$\rho\,(P^2\cos^2\psi + Q^2\sin^2\psi)(aP^2\cos^2\psi + Q^2\sin^2\psi)^{\frac{1}{2}} = 2P^2Q^2(a-1)^{\frac{1}{2}}$$

where ρ is the radius of curvature and ψ the inclination of the normal to Ox.

t, and then giving to P and Q the values of (2 . 52). Equation (2 . 32) can be integrated

$$z = 2P \cosh^{-1}(t^{\frac{1}{2}}) + 2Q \cosh^{-1}(t^{\frac{1}{2}}/a^{\frac{1}{2}}); \qquad (4 . 11)$$

with the origin at the centre of a bar no constant of integration is required.

I. *When the bars of the grating are charged*, we have by (2 . 22)

$$t = a \cosh^2 w - \sinh^2 w;$$

and

$$a^{\frac{1}{2}} = \cosh b = \sec \beta.$$

Substituting, we find after simplifying

$$z = 2P \sinh^{-1}(\sinh b \cosh w) + 2Q \sinh^{-1}(\sin \beta \sinh w) \qquad (4 . 12)$$

The lines of force and equipotentials, the distribution of the charge, &c., can now be determined if need be by the usual methods.

II. *When the grating is placed in a uniform field of force parallel to* Ox (*i.e.* the force is normal to the plane of the grating), we have by (2 . 24)

$$t = a \cosh^2 w,$$

$$z = 2P \cosh^{-1}(\sec \beta \cosh w) + 2Qw. \qquad (4 . 13)$$

III. *When the grating is placed in a uniform field of force parallel to* Oy (*i.e.* the force is parallel to the plane of the grating and perpendicular to the bars), we have by (2 . 26)

$$t = \cosh^2 w,$$

$$z = 2Pw + 2Q \cosh^{-1}(\cos \beta \cosh w). \qquad (4 . 14)$$

4 . 2. In each of these solutions the right-hand side is a sum of two terms, one multiplied by P and one by Q. If Q is put $= 0$ we have solutions of the case when the bars are flattened into strips parallel to Ox; if P is put $= 0$ we have the solution of the other extreme case mentioned in § 2 . 1, when the bars are flattened into strips lying in Oy. Adding the two results (or superposing two fields) and then taking a particular value of P/Q we could in either problem arrive at bars which are practically circles. But it appears clearer and more scientific to derive the solution of all the problems from the modified Schwarz-Christoffel relation (2 . 32), which in the first instance defines the geometrical character of the grating, and afterwards leads to physical properties, including the three we have discussed. A fact which, though obvious, may be worth mention

is that whatever physical problem is solved by replacing t in (2.32) by a function of w, the result is

$$z = 2Pp + 2Qq,$$

where (since q and p are both functions of w) q is a function of p. *But q is always the same function of p.* This is true in the three problems above, as well as others (*e.g.* if parts of the line Ax, BC, CX were conductors, or barriers in hydrodynamical problems).

4.3. Given a plane grating with bars of a certain oval shape, we have (by means of a relation between the complex variables z and w) determined the electrostatic field under three sets of conditions. But these fields may be superposed.* Hence we can deduce the electrostatic field when any of the gratings is placed with its bars bearing any charge, in any uniform field of force whose direction is at right angles to the axis of a bar.

Problems II and III have a hydrodynamical interpretation. If the grating is placed in a uniform stream of perfect liquid, in any orientation, the flow of the liquid through it can be found.

But the case of interest is when the bars are so nearly circular that the discrepancy can be ignored. That such a case should be included is a piece of good fortune: indeed, it appears to the writer almost incredible that the unpromising curves of equations (2.53) and (2.51) should, for all values of β from 0° to 89°, bear so close a resemblance to circles as is proved by Table 3.14.

5.1. *Note upon Problems II and III.* It is understood that u is positive within the region $xABCX$, so that $|e^{-w}| < 1$. In Problem II, z can be expanded in a series of powers of e^{-w} convergent throughout the region

$$z = 2(P+Q)w + C + P\left[(1-P_1)e^{-2w} + \frac{P_1-P_2}{2}e^{-4w} + \frac{P_2-P_3}{3}e^{-6w} + \dots\right] \tag{5.11}$$

where $P_1, P_2, P_3, \dots$ are the Legendre polynomials in $\cos 2\beta$. The value of C is $2P \log \sec \beta$. The expansion makes it clear that as we recede from the grating the lines of force ($v =$ constant) become straight lines parallel to Ox.

* Jeans, *Electricity and Magnetism*, §§ 100, 215–217. It would be necessary to express w as a function of z, which is possible numerically (or u as a function of x and y).

In Problem III we have a similar expansion, valid when $u > b$, which again shows that the equipotentials ($v = \text{const.}$) and lines of force ($u = \text{const.}$) behave as they should at a distance from the grating, viz.

$$z = 2(P+Q)w - C' + Q\left[(1-P_1)e^{-2w} + \frac{P_1-P_2}{2}e^{-4w} + \ldots\right], \quad (5.12)$$

$P_1, P_2, P_3, \ldots$ being now Legendre polynomials of $\cosh 2b$, a quantity greater than unity. In each problem the intensity at distant points is

$$1/(2P+2Q).$$

5.2. The lines $u = \text{const.}$ (equipotential in II, lines of force in III) are straight and parallel to the grating at a distance where u is large. They become bent near the grating, and the extent of their undulations enables us to estimate to what distance the field is appreciably disturbed by the presence of the grating. Suppose that a line $u = \text{constant}$ cuts Ox, CX, $O'x'$ in points E, F, E' (see Fig. 2.5) and that EE' cuts CX in H. The ratio of FH to EE', which we shall denote by f, will serve as a measure of the amount of bending of the line $u = \text{const.}$

On Ox, $v = 0$; and on Cx, $v = \frac{1}{2}\pi$. Hence, in Problem II,

$$OE = 2Qu + 2P\cosh^{-1}(\cosh b \cosh u),$$

$$CF = 2Qu + 2P\sinh^{-1}(\cosh b \sinh u).$$

In II, $$f = FH/EE' = (OE-CF)/2D$$

$$= (P/D)[\cosh^{-1}(\cosh b \cosh u) - \sinh^{-1}(\cosh b \sinh u)].$$

Similarly in III,

$$f = (Q/D)[\sinh^{-1}(\sinh u \cos\beta) - \cosh^{-1}(\cosh u \cos\beta)],$$

so long as $u > b$. Table 5.2 shows the distance (d) from the grating at which f has the value 1/100 for the six gratings of Table 3.12; the values of u are also shown.

TABLE 5.2.

Showing the distance d from the grating at which the curves $u = \text{const.}$ are so straight that $f = 1/100$.

	R/D	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$
II	u	0·576	1·154	1·482	1·575	1·480	1·369
	$d/2D$	·19	·40	·55	·67	·74	·76
III	u	0·658	1·392	2·13	3·10	4·64	6·10
	$d/2D$	·20	·41	·58	·73	·87	·94

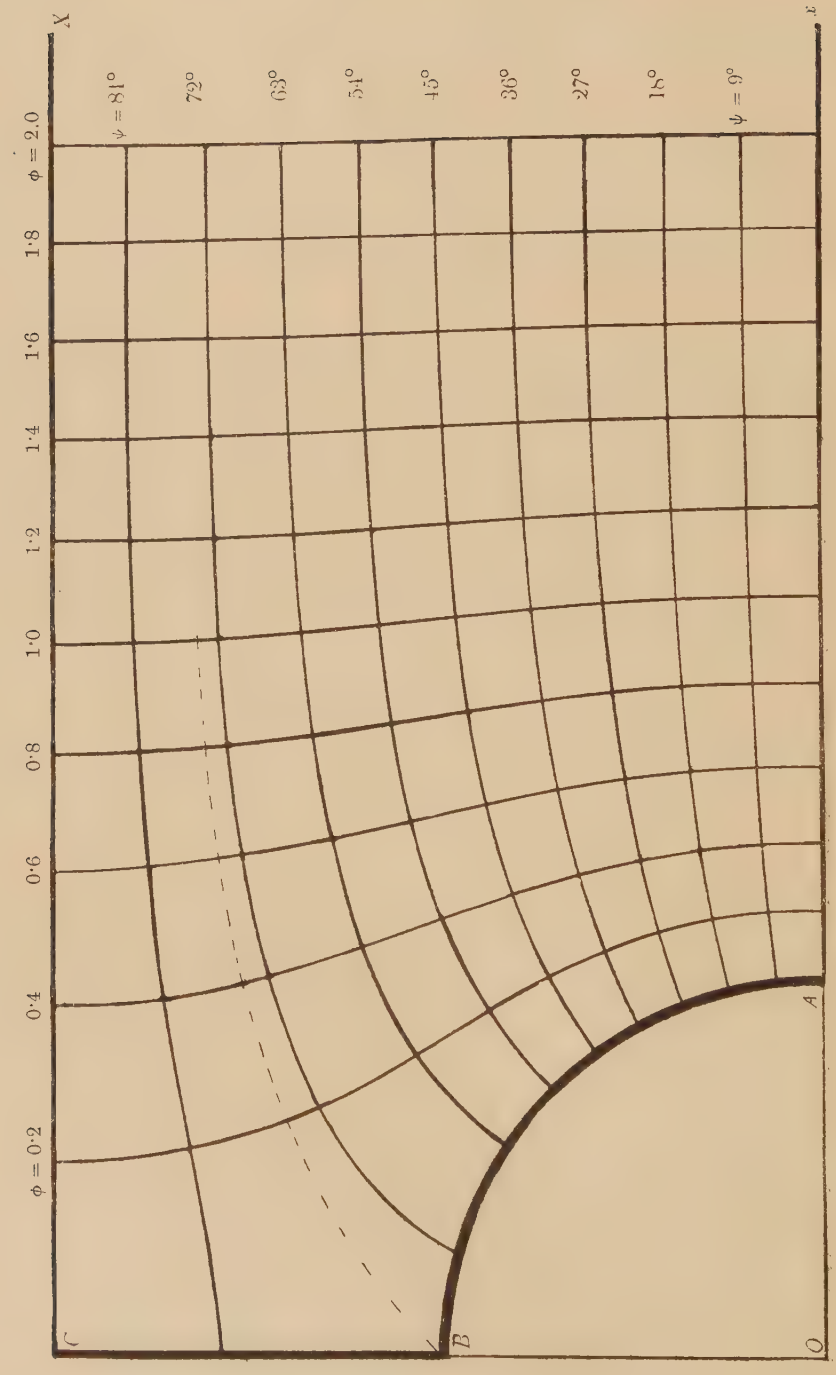

Fig. I.

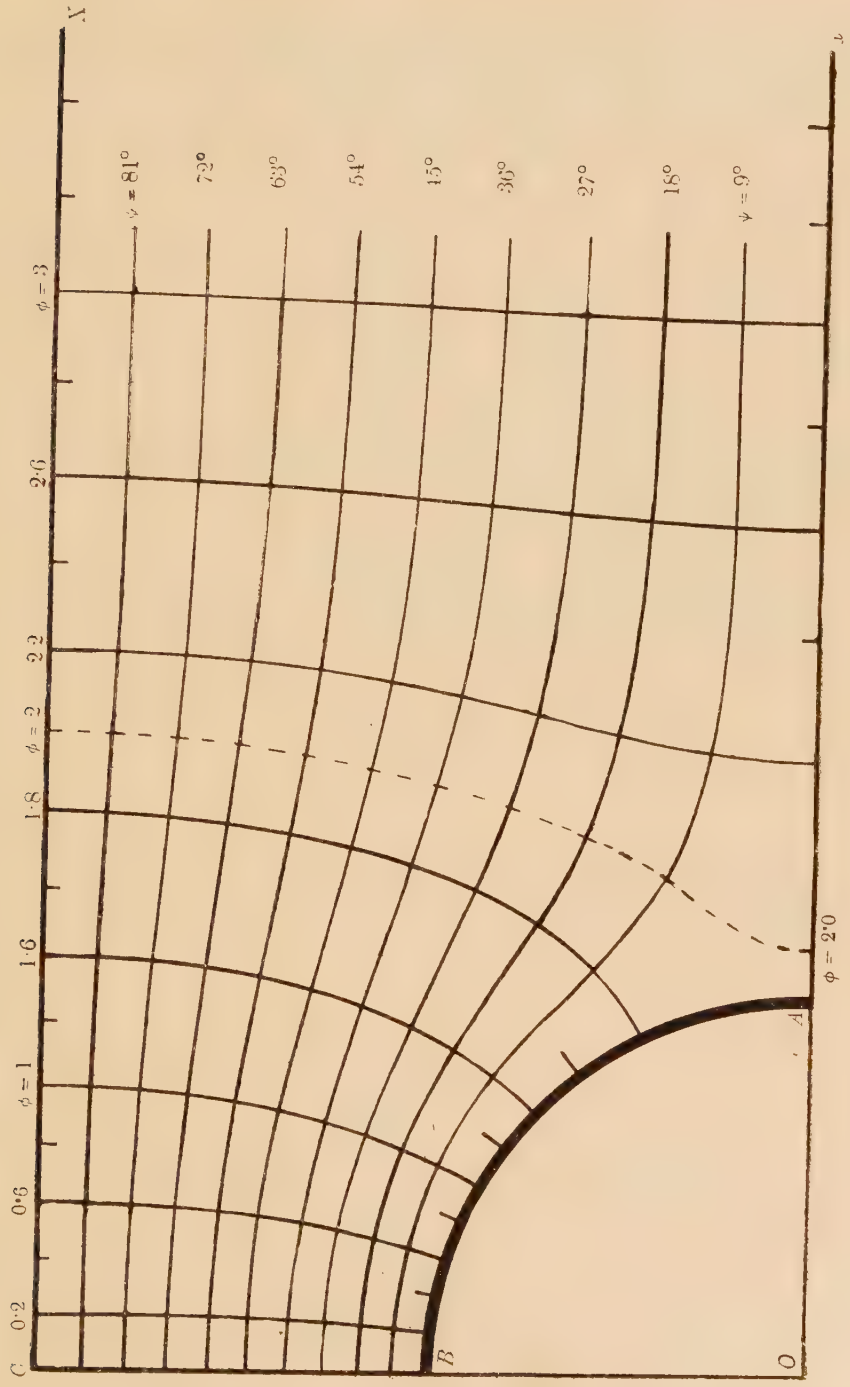

Fig. II.

The effect of the presence of the grating in deflecting the lines of force out of the straight is thus shown to be extremely limited in extent. Other numerical results, such as the proportion of the lines of force which do or do not meet the bars, can be considered in fields compounded of Cases I, II, III. The efficacy of a grating with thick bars in screening the fields on opposite sides of it from one another can also be estimated.

5.3. *Figs. I and II* show the field when a plane grating formed of round (practically circular) bars mounted so that the gap between two bars is equal to the diameter of each bar and placed in a uniform field of electric force. Fig. I illustrates Problem II, in which the direction of the field is normal to the plane of the grating; Fig. II illustrates Problem III, in which the direction is parallel to the plane (and shows also the flow of perfect liquid through the grating). In these figures ϕ and ψ are used in place of u and v.

6.1. *The properties of a grating formed of n rounded (to all intents circular) bars placed evenly round a circle* may be studied by a slight modification of these formulæ. In place of z write $\log(z/c)$ in §§ 2, 3, 4; thus instead of (4.11) write

$$\log(z/c) = 2P\cosh^{-1}(t)^{\frac{1}{2}} + 2Q\cosh^{-1}(t/a)^{\frac{1}{2}}, \qquad (6.11)$$

P and Q now being positive fractions, such that

$$P+Q = 1/n. \qquad (6.12)$$

The arguments of §§ 2, 3, 4 are repeated in polar coordinates, $\log(r/c)$ and θ playing the parts of x and y, As t diminishes through real values, z describes a path $xABCX$ (see Fig. 6.1), and θ (like y) increases by

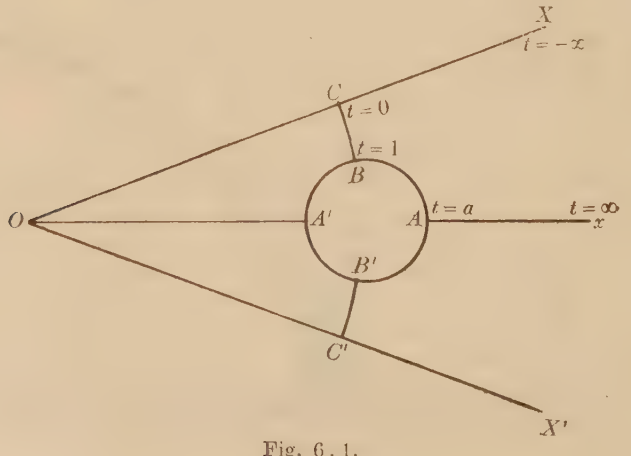

Fig. 6.1.

$\pi(P+Q)$, *i.e.* π/n. The sections of the bars of the grating are ovals, $ABA'B'A$, cut orthogonally by the circle with centre O and radius c; the parts of an oval which lie within and without the circle are inverse with respect to it. If, as before, we write

$$a^{\frac{1}{2}} = \cosh b = \sec \beta, \tag{2.51}$$

each oval subtends at O an angle $4Q\beta$, and is very nearly a circle of radius R provided that

$$R/c = \tan(2Q\beta) = \sinh(2Pb) \tag{6.13}$$

which implies that

$$\cosh(2Pb) = \sec(2Q\beta). \tag{6.14}$$

When these conditions are satisfied, by combining (6.11) with (2.26) we find the flow of liquid outwards between the bars from a line-source through O perpendicular to the plane of the diagram. Combining (6.11) with (2.24) we have the field when this line is charged and the bars are not. The combination of (6.11) with (2.22) gives the field when the line through O and the bars bear charges of opposite sign, the total charge per unit length on all the bars being double that on the line.

DIRICHLET SERIES AND THE THEORY OF PARTITIONS

By Major P. A. MacMahon.

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1. Consider the infinite product

$$\prod_p \frac{1}{1-b^p p^{-s}}$$

where there is a factor for every prime number p. b and s are numerical magnitudes which are not further specified for the moment.

On development we have

$$\prod_p \frac{1}{1-b^p p^{-s}} = \sum_n b^{\pi_1 p_1+\pi_2 p_2+\pi_3 p_3+\dots} n^{-s}$$

where the integer n has the prime number expression

$$n = p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \dots .$$

It is convenient to term the magnitude

$$\pi_1 p_1+\pi_2 p_2+\pi_3 p_3+\dots,$$

the "potency" of the integer n, and to write

$$pt(n) = \Sigma \pi p.$$

The magnitude $$\pi_1+\pi_2+\pi_3+\dots$$

will be termed the "multiplicity" of the integer n, and we write

$$mt(n) = \Sigma \pi.$$

The succession of integers $\pi_1\pi_2\pi_3\ldots$, *when arranged in descending order of magnitude*, will be held to specify a multipartite number

$$\overline{\pi_1\pi_2\pi_3\ldots.}$$

In the Theory of Partitions of numbers*

$$(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots)$$

usually denotes a partition of the number

$$\Sigma\pi p$$

into π_s parts equal to p_s; $s = 1, 2, 3, \ldots.$

1.1. Hence every integer n may be regarded as specifying a partition of its potency $pt(n)$ into parts which are primes, the number of parts being equal to its multiplicity, viz. $mt(n)$. There is here a one-to-one correspondence.

Conversely every partition of an integer ν into prime parts, $\Sigma\pi$ in number, connotes an integer n of potency ν and multiplicity $\Sigma\pi$, which is associated with a definite multipartite number $\overline{\pi_1\pi_2\pi_3\ldots.}$. If the integer ν be given, every solution in integers of the equation

$$\Sigma\pi p = \nu$$

connotes an integer which has the potency ν.

Further, if $\Sigma\pi = \mu$ be given, every solution in integers of the simultaneous equations

$$\Sigma\pi p = \nu$$

$$\Sigma\pi = \mu$$

connotes an integer which has the potency ν and the multiplicity μ. Again, if the integers $\pi_1, \pi_2, \pi_3, \ldots$ be fixed, the indeterminate equation

$$\pi_1 p_1 + \pi_2 p_2 + \pi_3 p_3 + \ldots = \nu,$$

connotes integers which have the potency ν, the multiplicity μ, and are associated with the multipartite number $\overline{\pi_1\pi_2\pi_3\ldots.}$

* In the Theory of the Partitions of Numbers what is termed "the partition notation" was initiated by Meyer Hirsch more than a century ago and has been developed by Cayley and other since that date.

1.2. The annexed table will make the correspondence clear.

Potency ν	Partition of potency	Integer n	Multiplicity $\Sigma\pi$	Multipartite number $\overline{\pi_1 \pi_2 \pi_3 \ldots}$
2	2	2	1	1
3	3	3	1	1
4	2^2	4	2	2
5	23	6	2	$\overline{11}$
,,	5	5	1	1
6	2^3	8	3	3
,,	3^2	9	2	2
7	$2^2 3$	12	3	$\overline{21}$
,,	25	10	2	$\overline{11}$
,,	7	7	1	1
8	2^4	16	4	4
,,	23^2	18	3	$\overline{21}$
,,	35	15	2	$\overline{11}$
9	$2^3 3$	24	4	$\overline{31}$
,,	3^3	27	3	3
,,	$2^2 5$	20	3	$\overline{21}$
,,	27	14	2	$\overline{11}$
10	2^5	32	5	5
,,	$2^2 3^2$	36	4	$\overline{22}$
,,	235	30	3	$\overline{111}$
,,	5^2	25	2	2
,,	37	21	2	$\overline{11}$
11	$2^4 3$	48	5	$\overline{41}$
,,	23^3	54	4	$\overline{31}$
,,	$2^3 5$	40	4	$\overline{31}$
,,	$3^2 5$	45	3	$\overline{21}$
,,	$2^2 7$	28	3	$\overline{21}$
,,	11	11	1	1

1.21. It will be noted, from the central column of numbers, that the integers 8, 9 have the same potency, viz. 6, but that their multiplicities are different. Also that the integers 27, 20 have the same potency, viz. 9, and the same multiplicity, viz. 3, but that they are associated with different multipartite numbers. Also that the integers 54, 40 have the same

potency, viz. 11, the same multiplicity, viz. 4, and are associated with the same multipartite number, viz. $\overline{31}$.

1.22. In this case we observe that the multipartite number arises from the two partitions 23^3, $2^3 5$ of the potency, and we have agreed above to regard both multipartites $\overline{13}$, $\overline{31}$ as included in the single multipartite $\overline{31}$. If we do not so agree, then a further classification takes place in regard to each permutation of the constituents $\pi_1, \pi_2, \pi_3, \ldots$ of the multipartite number $\overline{\pi_1 \pi_2 \pi_3 \ldots}$. Thus the next following integers 45, 28 are not only associated with the same potency, viz. 11, and the same multiplicity, viz. 3, but also, through the partitions $3^2 5$, $2^2 7$, in a stricter sense, with the same multipartite, viz. $\overline{21}$.

It will be gathered from the above that we may group all positive integers according to potency, multiplicity, and multipartite number (in the two ways specified) or according to compatible combinations of these characteristics.

1.23. Goldbach's celebrated conjecture "Every even number is the sum of two primes" amounts, from the present point of view, to showing that every even potency > 4 is compatible with association with a multipartite number $\overline{11}$.

1.24. The potency of an integer n is $\not< n$.

If n is the power of a prime and we put

$$p_1^{\pi_1} = \pi_1 p_1 \quad \text{or} \quad p_1^{\pi_1 - 1} = \pi_1$$

we must have $(p_1, \pi_1) = (p_1, 1)$ or $(2, 2)$.

This shows that the integer n has a potency n

(i) when n is a prime,

(ii) when $n = 4$.

It may be recalled that in other parts of the Theory of Numbers, *e.g.* in the theory of primitive roots of numbers, the number 4 is exceptionally associated with prime numbers.

If we supposed integers to be grouped according to potency, we expand

$$\Pi \frac{1}{1 - b^{\nu}} = \sum_{\nu} N_{\nu} b^{\nu}$$

and observe that N_ν integers are in the group of potency ν.

If we take any partition of ν, the potency, into primes and multiply the parts together, we obtain an integer whose potency is ν.

It is obvious that ν is a lower limit of the magnitude of the numbers which have a potency ν.

1.25. It is easy to find an upper limit because if ν be any real numerical quantity and it be separated into any number of parts, the continued product of such parts has an upper limit

$$e^{\nu/e}.$$

This fact is readily established by the ordinary process of the differential calculus.

When $\nu = e$, the upper limit is e. In general the upper limit is $>$ or $< e$ according as $\nu >$ or $< e$, an interesting property of the Napierian base. When ν is an integer, we note that 3 is the nearest integer to e. If $[\nu/3]$ denote the greatest integer in $\nu/3$, we find that the highest integer in the group is

$3^{[\nu/3]}$ if ν be of form 0 (mod 3),

$2^2.3^{[\nu/3]-1}$,, ,, 1 ,,

$2.3^{[\nu/3]}$,, ,, 2 ,,

When ν is a prime, the number ν is invariably the smallest number in the group.

The lower limits can be raised when ν is not a prime.

If p be the smallest prime such that $\nu - p$ is also prime,

$$p(\nu - p)$$

is a lower limit.*

1.3. A group of integers, which has the potency ν, involves a prime number ν if ν be prime. The $N_\nu - 1$ other integers of the group are composite.

If ν be composite the N_ν integers in the group are all composite.

If the integer n be $p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots$,

and π_s involves a_s primes, in its expression in terms of prime factors, it is clear that p_s may be interchanged with any one of the a_s primes without altering the potency.

---

* In this connexion the inequality

$$\frac{pt(n)}{mt(n)} \leqslant n^{1/[mt(n)]},$$

derived from a well-known formula, may be noted.

In such a case the above expression of n may connote as many as

$$(a_1+1)(a_2+1)(a_3+1)\ldots$$

integers which have the same potency.

Thus, *e.g.* the integer $5^{2^{2.3}}$

has the same potency as each of the integers

$$2^{2.3.5}. \quad 3^{2^{2.5}}$$

and so forth.

2. In the next place I propose, apart from the question of convergence, to discuss briefly a Dirichlet series

$$f(s) = \sum_1^\infty a_n n^{-s}$$

from the point of view of the partition of numbers.

The summation is in respect of all positive integers.

I suppose that the integer n, expressed in terms of primes, is

$$p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots.$$

I take a_n, in the series, to be the number which enumerates the partitions of the multipartite number

$$\overline{\pi_1 \pi_2 \pi_3 \ldots.}$$

2.1. Then I say that

$$\prod_{n=2}^{\infty} \frac{1}{1-n^{-s}} = \sum_1^\infty a_n n^{-s}$$

that is to say that the infinite product on the left is the generating function which enumerates the partitions of all multipartite numbers.

It is in fact obvious, from the constructon of the infinite product, that a_n enumerates the ways in which the integer n can be expressed as a *product of positive integers.*

Moreover, since $n = p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots,$

we find one such expession for each partition of the multipartite number

$$\overline{\pi_1 \pi_2 \pi_3 \ldots.}$$

2.11. *E.g.* $n = 2^2 . 3 . 5 = 60.$

The multipartite number is $\overline{211}$.

Partitions of $\overline{211}$	*Partition of* $2^2.3.5$	*Expression as a Product of Positive Integers*
$\overline{211}$	$2^2.3.5$	60
$\overline{210}$ $\overline{001}$	$2^2.3\times5$	12.5
$\overline{201}$ $\overline{010}$	$2^2.5\times3$	20.3
$\overline{111}$ $\overline{100}$	$2.3.5\times2$	30.2
$\overline{200}$ $\overline{011}$	$2^2\times3.5$	4.15
$\overline{110}$ $\overline{101}$	2.3×2.5	6.10
$\overline{200}$. $\overline{010}$. $\overline{001}$	$2^2\times3\times5$	4.3.5
$\overline{110}$ $\overline{100}$ $\overline{001}$	$2.3\times2\times5$	6.2.5
$\overline{101}$ $\overline{100}$ $\overline{010}$	$2.5\times2\times3$	10.2.3
$\overline{011}$ $\overline{100}$ $\overline{100}$	$3.5\times2\times2$	15.2.2
$\overline{100}$ $\overline{100}$ $\overline{010}$ $\overline{001}$	$2\times2\times3\times5$	2.2.3.5

So that $a_{60} = 11$, equal to the number of partitions of the multipartite $\overline{211}$.

On the right-hand side of the identity we regard the series of positive integers raised to the power $-s$ as arranged, it may be, into an infinite number of groups; for one group corresponds to every multipartite number; and each group contains an infinite number of members. For this purpose we may regard the different permutations of the constituents of the multipartite number

$$\overline{\pi_1 \pi_2 \pi_3 \cdots}$$

as denoting the same or different multipartites.

2.2. How many integers of a given group

$$\overline{\pi_1 \pi_2 \pi_3 \cdots}$$

are there which do not exceed a given magnitude?

This and a large number of other questions present themselves. The group which is associated with the unipartite number which is merely the unipartite number

unity,

includes the whole of the prime numbers exclusively.

The generating function

$$\prod_n \frac{1}{1-n^{-s}}$$

may be split up into an infinite number of factors of which the first now written down is the Riemann Zeta Function

$$\prod_n \frac{1}{1-n^{-s}} = \prod_p \frac{1}{1-p^{-s}} \times \prod_p \prod_{p'} \frac{1}{1-(p\,p')^{-s}} \times \prod_p \prod_{p'} \prod_{p''} \frac{1}{1-p\,p'\,p'')^{-s}} \times \ldots, \text{ ad inf.}$$

The functions herein appearing seem to be worthy of study by the powerful analytical theory of numbers that has been successfully applied to the Zeta Function.

2.21. The function $\prod_{n=1}^{\infty} \frac{1}{1-(2n+1)^{-s}}$

may be separately studied. We are concerned with uneven numbers exclusively.

2.22. Similarly the infinite product

$$\Pi(1+n^{-s})$$

is a generating function which enumerates the partitions of the multipartite number

$$\overline{\pi_1 \pi_2 \pi_3 \ldots}$$

into unrepeated parts.

Further, the partitions into a given number of parts may be studied by means of the functions

$$\Pi(1-an^{-s})^{-1},$$

$$\Pi(1+an^{-s}).$$

SUR L'EQUIVALENCE DE CERTAINES MÉTHODES DE SOMMATION.

Par MARCEL RIESZ.*

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. . . J'ai indiqué la démonstration de l'équivalence des moyennes $\omega^{-k} \sum_{r<\omega} u_r(\omega-r)^k$ et des moyennes de Cesàro dans les *Comptes Rendus* ("Une méthode de sommation équivalente à la méthode des moyennes arithmétiques", 12 juin 1911).

Permettez-moi de vous indiquer quelques résultats qui, si insignifiants qu'ils soient, donnent l'explication de quelques difficultés concernant les moyennes dont je viens de parler. Il s'agit en premier lieu d'un théorème d'équivalence assez général, relatif à certaines méthodes de sommation. Je ne sais pas si ce théorème est nouveau; en tout cas, il découle immédiatement d'un théorème connu de M. Toeplitz.

Dans ce que suit, je désignerai la série qu'on veut sommer par Σu_n et je poserai $s_n = u_0+u_1+\ldots+u_n$.

Soit $a_0, a_1, \ldots a_n, \ldots$ une suite indéfinie de nombres positifs, et posons $A_n = a_0+a_1+\ldots+a_n$. Admettons que

$$\lim_{n\to\infty} \frac{a_n}{a_0+a_1+\ldots+a_n} = \lim_{n\to\infty} \frac{a_n}{A_n} = 0. \tag{1}$$

Alors on sait que la formule

$$T_n^{(a)} = \frac{a_n s_0+a_{n-1}s_1+\ldots+a_0 s_n}{A_n} \tag{2}$$

* Extrait d'une lettre adressée à M. G. H. Hardy.

définit une méthode de sommation dans le sens que chaque fois que $\lim_{n\to\infty} s_n$ existe, il en est de même de $\lim_{n\to\infty} T_n^{(a)}$ et les deux limites coïncident.

Soit $b_0, b_1, \ldots, b_n, \ldots$ une autre suite de nombres positifs. Posons d'une façon analogue $B_n = b_0+b_1+\ldots+b_n$ et admettons encore que b_n et B_n remplissent une relation analogue à (1). On aura encore une méthode de sommation $T_n^{(b)}$. Evidemment, $T_n^{(a)}$ et $T_n^{(b)}$ pourront tendre vers des limites sans qu'il en soit ainsi pour s_n.

Au sujet de telles méthodes de sommation, M. Nörlund* a démontré que *chaque fois que* $\lim T_n^{(a)}$ *et* $\lim T_n^{(b)}$ *existent toutes les deux, ces limites seront égales.*

M. Nörlund considère à cet effet la méthode de sommation $T_n^{(c)}$ où $c_n = a_0b_n+a_1b_{n-1}+\ldots+a_nb_0$. Il est alors facile de démontrer que chaque fois que p. ex. $T_n^{(a)}$ tend vers une limite déterminée, $T_n^{(c)}$ tendra vers la même limite. Par cela, le théorème est établi.

Grâce aux hypothèses ci-dessus, on aura

$$\frac{A_{n+1}}{A_n}\to 1, \qquad \frac{B_{n+1}}{B_n}\to 1. \tag{3}$$

Les séries $\Sigma a_n x^n$, $\Sigma b_n x^n$, $\Sigma A_n x^n$, $\Sigma B_n x^n$ seront donc convergentes pour $|x|<1$. Posons

$$\frac{\Sigma b_n x^n}{\Sigma a_n x^n} = \frac{\Sigma B_n x^n}{\Sigma A_n x^n} = \Sigma p_n x^n, \qquad \frac{\Sigma a_n x^n}{\Sigma b_n x^n} = \frac{\Sigma A_n x^n}{\Sigma B_n x^n} = \Sigma q_n x^n. \tag{4}$$

Je dirai que les méthodes (a) et (b) sont *équivalentes* si les limites $\lim T_n^{(a)}$ et $\lim T_n^{(b)}$ existent en même temps. (Grâce au théorème de M. Nörlund, on est dispensé de s'occuper de l'égalité.) Je vais indiquer maintenant la démonstration du théorème suivant.

Pour que les méthodes (a) *et* (b) *soient équivalentes, il faut et il suffit que les séries* $\sum_0^\infty |p_n|$ *et* $\sum_0^\infty |q_n|$ *soient convergentes.*

La démonstration directe est très facile. Je préfère pourtant m'appuyer sur le théorème bien connu de M. Toeplitz que voici.

Posons $\qquad y_n = a_{n0}x_0+a_{n1}x_1+\ldots+a_{nn}x_n \quad (n=0, 1, 2, \ldots).$

* "Sur une application des fonctions permutables", *Lunds Universitets Årsskrift*, N. F. avd. 2, bd. 16, no. 3 (1919), pp. 1–10, *voir* p. 8.

Pour que l'existence de la limite $\lim x_n$ *entraîne celle de la limite* $\lim y_n$ *et de plus que ces deux limites soient égales, il faut et il suffit que*

$$(1) \quad \lim_{n\to\infty} (a_{n0}+a_{n1}+\ldots+a_{nn}) = 1;$$

$$(2) \quad \lim_{n\to\infty} a_{nv} = 0 \quad (v = 0, 1, 2, \ldots);$$

$$(3) \quad |a_{n0}|+|a_{n1}|+\ldots+|a_{nn}| < \text{const.} \quad (n = 0, 1, 2, \ldots).$$

De ce théorème, on tire immédiatement le lemme suivant qui nous sera très utile.

Pour que toute série sommable par la méthode (a) *soit aussi sommable par la méthode* (b), *il faut et il suffit que*

$$(5) \quad \frac{|p_0|A_n+|p_1|A_{n-1}+\ldots+|p_n|A_0}{B_n} < C \quad (n = 0, 1, 2, \ldots),$$

*C étant une constante, et que**

$$(6) \quad \lim_{n\to\infty} \frac{p_n}{B_n} \to 0.$$

En effet (à un point de vue tout à fait formel) on a

$$\Sigma B_r T_r^{(b)} x^r = \Sigma b_r x^r . \Sigma s_r x^r = \Sigma p_r x^r . \Sigma a_r x^r . \Sigma s_r x^r = \Sigma p_r x^r . \Sigma A_r T_r^{(a)} x^r.$$

Par conséquent

$$T_n^{(b)} = \frac{p_0 A_n T_n^{(a)}+p_1 A_{n-1} T_{n-1}^{(a)}+\ldots p_n A_0 T_0^{(a)}}{B_n}$$

Remarquons d'abord que l'on a évidemment

$$\frac{p_0 A_n+p_1 A_{n-1}+\ldots+p_n A_0}{B_n} = 1,$$

et alors la condition (1) de M. Toeplitz est toujours remplie. D'autre part les conditions (2) et (3) de M. Toeplitz reviennent dans le cas actuel aux conditions que nous venons de poser. Remarquons encore que la condition (5) entraîne que A_n/B_n *reste borné.*

Admettons maintenant que les méthodes (a) et (b) soient équivalentes. Alors, d'après ce que nous venons de dire, A_n/B_n et B_n/A_n resteront bornés, c'est-à-dire que

$$\frac{A_n}{B_n} < \mu_1, \qquad \frac{B_n}{A_n} < \mu_2,$$

* Je ne m'arrête pas sur la question si ces deux conditions sont indépendantes.

μ_1 et μ_2 étant des constantes. On aura de plus d'après (5)

$$\frac{p_0 A_n + |p_1| A_{n-1} + \ldots + |p_j| A_{n-j}}{B_n} = \frac{A_n}{B_n}\left(|p_0| + |p_1|\frac{A_{n-1}}{A_n} + \ldots + p_j \frac{A_{n-j}}{A^n}\right) < C,$$

et par conséquent

$$|p_0| + |p_1|\frac{A_{n-1}}{A_n} + \ldots + |p_j|\frac{A_{n-j}}{A_n} < C\mu_2.$$

Faisons tendre n vers l'infini ; il en vient en vertu de (3)

$$|p_0| + |p_1| + \ldots + |p_j| \leqslant C\mu_2.$$

L'indice j étant quelconque, il en résulte la convergence de la série $\Sigma|p_n|$ et alors, par raisons de symétrie, aussi la convergence de la série $\Sigma|q_n|$.

Admettons inversement que ces deux séries soient convergentes et posons $\Sigma|p_n| = P$, $\Sigma|q_n| = Q$. Alors la condition (6) sera manifestement remplie. De plus

$$A_n = B_0 q_n + B_1 q_{n-1} + \ldots + B_n q_0 \leqslant B_n(|q_0| + |q_1| + \ldots + |q_n|) \leqslant B_n Q,$$

et alors

$$\frac{|p_0| A_n + |p_1| A_{n-1} + \ldots + |p_n| A_0}{B_n} \leqslant \frac{A_n}{B_n} P \leqslant PQ,$$

c'est-à-dire que la sommabilité (a) entraîne la sommabilité (b). Par les mêmes raisons, la sommabilité (b) entraîne la sommabilité (a) et alors les méthodes sont équivalentes.

Dans les applications les plus importantes, A_n et B_n seront d'un type de croissance régulier et alors les fonctions représentées par les séries $\Sigma A_n x^n$ et $\Sigma B_n x^n$ auront en général pour $|x| \leqslant 1$ une singularité unique au point $x = 1$. L'équivalence reviendra donc à ce que les quotients $\Sigma B_n x^n : \Sigma A_n x^n$ et $\Sigma A_n x^n : \Sigma B_n x^n$ aient une singularité assez faible au point $x = 1$ et à ce que ni $\Sigma A_n x^n$ ni $\Sigma B_n x^n$ n'aient de zéros pour $|x| \leqslant 1$ (ou encore, mais ce sera là un cas très exceptionnel, à ce que ces zéros soient communs).

Avant de passer aux applications de ce théorème d'équivalence, je veux indiquer une application du lemme qui précède.

Je désignerai comme *sommation harmonique* la sommation définie par la formule

$$T_n = \frac{\frac{s_0}{n+1} + \frac{s_1}{n} + \ldots + \frac{s_n}{1}}{\log n}.$$

Démontrons que chaque fois que T_n tend vers une limite déterminée, la série en question sera aussi sommable (C, ϵ), ϵ étant un nombre positif quelconque.

On aura dans ce cas

$$\sum_0^\infty a_n x^n = \sum_0^\infty \frac{x^n}{n+1} = -\frac{\log(1-x)}{x},$$

$$\sum_0^\infty b_n x^n = (1-x)^{-\epsilon},$$

$$A_n = O(\log n), \qquad n^{-\epsilon} B_n \to \frac{1}{\Gamma(\epsilon+1)}.$$

D'autre part

$$\sum_0^\infty p_n x^n = \frac{\sum_0^\infty b_n x^n}{\sum_0^\infty a_n x^n} = \frac{-x(1-x)^{-\epsilon}}{\log(1-x)}.$$

Or on sait que les coefficients du développement de

$$\frac{x}{\log(1-x)}$$

sont $$O\left(\frac{1}{n \log^2 n}\right)$$

et que leur somme est zéro. On en conclut immédiatement

$$p_n = O\left(\frac{n^{\epsilon-1}}{\log n}\right).$$

[D'ailleurs, le même résultat, on le trouve en calculant p_n par le théorème de Cauchy, le chemin d'intégration étant composé d'une circonférence de rayon $R > 1$ et le segment double $(1, R)$.] Par les évaluations qu'on vient d'écrire, on vérifie facilement que les conditions (5) et (6) sont remplies.

Passons maintenant aux applications du théorème d'équivalence. Je vous ai déjà dit que ce théorème élucide les questions concernant l'équivalence des moyennes $\omega^{-k} \sum_{r<\omega} u_r(\omega - r)^k$ et des moyennes (C, k). Permettez-moi de rappeler quelques faits. D'abord (*Comptes Rendus*, 5 juillet 1909) j'ai défini ces moyennes en ne faisant prendre à ω que des valeurs entières et j'ai dit en passant que ces moyennes étaient équivalentes aux moyennes (C, k). Je me suis bientôt aperçu de ce que cet énoncé était inexact du moins dans le cas général, mais qu'il devenait exact,

lorsqu'on faisait varier ω d'une façon continue. J'ai rectifié l'erreur dans la Note des *Comptes Rendus* citée au début de cette lettre.

En quoi consistait alors cette erreur? J'ai constaté (sans me soucier beaucoup de tous ces détails) qu'il était facile de donner une série Σu_n pour laquelle la limite

$$\lim_{n\to\infty} \frac{\sum_{r=0}^{n-1} u_r(n-r)^2}{n^2}$$

existe et qui pourtant n'est pas sommable $(C, 2)$. Il est d'autre part facile de voir qu'une telle série est toujours sommable $(C, 3)$ [mais pas nécessairement $(C, 3-\epsilon)$]. En outre, on pouvait donner une série Σv_n, pour laquelle la limite

$$\lim_{n\to\infty} \frac{\sum_{r=0}^{n-1} v_r(n-r)^3}{n^3}$$

existe et qui n'est sommable (C, k) pour aucune valeur de k.

En voici maintenant l'explication. On a pour $k = 2$

$$\sum_0^\infty A_n x^n = \sum_0^\infty (n+1)^2 x^n = \sum_1^\infty n^2 x^{n-1} = \frac{1+x}{(1-x)^3},$$

et alors la fonction génératrice s'annule pour $x = -1$, c'est-à-dire en un point situé sur la circonférence $|x| = 1$. Pour $k = 3$, on a

$$\sum_0^\infty A_n x^n = \sum_0^\infty (n+1)^3 x^n = \sum_1^\infty n^3 x^{n-1} = \frac{1+4x+x^2}{(1-x)^4},$$

et alors la fonction génératrice s'annule pour $x = -2\pm\sqrt{3}$. Or $x = -2+\sqrt{3}$ est situé à l'intérieur du cercle de rayon *un*. Ces faits expliquent tout ce que j'ai dit au sujet de ces moyennes et expliquent aussi pourquoi les méthodes de sommation qui s'y rattachent, et en particulier la seconde, manquent tout à fait d'intérêt.

Or une modification légère de ces méthodes conduit à deux méthodes équivalentes aux méthodes $(C, 2)$ et $(C, 3)$ respectivement.

On a en effet $$\sum_1^\infty n^2 x^{n-2} = \frac{x^2-3x+4}{(1-x)^3}.$$

Le numérateur de cette expression ne s'annule que pour

$$x = \frac{3\pm i\sqrt{7}}{2};$$

c'est-a-dire qu'il ne s'annule pas pour $|x| \leqslant 1$.

De même $$\sum_{2}^{\infty} n^3 x^{n-2} = -\frac{x^3-4x^2+5x-8}{(1-x)^4}$$

et le numérateur ne s'annule que pour les valeurs

$$x = 3{\cdot}2\ldots, \qquad x = 0{\cdot}39\ldots \pm i1{\cdot}5,$$

situées toutes à l'extérieur du cercle de rayon *un*.

On en conclut immédiatement que *les moyennes*

$$\frac{\sum_{r=0}^{n-2} u_r(n-r)^2}{n^2}, \qquad \frac{\sum_{r=0}^{n-2} u_r(n-r)^3}{n^3}$$

sont équivalentes aux moyennes $(C, 2)$ *et* $(C, 3)$ *respectivement.*

Les indices $k = 2$ et $k = 3$ sont les seuls indices $k > 1$ que j'ai examinés. Je me hâte d'ajouter que je ne crois pas que ces résultats aient aucun avantage sur l'ancien théorème avec le paramètre continu ω*. En particulier, pour les séries de Dirichlet $f(s) = \Sigma \mu_n n^{-s}$ ces nouvelles moyennes sont bien plus difficiles à exprimer par la fonction $f(s)$ que celles avec le paramètre continu.

Pour $k = 0$ et $k = 1$, aucune question d'équivalence ne se pose, les moyennes

$$\frac{\sum_{r=0}^{n-1} u_r(n-r)^k}{n^k} \tag{7}$$

étant identiques aux moyennes $(C, 0)$ et $(C, 1)$ respectivement.

Pour $-1 < k < 1$, *je peux démontrer maintenant un fait que j'ai toujours présumé, c'est que les moyennes* (7) *sont équivalentes aux moyennes* (C, k).

Premièrement, k étant tel qu'on vient de dire, la fonction génératrice

$$\sum_{0}^{\infty} A_n x^n = \sum_{1}^{\infty} n^k x^{n-1}$$

ne s'annule pas pour $|x| \leqslant 1$. Pour $k < 0$, on le voit immédiatement par un théorème connu de M. Kakeya (*The Tôhoku Math. Journ.*, 1912, t. 2,

* Il n'est pas sans intérêt de remarquer que pour toute valeur < 1 du paramètre positif τ, la fonction $\sum_{n=0}^{\infty} (\tau+n)^2 x^n$ aura toujours un zéro à l'intérieur du cercle de rayon *un*, c'est-à-dire que le théorème d'équivalence relatif aux moyennes avec le paramètre continu n'est pas une conséquence du théorème que nous venons de démontrer.

pp. 140–142), les n^k étant positifs et décroissants avec $1/n$. Pour $0<k<1$, les coefficients de $(1-x)\sum_1^\infty n^k x^{n-1}$ sont positifs et décroissants et alors le théorème de M. Kakeya s'applique encore.

Cela étant, pour déterminer l'ordre de grandeur des coefficients des séries

$$(1-x)^{k+1}\sum_{n=1}^{\infty} n^k x^{n-1} = \sum_{n=0}^{\infty} p_n x^n, \qquad \frac{1}{(1-x)^{k+1}\sum_{n=1}^{\infty} n^k x^{n-1}} = \sum_{n=0}^{\infty} q_n x^n,$$

on n'aura qu'à étudier les singularités au point 1.

En utilisant alors le chemin d'intégration mentionné plus haut et en intégrant par parties, on montre facilement par une formule qu'on trouve dans le "Calcul des résidus" de M. Lindelöf (p. 139) que l'ordre de grandeur des coefficients p_n et q_n est $O(n^{-k-2})$ et, par conséquent, les séries $\Sigma|p_n|$ et $\Sigma|q_n|$ sont convergentes. Le théorème d'équivalence général qui précède pourra alors immédiatement être appliqué pour $0<k<1$ et après des modifications légères aussi pour $-1<k<0$. . .

Stockholm, le 6 février 1923.

ON THE SUMMABILITY OF GENERALIZED FOURIER'S SERIES

By E. W. Hobson.

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The theorem of Prof. G. H. Hardy that a Fourier's series is summable (C, k), where $k > 0$, almost everywhere in the interval $(-\pi, \pi)$, does not appear to admit of complete extension to the case of the generalized Fourier's series which correspond to function possessing Denjoy integrals. It has, however, been shown* by I. Privaloff that such a generalized Fourier's series is summable (C, k), where $k > 1$, almost everywhere in the interval of representation.

In the present communication it is shown that the generalized Fourier's series corresponding to a function which possesses a Denjoy integral is summable $(C, 1)$ almost everywhere in each interval contiguous to the non-dense closed set of points of non-summability of the function.

In the case of a function possessing a Denjoy integral for which the points of non-summability of the function has content zero, it follows that the series is summable $(C, 1)$ almost everywhere in the interval $(C, 1)$.

It is further shown that, whatever be the content of the set of points of non-summability of the function, the Cesàro sum $(C, 1)$ converges uniformly to the value of the function, in any interval in which the function is continuous, the continuity at the end-points of the interval being supposed to hold on both sides. Such an interval is necessarily interior to an interval contiguous to the set of points of non-summability of the function.

* *Rendiconto di Palermo*, Vol. 41 (1916), p. 202. I am, however, unable to follow some of the reasoning in Privaloff's memoir.

Let $f(x)$ have a Denjoy integral in the interval $(-\pi, \pi)$, then the nth partial Cesàro sum, of order 1, of the generalized Fourier's series corresponding to $f(x)$ is

$$\frac{1}{2n\pi}\int_{-\pi}^{\pi} f(x')\left[\frac{\sin\frac{1}{2}n(x'-x)}{\sin\frac{1}{2}(x'-x)}\right]^2 dx'.$$

Let (a, b) be an interval which is interior to an interval contiguous to the set H of points of non-summability of the function $f(x)$; and let $f(x) = f_1(x) + f_2(x)$, where $f_1(x) = f(x)$ at all points in the interval (a, b), and $f_1(x) = 0$ at all other points of $(-\pi, \pi)$. Clearly the function $f_1(x)$ is summable in the interval $(-\pi, \pi)$, and, in accordance with Hardy's theorem,

$$\frac{1}{2n\pi}\int_{-\pi}^{\pi} f_1(x')\left[\frac{\sin\frac{1}{2}n(x'-x)}{\sin\frac{1}{2}(x'-x)}\right]^2 dx$$

converges to $f(x)$ almost everywhere in the interval (a, b), and to zero almost everywhere in the remaining parts of the interval $(-\pi, \pi)$. Also the convergence takes place everywhere, and is uniform, in any interval in which $f_1(x)$ is continuous, the continuity at the end-points of the interval being on both sides.

The convergence of

$$\frac{1}{2n\pi}\int_{-\pi}^{\pi} f_2(x')\left[\frac{\sin\frac{1}{2}n(x'-x)}{\sin\frac{1}{2}(x'-x)}\right]^2 dx'$$

will be examined, in the interval $(a+\mu, b-\mu)$, where μ is an arbitrarily chosen positive number $\{< (b-a)/2\}$. The function $f_2(x')$ has the value zero within the interval (a, b) and elsewhere it has the values of $f(x)$. The expression to be examined is equivalent to

$$\frac{1}{2n\pi}\int_{-\pi}^{a} f(x')\left[\frac{\sin\frac{1}{2}n(x'-x)}{\sin\frac{1}{2}(x'-x)}\right]^2 dx' + \frac{1}{2n\pi}\int_{b}^{\pi} f(x')\left[\frac{\sin\frac{1}{2}n(x'-x)}{\sin\frac{1}{2}(x'-x)}\right]^2 dx'.$$

Let us consider $$\int_{-\pi}^{a} f(x')\,F(x'-x, n)\,dx',$$

where $$F(x'-x, n) \equiv \frac{1}{2n\pi}\left[\frac{\sin\frac{1}{2}n(x'-x)}{\sin\frac{1}{2}(x'-x)}\right]^2.$$

The number x being confined to the interval $(a+\mu, b-\mu)$, we see that, in the integrand, $x-x' \geqslant \mu$, $x-x' \leqslant b-\mu+\pi < 2\pi-\mu$, thus

$$\left|\sin\tfrac{1}{2}(x'-x)\right| \geqslant \sin\tfrac{1}{2}\mu$$

for all the values of x or x' concerned.

The method of integration by parts being valid* for the product of a function that is integrable (D) and a monotone bounded function, it can be applied to the above integral since, for fixed values of x and n, the function $F(x'-x, n)$ is expressible as the difference of two bounded monotone functions of x'. We thus have, for fixed values of n and x,

$$\int_{-\pi}^{a} f(x')\,F(x'-x, n)\,dx' = \Big[\phi(x')\,F(x'-x, n)\Big]_{-\pi}^{a} - \int_{-\pi}^{a} \phi(x')\,\frac{dF(x'-x, n)}{dx'}\,dx',$$

where $\phi(x')$ denotes the continuous function

$$\int_{-\pi}^{x'} f(x')\,dx'.$$

Now $$|\,\phi(x')\,F(x'-x, n)\,| \leqslant \frac{M}{2\pi n}\,\operatorname{cosec}^2\frac{\mu}{2},$$

where M is the upper limit of $|\,\phi(x')\,|$ in $(-\pi, \pi)$; hence

$$\Big[\phi(x')\,F(x'-x, n)\Big]_{-\pi}^{a}$$

converges to zero, as $n \sim \infty$, uniformly for all points x in $(a+\mu, b-\mu)$.

In order to show that

$$\int_{-\pi}^{a} \phi(x')\,\frac{dF(x'-x, n)}{dx'}\,dx'$$

converges to zero, as $n \sim \infty$, uniformly for all values of x in $(a+\mu, b-\mu)$, the fundamental convergence theorem† established for such integrals will be employed.

It is necessary to prove (1), that

$$\left|\frac{dF(x'-x, n)}{dx'}\right|$$

is less than a fixed positive number, for all values of n, and for all values of x' in $(-\pi, a)$, and of x in $(a+\mu, b+\mu)$, and further (2), that, for any interval (α, β) in $(-\pi, a)$,

$$\int_{\alpha}^{\beta} \frac{dF(x'-x, n)}{dx'}\,dx'$$

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* See *Theory of Functions of a Real Variable*, Vol. 1, p. 648.

† *Proc. London Math. Soc.*, Ser. 2, Vol. 12, p. 166.

converges to zero, as $n \sim \infty$, uniformly for all values of x in $(a+\mu, b-\mu)$ and all values of α and β in $(-\pi, a)$.

We have

$$\frac{dF(x'-x, n)}{dx'} = \frac{1}{2n\pi}\left[\frac{n \sin n(x'-x)}{2 \sin^2 \frac{1}{2}(x'-x)} - \frac{\sin^2 \frac{1}{2}n(x'-x) \cos \frac{1}{2}(x'-x)}{2 \sin^3 \frac{1}{2}(x'-x)}\right],$$

thus
$$\left|\frac{dF(x'-x, n)}{dx'}\right| \leqslant \frac{1}{4\pi} \operatorname{cosec}^2 \frac{\mu}{2} + \frac{1}{4\pi} \operatorname{cosec}^3 \frac{\mu}{2},$$

which shows that the condition (1) is satisfied.

Again
$$\int_\alpha^\beta \frac{dF(x'-x, n)}{dx'}\, dx' = F(\beta-x, n) - F(\alpha-x, n);$$

now
$$|F(\beta-x, n)| \leqslant \frac{1}{2n\pi} \operatorname{cosec}^2 \frac{\mu}{2},$$

and thus $F(\beta-x, n)$ converges uniformly to zero, as also does $F(\alpha-x, n)$. Thus the condition (2) is satisfied. It has now been proved that

$$\int_{-\pi}^{a} \phi(x') \frac{dF(x'-x, n)}{dx'}\, dx'$$

converges to zero, as $n \sim \infty$, uniformly for all values of x in $(a+\mu, b-\mu)$; and therefore that

$$\int_{-\pi}^{a} f(x')\, F(x'-x, n)\, dx'$$

also converges to zero, uniformly for all points x in $(a+\mu, b-\mu)$. In the same manner the corresponding result can be established for the integral

$$\int_{b}^{\pi} f(x')\, F(x'-x, n)\, dx'.$$

It has now been shown that, for any interval (a, b) interior to an interval contiguous to H, the partial Cesàro sum of the generalized Fourier's series converges to $f(x)$ almost everywhere in the interval $(a+\mu, b-\mu)$, where $\mu > 0$; and consequently the series converges to $f(x)$ almost everywhere in (a, b), and therefore also almost everywhere in the contiguous interval itself. The points of convergence depend merely on the nature of the function $f(x)$ in that contiguous interval, being the same in each interior interval as those corresponding to a summable function equal to $f(x)$ in the interval, and zero outside it.

For that class of Denjoy integrals for which $m(H) = 0$, which class includes the Harnack-Lebesgue integrals as a special class, it is seen at

once that the Cesàro sum $(C, 1)$ is equal to $f(x)$ almost everywhere in $(-\pi, \pi)$.

The following theorems have thus been established :—

If $f(x)$ have a Denjoy integral in the interval $(-\pi, \pi)$, then, almost everywhere in an interval contiguous to the set H, of points of non-summability of $f(x)$, the Cesàro sum $(C, 1)$ of the generalized Fourier's series corresponding to $f(x)$ exists, and is equal to $f(x)$.

In case the set H have content zero, and in particular for all Harnack-Lebesgue integrals, the series is summable $(C, 1)$ almost everywhere in the interval $(-\pi, \pi)$.

In any interval of continuity of $f(x)$, provided the continuity at the end-points of the interval is on both sides, the Cesàro partial sums $(C, 1)$ converge uniformly to the values of $f(x)$.

It should be observed that Privaloff's theorem is a necessary consequence of the result here obtained, in the case in which H has the measure zero, and thus in particular for the series corresponding to functions with Harnack-Lebesgue integrals. In the general case in which $m(H) > 0$, the above result does not show that the series is summable $(C, 1)$ almost everywhere in $(-\pi, \pi)$, and thus Privaloff's result is not included in the one here given.

ON THE DISCONTINUOUS MOTION PRODUCED IN AN INFINITE STREAM BY TWO PLANE OBSTACLES

By E. G. C. POOLE.

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1. Introduction.
2. CASE I (i). Criticism of Riabouchinsky's solution.
3. CASE I (ii). Generalization of the symmetrical case.
4. CASE I (iii). Generalization for a channel of finite breadth.
5. CASE II. One simply connected wake.
6. CASE III. Cascades and a barrier across the wake.
7. CASE IV. Two-wake problem (i) Symmetrical case.
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Preface.

My warmest thanks are due to Prof. Love, who first interested me in the theory of discontinuous fluid motion, and without whose generous help and encouragement the following paper would not have been written. It will appear in the sequel that two out of the four problems here treated turn on a difficult point in conformal representation, which was solved by Prof. Love, who has allowed his work to be incorporated in this paper. I have also to thank Mr. Littlewood for a valuable note elucidating the theory of the same problem.

1. *Introduction.*

In two recent papers, Riabouchinsky has examined a curious case of discontinuous fluid motion in two dimensions. An infinite stream is obstructed by two plane obstacles, lying one behind the other, which do not leave any infinite wake, but which have between them a finite area of dead water, bounded on the remaining sides by free surfaces. Riabouchinsky's solution of the case of complete symmetry was correct,* but he subsequently attempted a generalization† which contains serious

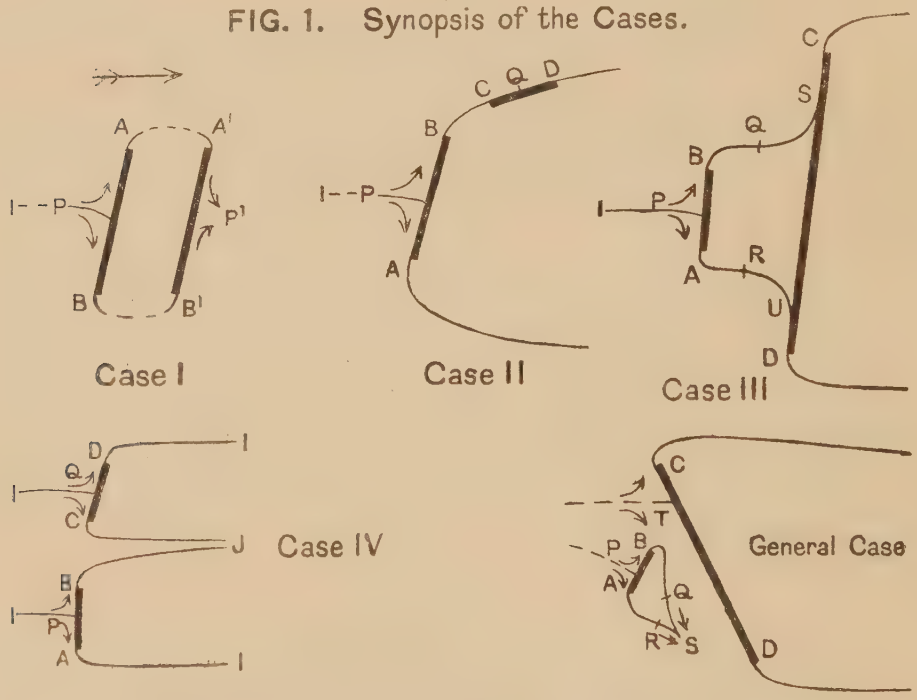
* *Proc. London Math. Soc.*, Vol. 19 (1920), p. 206.

† *Comptes Rendus du Congrès International des Mathématiciens, Strasbourg* (1920).

errors. In undertaking the present investigation, I was not originally actuated by any desire to repeat Riabouchinsky's work, which I had not as yet verified, but rather to extend it to a discussion of the general configuration, of which it is such an odd and interesting example.

The two-obstacle problem has five degrees of freedom; for if AB and CD are the obstacles, we require to know the relative lengths of AB, BC, and CD, and the inclination of these lines to the direction of flow at infinity. The problem of discontinuous motion is *a priori* easier than that of continuous motion, for, in the former, the region occupied by moving fluid *may* be simply connected, but, in the latter, it cannot be so, and, if it is not, conformal representation on a half-plane is out of the

FIG. 1. Synopsis of the Cases.



question.* We must begin with some hypothesis about the general type of motion; and we can test our hypothesis, and reject many geometrically conceivable configurations as physically impossible, by applying a principle

* The solution of the problem of continuous flow can be obtained if we know how to represent the w-plane and the z-plane separately on a standard region, *e.g.* a ring bounded by concentric circles. This problem has been solved by Mr. Hodgkinson and myself, in a paper accepted for publication by the Society. [Note added January 23rd, 1924.]

due to Brillouin,* which asserts that the dead space where the pressure is an absolute minimum must be bounded by a convex curve. For since the pressure increases as we move along the normal to the boundary into the stream, the normal acceleration of the particles in the free surface is directed towards the dead space, and the centre of curvature of the free surface must therefore lie towards the dead space. The present paper discusses four different configurations, which do not cover the case of general position, though one of them might be expected to do so.

Referring to Fig. 1, we shall examine the following cases :—

I. *One finite simply connected dead space.*—This is essentially Riabouchinsky's problem. The boundary of the dead space is convex, with one obstacle in front and the other behind, the edges being smoothly connected by free stream-lines. The branch points where the stream separates and reunites must lie against the obstacles, and not on the free surfaces, because the velocity must be zero at the branch points, and is constant on the free surfaces.

II. *One infinite simply connected wake.*—The boundary is convex. There is a branch point on one obstacle and the other is interpolated, without any discontinuity in the direction of flow, in one of the free surfaces. We must have a point of minimum velocity on this obstacle, as the velocity in the free stream-line is not maintained along the rigid wall.

III. *One infinite wake not simply connected.*—The circumstances contemplated here occur when the stream breaks on the first obstacle, falls in two cascades on to the second, and then flows off to infinity on either side of an infinite wake. The free surfaces forming the cascades have points of inflexion, so that the forward compartment of dead water must be at a pressure above the minimum. The infinite part of the wake is the region of minimum pressure and has a convex outline.

We need not discuss the trivial case where one obstacle lies completely sheltered from the stream in the wake of the other, as may well occur in practice.

IV. *Two infinite wakes.*—If each obstacle has an infinite wake, it can be proved that both are at the same pressure, which is a minimum, and therefore both are convex.

The solutions of Cases II and III are readily obtained by the standard

* *Annales de Physique et de Chimie* (1911).

method, and have each three degrees of freedom, so that neither of them can be stable. As Case I had apparently been solved by Riabouchinsky, my attention was mainly directed to Case IV. An interesting novelty was the overlapping Ω-region, necessitating the use of a Riemann surface. The fundamental polygon has external angles whose sum is eight right angles, a case not contemplated in the ordinary discussions of the Schwarz-Christoffel method.* It is possible to find a solution nominally agreeing with the ordinary formula, but this is a mere degenerate case of a uniform stream, with two thin obstacles lying along the stream-lines. I was unable to elucidate the whole of the physical circumstances or to make any progress with the analytical problem, and referred the question to Prof. Love, who took the most active interest in it. He devised the method of an indirect attack by way of the case of symmetry, established the existence of a point of stationary velocity corresponding to the winding point of the Riemann surface, and worked out the solution completely. I have reproduced his results here without any further modification than the use of elliptic function notation.

Nevertheless, even this success left two things to be desired. The method was essentially empirical, but a valuable note by Mr. Littlewood has helped to remove this reproach. The other objection was that the solution has only four degrees of freedom, so that either the solution is analytically imperfect, or the physical hypothesis does not fit the general case. There are good reasons for thinking that the latter is the true explanation; it is essential for the existence of the configuration that the jet issuing from between the obstacles should have asymptotes parallel to the stream at infinity, but by altering the obstacles, we can presumably control the direction of the jet, so that the restriction is an essential one. From this we get a hint as to what we may expect in general.

The examination of Case I was undertaken last, merely out of curiosity to see the results of Riabouchinsky's work expressed in more ordinary language. But the discovery of errors in his work made it desirable to repeat his investigation, and Prof. Love's method was clearly applicable. With his valuable collaboration, I worked out the details of this case, finding a solution with two degrees of freedom, in which the obstacles must be equal and parallel, besides verifying another condition which I can only express analytically. The fundamental Ω-polygon is drawn on a two-sheeted surface, whose winding point corresponds to the point at infinity. Prof. Love also suggested that the result might be tested by taking the corresponding symmetrical problem in a channel of finite breadth.

* Cf. Picard, *Traité d'Analyse*, t. 2, p. 313.

While adhering to the notations most familiar to British writers, I have found myself embarrassed by the superfluous constants of the Schwarz-Christoffel formula. I have therefore systematically eliminated them by the introduction of elliptic functions, following in this respect the example of Villat, in his discussion of the problem of the wedge pointing upstream. The result seems justified by the comparative simplicity of the formulæ, which express the angles of the figure, in most of the problems.

Since none of these four cases admits five degrees of freedom, we are left to conjecture what is the configuration of the general case. Remembering that problems in discontinuous motion may be indeterminate,* we might try to elaborate some of the above cases. Thus in Case I, we might introduce a triangular spit of dead water, with one straight line and two concave curved sides meeting in three cusps, at one or both of the places where there is a branch point. The velocity along these free stream-lines would be less than the maximum, and the effect on the Ω-diagrams would be that the rectangles would not be open to the right, but would be closed by an additional side. Every such complication, however, would imply further conditions of alignment, etc., and it is probable that, if we could complete the solution, the final number of degrees of freedom would not be much increased. In Case III, we might make the forward compartment a convex region, as in Case I, and the region extending to infinity might have concave boundaries and form a thin column of dead-water sheltered behind the second obstacle. But none of these solutions and others of a like nature seem very convincing. It is more reasonable to build on the foundation of our best solution, that of Case IV, and to inquire what will happen when the condition imposed on the jet is not satisfied. The direction of the jet making a finite angle with that of the infinite stream, it will inevitably interfere with the outer free surface of one of the wakes. One obstacle will have a wake extending to infinity, with a convex boundary. The other will have a finite dead-water behind it, at a higher pressure, with inflexions in the free stream-lines ; these will leave the edges of that obstacle and meet in a cusp behind it. The case of two separate and finite dead spaces of this type is excluded by Brillouin's principle, for one of them must be convex, and this will require both the obstacles to maintain its shape and to take the pressure at the branch points. The area occupied by the fluid is now doubly connected, so that conformal representation on a half-plane is impossible, and a new and more serious difficulty is added to the problem, the solution of which must be left to the future.† '

* Villat, *Aperçus théoriques sur la Résistance des Fluides*, Ch. 9, Gauthier Villars (1920).

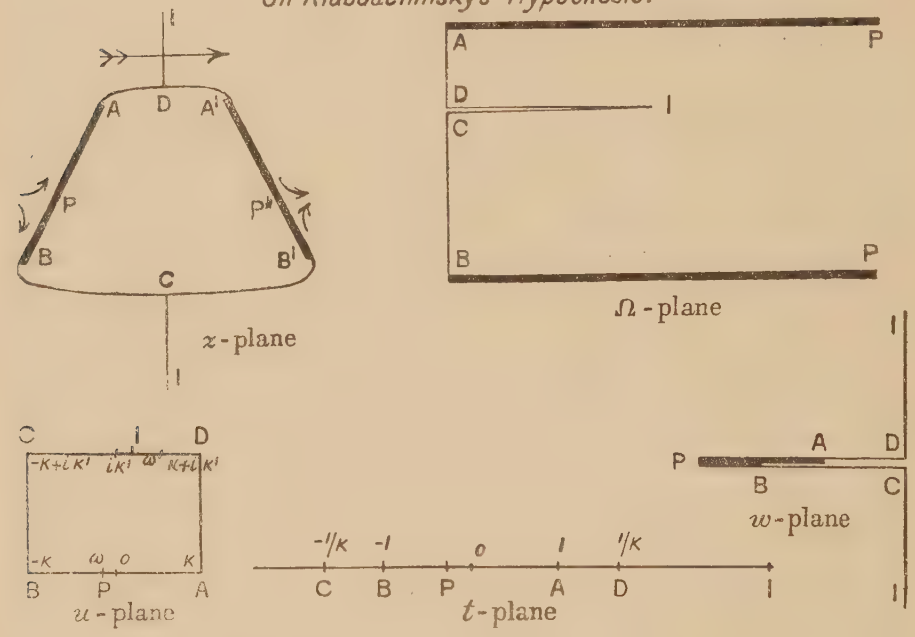
† I am at present verifying what I believe may be the solution of this case. [Note added January 23rd, 1924.]

2. Case I (i). *Criticism of Riabouchinsky's Solution.*

Accepting as valid Riabouchinsky's solution of the case of complete symmetry, we shall examine his solution of the anti-symmetrical case. The two obstacles AB, $A'B'$ are images of one another in a line perpendicular to the direction of the stream at infinity (see Fig. 2). The edges

FIG. 2. CASE I.

On Riabouchinsky's Hypothesis.



AA', BB' are connected by free stream-lines which, together with the obstacles themselves, enclose a space of dead-water. This is a region of minimum pressure, and the velocity along the stream lines is greater than that of the stream at infinity. The point at infinity is an ordinary point, and (by inversion or otherwise) it can be seen that the region occupied by the moving liquid is simply connected.

We write $z = x+iy$ where (x, y) are rectangular coordinates in the plane of the motion. The velocity potential ϕ and stream-function ψ are conjugate functions, so that $w = \phi+i\psi$ is an analytic function of z. The line of symmetry is the axis of y. The liquid, flowing at infinity along the positive direction of the axis of x, impinges on the obstacle AB and a particular stream-line, say $\psi = 0$, has a branch point at P on AB. The moving liquid leaps the gaps AA', BB' along free stream-lines where the velocity remains constant and equal to unity (say). The divided stream reunites symmetrically at the branch point P' on $A'B'$. Now if

the free stream-lines AA', BB' cut Oy at C and D respectively, then the value of ϕ at C or D is clearly (by symmetry) half the sum of its values at P and P'. Hence the portions of Oy lying in the moving liquid, which are cut orthogonally (again by symmetry) by every stream-line, must form part of the same equipotential $\phi = 0$ (say). Apart from several analytical errors, the neglect of this condition totally vitiates Riabouchinsky's generalized solution. Starting with his hypothesis, we shall prove that the case necessarily reduces to that of complete symmetry about both Ox and Oy.

Taking the standard notation, we write q for the magnitude and θ for the direction of the velocity at any point. Putting Ω for the logarithm of dz/dw, we have $\Omega = -\log q + i\theta$. Along a free stream-line q is constant, and along a rigid boundary θ is constant, so that the Ω-region has straight boundaries parallel to the axes. This reduces the problem to one of conformal representation. In the present problem we may confine our attention to the half-planes in which x and ϕ are negative. The regions in the different planes are shown in Fig. 2. The Ω- and w-regions are simply connected, and we apply the Schwarz-Christoffel formula to represent each on a half-plane of s (say). We have for w

$$\frac{dw}{ds} = \text{const.} \times \frac{s-p'}{\sqrt{\{(s-c')(s-d')\}}} \tag{1}$$

where the values $s = p'$, c', d', ∞ correspond to P, C, D, and I (at infinity) in the w-plane. This is an elementary integral, but a further condition to be imposed is that w should vanish at *both* C and D. This requires

$$\int_{c'}^{d'} \frac{(s-p')ds}{\sqrt{\{(s-c')(s-d')\}}} = 0, \tag{2}$$

and therefore

$$2p' = c' + d' \tag{3}$$

$$w = \text{const.} \times \sqrt{\{(s-c')(s-d')\}}. \tag{4}$$

Now by a homographic transformation of the s-plane into the plane of t, we can make A, B, C, D correspond to the values $t = 1$, -1, $-1/k$, $1/k$ respectively, where $0 < k < 1$, k being chosen to give the correct anharmonic ratio. Let p, j be the values of t corresponding to P and I. Then we find

$$w = \text{const.} \times \sqrt{\{(1-k^2t^2)\}}/(t-j). \tag{5}$$

The value of p is found by considering that (CD, PJ) is a harmonic range before and after the homographic transformation. Hence

$$k^2pj = 1. \tag{6}$$

Further, since P lies on AB and I outside CD on the t-plane, we have $p^2 < 1$, $k^2 j^2 > 1$, and from (6)

$$k^4 j^2 > 1. \tag{7}$$

The formula for Ω may now be written down, and is

$$\frac{d\Omega}{dt} = \frac{A(t-j)}{(k^2 jt-1)\sqrt{\{(1-t^2)(1-k^2 t^2)\}}}. \tag{8}$$

Putting $t = \operatorname{sn}(u, k)$, the half-plane of t is represented on a rectangle $ABCD$, whose corners are at $u = K, -K, -K+iK', K+iK'$ respectively

Now
$$\frac{d\Omega}{du} = \frac{A(\operatorname{sn} u - \operatorname{sn} \varpi')}{(\operatorname{sn} u - \operatorname{sn} \varpi)} \tag{9}$$

where $p = \operatorname{sn} \varpi$, $j = \operatorname{sn} \varpi'$. We see that ϖ is real and numerically less than K, while ϖ' has a real part numerically less than K and an imaginary part equal to iK'. Now from (6) we have

$$k^2 \operatorname{sn} \varpi \operatorname{sn} \varpi' = 1. \tag{10}$$

The factor A is next found by observing that the imaginary part of Ω increases by $i\pi$ as we pass over P in the sense BA; Ω must tend asymptotically to $-(\log u - \varpi)$, so that the residue at $u = \varpi$ of $(d\Omega)/(du)$ is -1. From (9), (10) we have

$$\frac{d\Omega}{du} = \frac{k^2 \operatorname{cn} \varpi \operatorname{sn} \varpi}{\operatorname{dn} \varpi} \cdot \frac{\operatorname{sn} u - \operatorname{sn} \varpi'}{\operatorname{sn} u - \operatorname{sn} \varpi} \tag{11}$$

$$= \frac{\operatorname{cn} \varpi \operatorname{dn} \varpi}{\operatorname{sn} \varpi - \operatorname{sn} u} + \frac{\operatorname{cn} \varpi \operatorname{dn} \varpi}{\operatorname{sn} \varpi' - \operatorname{sn} \varpi} \tag{12}$$

$$= \frac{\operatorname{cn} \varpi \operatorname{dn} \varpi}{\operatorname{sn} \varpi - \operatorname{sn} u} + \frac{k^2 \operatorname{cn} \varpi \operatorname{sn} \varpi}{\operatorname{du} \varpi}. \tag{13}$$

We now express the condition $\Omega(C) = \Omega(D)$. Substituting

$$\operatorname{sn}(v+iK') = 1/k \operatorname{sn} v,$$

and integrating (11) from $u = -K+iK'$ to $u = K+iK'$, we get [using (11)]:

$$\begin{aligned} 0 &= \frac{k^2 \operatorname{cn} \varpi \operatorname{sn} \varpi}{\operatorname{dn} \varpi} \int_{-K}^{K} \frac{1 - k \operatorname{sn} u \operatorname{sn} \varpi'}{1 - k \operatorname{sn} u \operatorname{sn} \varpi} du \\ &= \frac{2k^2 \operatorname{cn} \varpi \operatorname{sn} \varpi}{\operatorname{dn} \varpi} \int_{0}^{K} \frac{1 - k^2 \operatorname{sn}^2 u \operatorname{sn} \varpi \operatorname{sn} \varpi'}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 \varpi} du \\ &= \frac{2k^2 \operatorname{cn} \varpi \operatorname{sn} \varpi}{\operatorname{dn} \varpi} \int_{0}^{K} \frac{\operatorname{cn}^2 u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 \varpi} du, \end{aligned} \tag{14}$$

by using (10).

The integral being essentially positive, and $k^2 \neq 0$, we must have $\operatorname{sn}\varpi \operatorname{cn}\varpi = 0$. If $\operatorname{sn}\varpi = 0$, we get the case of complete symmetry, correctly solved in Riabouchinsky's first paper. If $\operatorname{cn}\varpi = 0$, $\Omega \equiv 0$, and the branch point P lies at the extremity of the obstacle AB. Hence both obstacles are parallel to Ox, *i.e.* no disturbance is produced in the stream by the two infinitely thin laminas placed parallel to it.

3. Case I (ii).—*Generalization of the symmetrical case.*

The symmetrical case is given by making $\varpi \to 0$, $\varpi' \to iK'$, so as to verify $k^2 \operatorname{sn}^2\varpi \operatorname{sn}^2\varpi' = 1$. The integrals are elementary and we restore the variable $t = \operatorname{sn} u$. There is no loss of generality in making w equal to -1 at P and $+1$ at P'. Thus

$$w = -\sqrt{(1-k^2t^2)}, \qquad (d\Omega)/(dt) = -1/t\sqrt{\{(1-t^2)(1-k^2t^2)\}}. \tag{15}$$

This is equivalent to

$$t = (1+k'^2\sinh^2\alpha)^{-\frac{1}{2}}, \qquad w = -k'(1-k^2\tanh^2\alpha)^{-\frac{1}{2}}, \qquad \Omega = \alpha + \tfrac{1}{2}\pi i. \tag{16}$$

At A we have $t = 1$, $\alpha = 0$, and $\Omega = \frac{1}{2}\pi i$. The elimination of α or t gives

$$\frac{d\Omega}{dw} = \frac{k}{(w^2-1)\sqrt{(w^2-k'^2)}} \tag{17}$$

or

$$k^2\coth^2\Omega + (k'^2/w^2) = 1. \tag{18}$$

Hence

$$\frac{dz}{dw} = e^{\Omega} = \frac{kw+\sqrt{(w^2-k'^2)}}{k'\sqrt{(w^2-1)}}. \tag{19}$$

The solution can be completed by a quadrature, depending on an elliptic integral. We need not complete this investigation, which would merely repeat Riabouchinsky's first paper; but we shall try to obtain a generalization by transforming the entire region outside the dead-water on to a half-plane. This is theoretically possible, since this region is simply connected, as we can see by projection on to a Neumann sphere, where the point at infinity is an ordinary point. We must make a homographic transformation of the w-plane, so that the two edges of the cut from -1 to $+1$ become the two edges of a semi-infinite cut along the positive real axis of a new plane of ξ, taking care to choose the signs so that the upper half-planes correspond to one another. By then putting $\xi = \eta^2$, our region is represented on the upper half-plane of η. We shall have

$$\eta^2 = \xi = \frac{1+w}{1-w}, \qquad w = \frac{\xi-1}{\xi+1} = \frac{\eta^2-1}{\eta^2+1}, \tag{20}$$

$$\frac{d\Omega}{d\eta} = \frac{-(1+\eta^2)}{\eta\sqrt{\{(\eta^2-l)(\eta^2-l^{-1})\}}}, \tag{21}$$

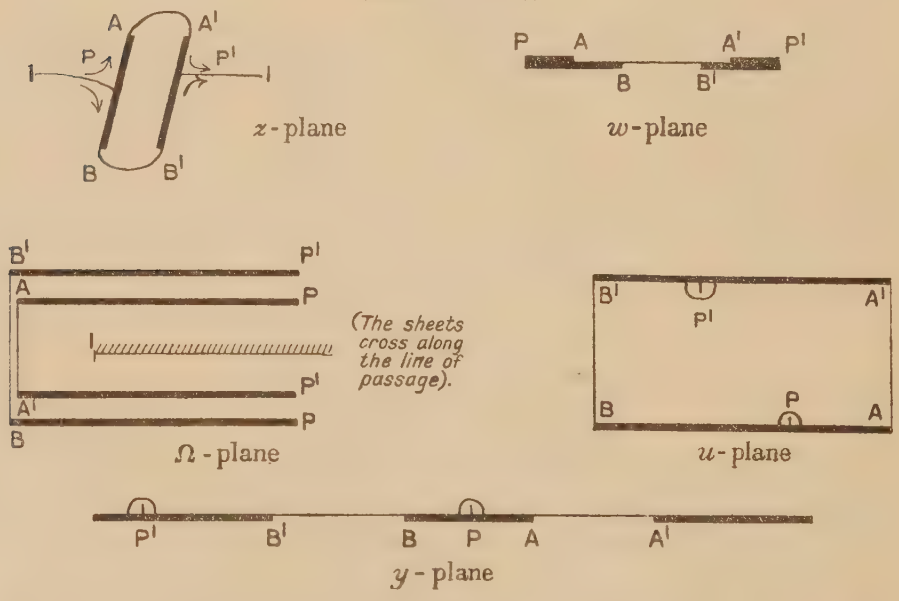
where $$l = \frac{1-k'}{1+k'}, \quad k' = \frac{1-l}{1+l}, \quad k = \frac{2\sqrt{l}}{1+l}, \quad l' = \frac{2\sqrt{k'}}{1+k'}, \tag{22}$$

We recognize the formula of Landen's transformation.

From Fig. 3, we see that the Ω-region is a double open rectangle. The sides AA', BB' must lie along the imaginary axis and the distances AB, $A'B'$ are equal to π, so that the outline of the figure is symmetrical. The numerator of (21) vanishes only at the point $\eta = i$, in the upper half-plane. This corresponds to an infinite value of w and of z, so that

FIG. 3. CASE I.

Generalization.



the velocity is stationary in magnitude and direction at infinity. The four factors under the radical in the denominator of (21) correspond to the right angles at A, B, C, D in the Ω-plane.

Since the form of (21) is invariant for homographic transformations of η, by the theory of conformal representation, we must regard the "η" in the denominator as a degenerate quadratic, with one root infinite. Another quadratic with conjugate imaginary roots appears both in the denominator of w in (20) and in the numerator of $(d\Omega)/(d\eta)$ in (21), and we saw that the root in the upper half-plane of η corresponds to the point at infinity. The numerator of w is a quadratic with real roots, and our three quadratics are so related that the one in the denominator in (21) is the Jacobian of the numerator and denominator in (20). This is because the branch points,

where the expression (21) becomes infinite, lie at the ends of the cut in the w-plane, where w attains its greatest and least values for points on the boundary, *i.e.* for real values of η. Geometrically, the branch points are given as the double points of an involution, two pairs of which are given by the quadratics in the numerator and denominator of w. By expressing the Jacobian as a product of two linear factors, we see that the other quadratics are sums of multiples of the squares of these factors.

It is obvious the latter property, of the three related quadratics, must be retained in any generalization; but it is not so clear that the numerator of $(d\Omega)/(d\eta)$ must always be the same as the denominator of w. Now, at infinite distances, we shall always have

$$z = aw + b + \frac{c}{w} + \frac{d}{w^2} + \dots,$$

$$\frac{dz}{dw} = a - \frac{c}{w^2} - \frac{2d}{w^3} + \dots,$$

$$\Omega = \alpha + \frac{\gamma}{w^2} + \frac{\delta}{w^3} + \dots,$$

$$\frac{d\Omega}{d\eta} = \frac{dw}{d\eta} O\left(\frac{1}{w^3}\right) = \frac{d}{d\eta}\left(\frac{1}{w}\right) O\left(\frac{1}{w}\right).$$

Now $1/w$ is holomorphic at the value of η corresponding to infinite values of z and w, and therefore $(d\Omega)/(d\eta)$ must vanish there. We shall therefore postulate that this property is to be retained in any generalized example of the configuration.

Let us now consider the problem of the conformal representation of the Ω-diagram on a half-plane. We see that over part of the figure every value of Ω in the overlapping figure corresponds to two distinct points in the fluid. We must therefore regard the figure as drawn on a two-sheeted Riemann surface, on which it is simply connected. Since we can pass continuously from any point to any other point by the analytical continuation of a single function, there must be a branch-cut, ending at a winding-point I where the two values of η, corresponding to a given Ω, become equal. The point at infinity, PP', in the Ω-plane, lying on the boundary of the Ω-region in each sheet, may conveniently be taken as the other terminus of the branch-cut.

Since it is not true that the interior of the η- and Ω-regions are free from singular points, the conditions for applying the Schwarz-Christoffel method are not satisfied, and we must introduce a modification.* Con-

* I here follow Mr. Littlewood's note.

sider a polygon the sum of whose external angles is $2(\nu+1)\pi$ instead of 2π, which we may suppose to be drawn on an overlapping $(\nu+1)$-sheeted Riemann surface, with ν branch points. We take

$$\frac{dZ}{dz} = \int P(z)(z-a)^{\alpha-1}(z-b)^{\beta-1}\ldots dz$$

where $$P(z) = \lambda_0+\lambda_1 z+\ldots+\lambda_{2\nu} z^{2\nu}$$

and $\alpha\pi$, $\beta\pi$, ... are the internal angles of the polygon. The constants λ_r are all real, and the roots of $P(z)$ must all be imaginary and conjugate in pairs, so that ν of them lie in each half-plane. The affixes of these roots each give two real constants, which are the coordinates of the branch points. We have also the relation $\Sigma(1-\alpha) = 2(\nu+1)$. The formula now contains enough constants to determine the polygon and to assign the positions of the branch points.

The actual solution was obtained by following step by step the method of Prof. Love in Case IV, which was solved first, by generalizing the case of complete symmetry. We have in (20) and (21) a biquadratic, two independent quadratics, and the *Jacobian* of the latter. We keep the former in the canonical Legendre form, but make a slight *change of notation*, putting K, k instead of Landen's L, l. By the introduction of trivial constants, we can make $w = -1$ at P and $w = +1$ at P'. Using a variable y equivalent to $\eta/\sqrt{l}$, we have

$$w = \frac{-A(y-p)^2+A'(y-p')^2}{A(y-p)^2+A'(y-p')^2}, \quad \frac{d\Omega}{dy} = \frac{A(y-p)^2+A'(y-p')^2}{(y-p)(y-p')\sqrt{\{(1-y^2)(1-k^2y^2)\}}}. \tag{23}$$

The constants A, A' are real and have the same sign; $p^2 < 1$ because P is on AB, $k^2p'^2 > 1$ because P' is on $A'B'$. Put $y = \operatorname{sn}(u, k)$, $p = \operatorname{sn}\varpi$, $p' = \operatorname{sn}\varpi'$, we have ϖ and $(\varpi'-iK')$ real and numerically $\leqslant K$. Now consider the behaviour of Ω at P and P'. Describing the boundary in the positive sense, with a small indentation to exclude these poles, the imaginary part of Ω increases by $i\pi$. Hence Ω is asymptotically equivalent to $-\log(y-p) \sim -\log(u-\varpi)$ near P and so the residue of $(d\Omega)/(dy)$ or $(d\Omega)/(du)$ at P (and similarly at P') is -1. This fixes the values of A and A', and we find:

$$\frac{d\Omega}{du} = \frac{\operatorname{cn}\varpi\operatorname{dn}\varpi(\operatorname{sn}u-\operatorname{sn}\varpi')^2-\operatorname{cn}\varpi'\operatorname{dn}\varpi'(\operatorname{sn}u-\operatorname{sn}\varpi)^2}{(\operatorname{sn}\varpi'-\operatorname{sn}\varpi)(\operatorname{sn}\varpi-\operatorname{sn}u)(\operatorname{sn}u-\operatorname{sn}\varpi')}$$

$$= -\frac{\operatorname{cn}\varpi\operatorname{dn}\varpi}{(\operatorname{sn}u-\operatorname{sn}\varpi)} - \frac{\operatorname{cn}\varpi'\operatorname{dn}\varpi'}{(\operatorname{sn}u-\operatorname{sn}\varpi')} - \frac{(\operatorname{cn}\varpi\operatorname{dn}\varpi-\operatorname{cn}\varpi'\operatorname{dn}\varpi')}{(\operatorname{sn}\varpi-\operatorname{sn}\varpi')}. \tag{24}$$

Hence $$\Omega = -Pu - \log QR + \text{const.}, \tag{25}$$

where $$P = \frac{(\operatorname{cn}\varpi \operatorname{dn}\varpi - \operatorname{cn}\varpi' \operatorname{dn}\varpi')}{(\operatorname{sn}\varpi - \operatorname{sn}\varpi')} + Z(\varpi) + Z(\varpi') \tag{26}$$

$$= Z(\varpi+\varpi'-iK') - k \operatorname{sn}(\varpi+\varpi'-iK') - (i\pi)/(2K), \tag{27}$$

$$Q = \Theta^2(u) \div H(u+\varpi)\, H(u+\varpi'), \tag{28}$$

$$R = (\operatorname{cn} u \operatorname{dn}\varpi - \operatorname{dn} u \operatorname{cn}\varpi)(\operatorname{cn} u \operatorname{dn}\varpi' - \operatorname{dn} u \operatorname{cn}\varpi'). \tag{29}$$

Since the velocities along AA' and BB' are equal, the real part of Ω takes equal values at $u = \pm K$; and because $\log QR$ satisfies this condition separately, it follows that the real part of P must vanish. But we know that $(\varpi+\varpi'-iK')$ is real and numerically not greater than $2K$. Hence the value $u = \varpi+\varpi'-iK'$ must satisfy

$$f(u) \equiv Z(u) - k \operatorname{sn} u = 0. \tag{30}$$

This is an odd function of u and we will show that it has no real root, except zero, which is numerically less than $2K$. For $f(u)$ is continuous and differentiable in the interval $(0, 2K)$, and we have

$$f'(u) \equiv \operatorname{dn}^2 u - (E/K) - k \operatorname{cn} u \operatorname{dn} u, \qquad f''(u) \equiv k \operatorname{sn} u (\operatorname{dn} u - k \operatorname{cn} u)^2.$$

Since $f''(u)$ is positive in the interval, $f'(u)$ increases steadily. It is negative at $u = 0$ and $u = K$, but positive at $u = 2K$. [To verify this we observe that the mean-value theorem applied to $E(u)$ gives $k'^2 < (E/K)$.] Since $f'(u)$ vanishes once only in $(0, 2K)$, we see by the converse of Rolle's Theorem that $f(u)$ cannot have another real root between $u = 0$ and $u = 2K$.

Neglecting the degenerate cases $\varpi = \pm K$, $\varpi' = \pm K + iK'$, for which the branch points lie at the ends of the obstacles, which will lie parallel to the stream, we must have always

$$\varpi+\varpi'-iK' = 0,$$

which we take as the fundamental relation.
Now

$$H(u+\varpi)\, H(u-\varpi+iK')$$

$$= i\sqrt{k} \operatorname{sn}(u+\varpi)\, \Theta(u+\varpi)\, \Theta(u-\varpi) \exp\left[\frac{\pi K'}{4K} - \frac{\pi i(u-\varpi)}{2K}\right],$$

$$\Theta(u+\varpi)\, \Theta(u-\varpi)\, \Theta^2(0) = \Theta^2(u)\, \Theta^2(\varpi)(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 \varpi).$$

Hence

$$Q = (\operatorname{sn} u \operatorname{cn} \varpi \operatorname{dn} \varpi + \operatorname{sn} \varpi \operatorname{cn} u \operatorname{dn} u)^{-1} e^{(i\pi u)/(2K)} \times \text{const.}, \tag{32}$$

$$R = i(\operatorname{dn} u \operatorname{cn} \varpi - \operatorname{cn} u \operatorname{dn} \varpi)(\operatorname{dn} u \operatorname{dn} \varpi - k \operatorname{cn} u \operatorname{cn} \varpi)/k \operatorname{sn} \varpi. \tag{33}$$

The exponential in Q is cancelled by the imaginary part of P leaving

$$\Omega = \log \frac{(\operatorname{sn} u \operatorname{cn} \varpi \operatorname{dn} \varpi + \operatorname{sn} \varpi \operatorname{cn} u \operatorname{dn} u)}{(\operatorname{dn} u \operatorname{cn} \varpi - \operatorname{cn} u \operatorname{dn} \varpi)(\operatorname{dn} u \operatorname{dn} \varpi - k \operatorname{cn} u \operatorname{cn} \varpi)} + M + iN, \tag{34}$$

where M, N are constants as yet undetermined.

The corresponding form of w given by (23) is

$$w = \frac{(1+k \operatorname{sn} u \operatorname{sn} \varpi)^2 - k(\operatorname{sn} u - \operatorname{sn} \varpi)^2}{(1+k \operatorname{sn} u \operatorname{sn} \varpi)^2 + k(\operatorname{sn} u - \operatorname{sn} \varpi)^2}$$

$$= \frac{(1-k \operatorname{sn}^2 u)(1-k \operatorname{sn}^2 \varpi) + 4k \operatorname{sn} u \operatorname{sn} \varpi}{(1+k \operatorname{sn}^2 u)(1+k \operatorname{sn}^2 \varpi)} \tag{35}$$

The point at infinity in the w plane corresponds to $1+k \operatorname{sn}^2 u = 0$, which has a solution independent of ϖ. The only admissible root within the rectangle is $u = \frac{1}{2}iK'$. The values of M, N can be found by making the *real* part of Ω vanish along the free stream-lines, where the velocity attains its maximum value of unity, and by making its *imaginary* part vanish at infinity, *i.e.* when $u = \frac{1}{2}iK'$, because the stream is then parallel to Ox. Thus

$$M = 2 \log k' \tag{36}$$

and [since $\operatorname{sn} \frac{1}{2}iK' = i/\sqrt{k}$, $\operatorname{cn} \frac{1}{2}iK' = \sqrt{\{(i+k)/k\}}$, $\operatorname{dn} \frac{1}{2}iK' = \sqrt{(i+k)}$] we have

$$N = -\arctan \frac{\operatorname{cn} \varpi \operatorname{dn} \varpi}{(1+k) \operatorname{sn} \varpi}. \tag{37}$$

It is of some interest to carry the quadratures a stage further, because we can easily prove that the obstacles are necessarily parallel and of equal lengths. Thus two out of the three conditions imposed on the system can be expressed in an elegant intuitive form by saying that the corners of the obstacles form a parallelogram, about the centre of which the motion is completely symmetrical. Since we are not at present interested in complicated elliptic integrals for their own sake, however, we will only carry the analysis far enough to prove these statements.

By putting $u = K$ for A, and $u = K+iK'$ for A', we find that the slopes of the obstacles are

$$\theta_A = N + I \log (\operatorname{cn} \varpi \operatorname{dn} \varpi / k'^2 \operatorname{cn} \varpi \operatorname{dn} \varpi)$$

$$\theta_A = N + I \log \left(k^{-1} \operatorname{cn} \varpi \operatorname{dn} \varpi / k \operatorname{cn} \varpi \operatorname{dn} \varpi \operatorname{cn}^2 (K+iK') \right)$$

(where Ix means the imaginary part of x). Hence

$$\theta_{A'}-\theta_A = -I\log \mathrm{cn}^2(K+iK') = \pm\pi. \tag{39}$$

This proves the statement that the obstacles are parallel.
Again

$$\frac{dz}{dw} = e^{\Omega} = k'^2 e^{iN}\frac{(\mathrm{sn}\,u\,\mathrm{cn}\,\varpi\,\mathrm{dn}\,\varpi+\mathrm{sn}\,\varpi\,\mathrm{cn}\,u\,\mathrm{dn}\,u)}{(\mathrm{dn}\,u\,\mathrm{cn}\,\varpi-\mathrm{cn}\,u\,\mathrm{dn}\,\varpi)(\mathrm{dn}\,u\,\mathrm{dn}\,\varpi-k\,\mathrm{cn}\,u\,\mathrm{cn}\,\varpi)}. \tag{40}$$

Calling this $k'^2e^{iN}F(u)$, we have

$$F(iK'-u) = \frac{\left(-\dfrac{\mathrm{cn}\,\varpi\,\mathrm{dn}\,\varpi}{k\,\mathrm{sn}\,u}-\dfrac{\mathrm{sn}\,\varpi\,\mathrm{cn}\,u\,\mathrm{dn}\,u}{k\,\mathrm{sn}^2u}\right)}{\left(\dfrac{i\,\mathrm{cn}\,u\,\mathrm{cn}\,\varpi}{\mathrm{sn}\,u}-\dfrac{i\,\mathrm{dn}\,u\,\mathrm{dn}\,\varpi}{k\,\mathrm{sn}\,u}\right)\left(\dfrac{i\,\mathrm{cn}\,u\,\mathrm{dn}\,\varpi}{\mathrm{sn}\,u}-\dfrac{ik\,\mathrm{dn}\,u\,\mathrm{cn}\,\varpi}{k\,\mathrm{sn}\,u}\right)}$$

$$= F(u). \tag{41}$$

Again, $w=\Phi(u)$, where $\Phi(u)$ is defined by (35), and we have

$$\Phi(iK'-u) = \frac{\left(1-\dfrac{\mathrm{sn}\,\varpi}{\mathrm{sn}\,u}\right)^2-k\left(\dfrac{-1}{k\,\mathrm{sn}\,u}-\mathrm{sn}\,\varpi\right)^2}{\left(1-\dfrac{\mathrm{sn}\,\varpi}{\mathrm{sn}\,u}\right)^2+k\left(\dfrac{-1}{k\,\mathrm{sn}\,u}-\mathrm{sn}\,\varpi\right)^2} = -\Phi(u). \tag{42}$$

Now the lengths of the laminas are

$$l_1 = \int_{-K}^{K}\left|\frac{dz}{dw}\right|.\left|\frac{dw}{du}\right|du, \qquad l_2 = \int_{iK'-K}^{iK'+K}\left|\frac{dz}{dw}\right|.\left|\frac{dw}{du}\right|du. \tag{43}$$

Thus

$$l_2 = k'^2\int_{iK'-K}^{iK'+K}|F(u)\Phi'(u)|\,du$$

$$= k'^2\int_{-K}^{K}|F(iK'-u)\Phi'(iK'-u)|\,du$$

$$= k'^2\int_{-K}^{K}|F(u)\Phi'(u)|\,du = l_1. \tag{44}$$

This proves the statement that the laminas are equal.

Since there are only two parameters (k^2 and ϖ) at our disposal, a third restrictive condition is contained implicitly in our formulae, but I have not found any simple geometrical interpretation of it.

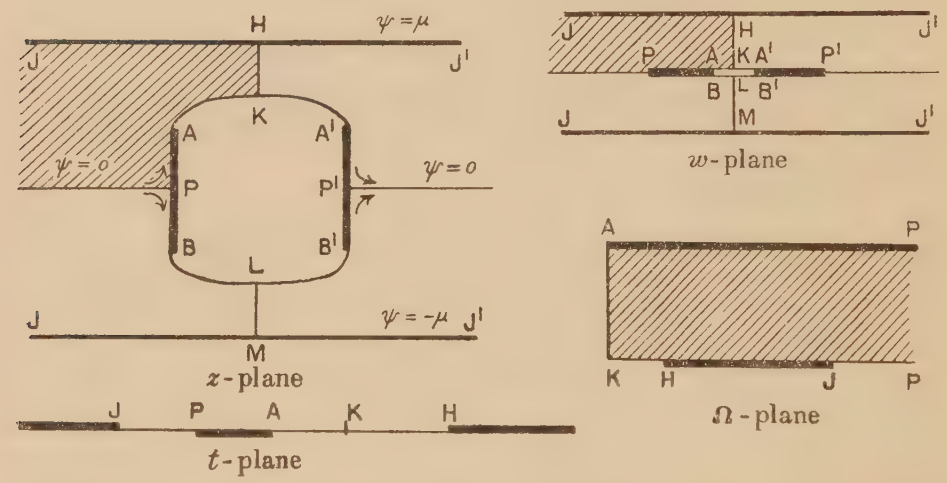
4. CASE I (iii). *Generalization for a Channel of Finite Breadth.*

In order to test and if possible to generalize the results of the last section, Prof. Love suggested that it would be of interest to solve the

corresponding symmetrical problem for a channel of finite breadth. The result did not throw any fresh light on the previous problem, but the investigation is so compact that a digression from the problem of the paper seems permissible in order to include it. Referring to Fig. 4, we see that

FIG. 4. CASE I.

Obstacles in a Channel of Finite Breadth.



only one quarter of the figure need be considered. Transforming to the half-plane of t, we have

$$\frac{d\Omega}{dt} = \frac{A}{(t-p)\sqrt{\{(t-a)(t-k)\}}}, \qquad \frac{dw}{dt} = \frac{B}{(t-j)\sqrt{\{(t-h)(t-k)\}}} \tag{45}$$

In the former, let t be transformed homographically so that the values $t = p, a, k$ correspond to $\xi = \infty, -1, 1$ respectively; in the latter, let t be transformed homographically so that $t = j, k, h$ correspond to $\eta = \infty, -1, 1$ respectively. We have

$$\frac{\xi+1}{\xi-1} = \frac{t-a}{t-k}\cdot\frac{p-k}{p-a}, \qquad \frac{\eta+1}{\eta-1} = \frac{t-k}{t-h}\cdot\frac{j-h}{j-k}, \tag{46}$$

$$\frac{d\Omega}{d\xi} = \frac{A'}{\sqrt{(\xi^2-1)}}, \qquad \frac{dw}{d\eta} = \frac{B'}{\sqrt{(\eta^2-1)}}, \tag{47}$$

$$\xi = \cosh(A''\Omega + C), \qquad \eta = \cosh(B''w + D), \tag{48}$$

where A, B, A', B', A'', B'', C, D are constants.

Now ξ is real when Ω is real or pure imaginary, and ξ varies from -1 to 1 as Ω varies from $\frac{1}{2}\pi i$ to zero. Hence

$$\xi = \cosh 2\Omega, \tag{49}$$

and η varies from -1 to 1 as w varies from zero to μi. Hence

$$\eta = -\cosh \pi w/\mu. \tag{50}$$

From (46)
$$\frac{t-a}{t-k} = \frac{p-a}{p-k} \coth^2 \Omega \tag{51}$$

and
$$\frac{t-h}{t-k} = \frac{j-h}{j-k} \coth^2 \frac{\pi w}{2\mu}. \tag{52}$$

Eliminating t, we find

$$\frac{(k-h)(p-a)}{(p-k)} \coth^2 \Omega - \frac{(k-a)(j-h)}{(j-k)} \coth^2 \frac{\pi w}{2\mu} = (a-h). \tag{53}$$

To pass from this result to the case of the infinite stream, let $\mu \to \infty$ and $(j-k) \to 0$ in such a manner that $4\mu^2(k-a)(j-h)/(j-k)\pi^2$ remains constant and equal to unity (say). We have in the limit

$$P \coth^2 \Omega + 1/w^2 = Q, \tag{54}$$

which is essentially the same as Riabouchinsky's result.

5. CASE II.—*One Simply Connected Wake.*

The next two cases are soluble by the ordinary method. Referring to Fig. 5 we make A, B, C, D correspond to $t = -1, 1, 1/k, -1/k$ respectively. Then

$$\frac{d\Omega}{dt} = \frac{A(t-q)}{(t-p)\sqrt{\{(1-t^2)(1-k^2t^2)\}}}, \qquad w = \frac{B(t-p)^2}{(t-j)^2}, \tag{55}$$

where p corresponds to the branch-point P on AB where the stream divides, q to the point of minimum velocity on CD, and j to the point at infinity. Hence

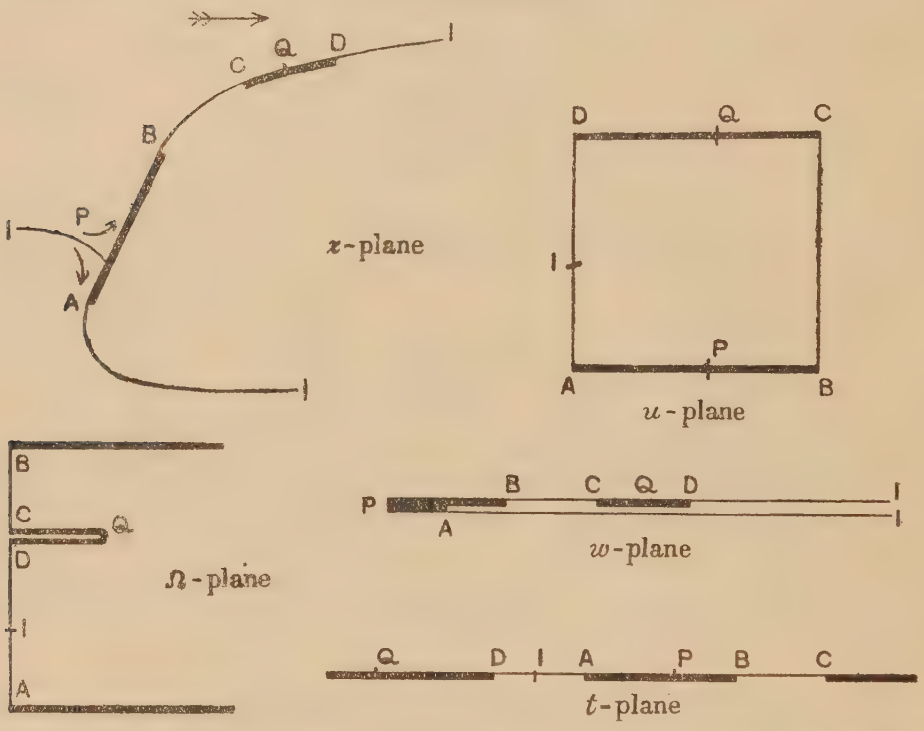
$$p^2 < 1, \qquad k^2q^2 > 1. \tag{56}$$

Putting $t = \mathrm{sn}(u, k)$, we have the half-plane of t represented on a rectangle whose corners $-K$, K, $K+iK'$, $-K+iK'$ correspond to A, B, C, D respectively. Let $j = \mathrm{sn}\,\rho$, $p = \mathrm{sn}\,\varpi$, $q = \mathrm{sn}\,\varpi'$, where ϖ and $\varpi' - iK'$ are real and numerically less than K, and where the real part of ρ is $\pm K$.

$$\frac{d\Omega}{du} = \frac{A(\mathrm{sn}\,u - \mathrm{sn}\,\varpi')}{(\mathrm{sn}\,u - \mathrm{sn}\,\varpi)}, \qquad w = \frac{B(\mathrm{sn}\,u - \mathrm{sn}\,\varpi)^2}{(\mathrm{sn}\,u - \mathrm{sn}\,\rho)^2}. \tag{57}$$

Now since the imaginary part of Ω remains constant along PA and PB, the constant A is real; and in order that the discontinuity may be $i\pi$ as

FIG. 5. CASE II.



u passes over P, we must have $\Omega \sim -\log(u-\varpi)$, so that the residue of $(d\Omega)/(du)$ is -1. Hence

$$\frac{d\Omega}{du} = -\frac{\operatorname{cn}\varpi \operatorname{dn}\varpi}{(\operatorname{sn} u - \operatorname{sn}\varpi)} + \frac{\operatorname{cn}\varpi \operatorname{dn}\varpi}{(\operatorname{sn}\varpi' - \operatorname{sn}\varpi)}. \tag{58}$$

Therefore
$$\Omega = -u\left\{Z(\varpi) + \frac{\operatorname{cn}\varpi \operatorname{dn}\varpi}{(\operatorname{sn}\varpi - \operatorname{sn}\varpi')}\right\} + \tfrac{1}{2}\log S + \text{const.} \tag{59}$$

where
$$S = \frac{H(u-\varpi)(\operatorname{cn}\varpi \operatorname{dn} u - \operatorname{cn} u \operatorname{dn}\varpi)}{H(u+\varpi)(\operatorname{cn}\varpi \operatorname{dn} u + \operatorname{cn} u \operatorname{dn}\varpi)}. \tag{60}$$

The velocities at C and D are equal in magnitude and direction, and the values of S at $\pm K + iK'$ are also equal. Hence the coefficient of u in (59) must vanish identically and we have the following equation to determine ϖ' in terms of ϖ

$$Z(\varpi) + \frac{\operatorname{cn}\varpi \operatorname{dn}\varpi}{(\operatorname{sn}\varpi - \operatorname{sn}\varpi')} = 0. \tag{61}$$

Reducing S by the addition theorem, we get

$$\Omega = \log\left\{\frac{\Theta(u)(\mathrm{cn}\,\varpi\,\mathrm{dn}\,u - \mathrm{cn}\,u\,\mathrm{dn}\,\varpi)}{H(u+\varpi)}\right\} + \text{const.} \tag{62}$$

Making $\Omega = 0$ at infinity, *i.e.* when $u = \rho$, we get

$$\frac{dz}{dw} = e^{\Omega} = \frac{\Theta(u)\,H(\rho+\varpi)(\mathrm{cn}\,\varpi\,\mathrm{dn}\,u - \mathrm{cn}\,u\,\mathrm{dn}\,\varpi)}{\Theta(\rho)\,H(u+\varpi)(\mathrm{cn}\,\varpi\,\mathrm{dn}\,\rho - \mathrm{cn}\,\rho\,\mathrm{dn}\,\varpi)}, \tag{63}$$

The parameter ϖ' has been completely eliminated by (61). There are three real parameters, k^2, ϖ, and the imaginary part of ρ. The expression (63) is not in general doubly periodic, but becomes so if $\varpi = 0$. This gives a two-parameter solution free from elliptic functions. Restoring t and j, we have

$$\frac{dz}{dw} = A\,\frac{\sqrt{(1-k^2t^2)} - \sqrt{(1-t^2)}}{t} \qquad w = B\,\frac{t^2}{(t-j)^2} \tag{64}$$

$$z = ABj\left\{\frac{-(1-jt)\sqrt{(1-t^2)}}{(1-j^2)(j-t)^2} + \frac{1}{(1-j^2)^{\frac{3}{2}}}\log\frac{(1-jt+\sqrt{(1-j^2)}\sqrt{(1-t^2)}}{(j-t)}\right.$$

$$\left. + \frac{(1-k^2jt)\sqrt{(1-k^2t^2)}}{(1-k^2j^2)(j-t)^2} - \frac{k^2}{(1-k^2j^2)^{\frac{3}{2}}}\log\frac{(1-k^2jt+\sqrt{(1-k^2j^2)}\sqrt{(1-k^2t^2)}}{(j-t)}\right\}. \tag{65}$$

Putting $t = \pm 1$, $t = \pm 1/k$, we have the corners of the obstacles. The evaluation of pressures and couples would require further integrations, but the intrinsic equation of the free stream-lines can be obtained at once from (64) since $dw = d\phi = ds$ and $\Omega = i\theta$. The inclinations of the obstacles can be found from (63) in the most general case. Putting $u = K$ for B and $u = K + iK'$ for C, we get

$$\theta_B = am\,\frac{H(\rho+\varpi)}{\Theta(\rho)(\mathrm{cn}\,\varpi\,\mathrm{dn}\,\rho - \mathrm{cn}\,\rho\,\mathrm{dn}\,\varpi)}, \tag{66}$$

$$\theta_C - \theta_B = \frac{\pi}{2}\left(1 + \frac{\varpi}{K}\right). \tag{67}$$

This gives the angle between the obstacles, and we see that the special case considered above occurs when the obstacles are at right angles to one another.

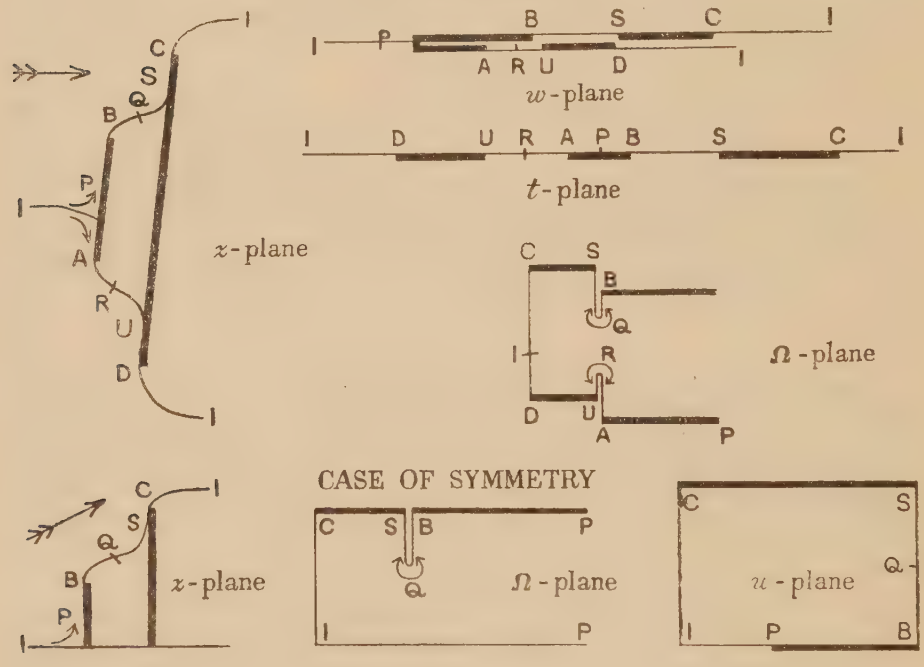
6. CASE III.—*Cascades and a Barrier across the Wake.*

Consider now the case where the stream breaks over the first obstacle and falls in two cascades on to the second. There is a dead space between the obstacles which need not be at the same pressure as the infinite wake

The velocity along the stream-lines of the finite dead space will be less than that along the boundaries of the wake, the latter having the velocity of the stream at infinity. We may therefore postulate the inflexions at Q, R without violating Brillouin's principle. We transform the figures (see Fig. 6) on to a half-plane of t, making $t = \infty$ correspond to $z = \infty$ and $w = \infty$. The formulæ are

$$\frac{d\Omega}{dt} = \frac{A(t-q)(t-r)}{(t-p)\sqrt{\{(t-a)(t-b)(t-c)(t-d)(t-s)(t-u)\}}}, \quad w = B(t-p)^2. \quad (68)$$

FIG. 6. CASE III.



Of these eleven constants, B is trivial, and three may be chosen at random; seven parameters remain, but these have to fulfil several conditions. The velocities at A and B must be equal in magnitude and opposite in direction. Hence we have $\Omega(B) = \Omega(A) + i\pi$, equivalent to two relations. Again, the velocities at C and D are also equal and opposite; but along CD, $(d\Omega)/(dt)$ (and therefore A) is purely imaginary, a fact we had already tacitly assumed in counting A as only one constant. Hence we have one relation equivalent to $\Omega(D) = \Omega(C) + i\pi$. We have reduced the number of parameters to *four*, but we have still to express that the segments SC, UD (which are necessarily parallel) shall lie in the

same straight line. This will be expressed by a condition imposed on $z = \int e^{\Omega} dw$, leaving finally only three degrees of freedom.

The integrals are hyper-elliptic, but reduce to elliptic integrals in the case of symmetry. Putting $p = 0$ and making the pairs of points AB, CD, US, QR lie symmetrically with respect to P, we can take t^2 or w as variable instead of t. Transforming the upper half-plane w-region homographically on to a new half-plane of ξ, so that B, S, C, I correspond respectively to the values 1, $1/k$, $-1/k$, -1 of ξ, and using new constants, we have

$$\frac{d\Omega}{d\xi} = \frac{A(\xi - q)}{(\xi - p)\sqrt{\{(1-\xi^2)(1-k^2\xi^2)\}}}, \qquad w = \frac{B(\xi - p)}{(\xi + 1)}. \tag{69}$$

Put $\xi = \operatorname{sn}(u, k)$, $p = \operatorname{sn} \varpi$, $q = \operatorname{sn} \varpi'$, then

$$\frac{d\Omega}{du} = \frac{A(\operatorname{sn} u \operatorname{sn} \varpi')}{(\operatorname{sn} u - \operatorname{sn} \varpi)}, \qquad w = \frac{B(\operatorname{sn} u - \operatorname{sn} \varpi)}{(\operatorname{sn} u + 1)}. \tag{70}$$

The residue of $(d\Omega)/(du)$ at $u = \varpi$ is $-\frac{1}{2}$, corresponding to the discontinuity $\frac{1}{2}\pi$ in the imaginary part of Ω; hence

$$\frac{d\Omega}{du} = -\frac{1}{2}\frac{\operatorname{cn} \varpi \operatorname{dn} \varpi}{(\operatorname{sn} u - \operatorname{sn} \varpi)} + \frac{1}{2}\frac{\operatorname{cn} \varpi \operatorname{dn} \varpi}{(\operatorname{sn} \varpi' - \operatorname{sn} \varpi)}, \tag{71}$$

$$-2\Omega = \left\{ Z(\varpi) + \frac{\operatorname{cn} \varpi \operatorname{dn} \varpi}{(\operatorname{sn} \varpi - \operatorname{sn} \varpi')} \right\} u + \log \left\{ \frac{H(u - \varpi)}{\Theta(u)(\operatorname{cn} u \operatorname{dn} \varpi + \operatorname{cn} \varpi \operatorname{dn} u)} \right\} + \text{const.} \tag{72}$$

Since Ω must take equal values at S and B, we find, on putting $u = K + iK'$ and $u = K$ and noticing that

$$\frac{H(K + iK' - \varpi)\Theta(K)}{H(K - \varpi)\Theta(K + iK')} = e^{(i\pi\varpi)/(2K)} \frac{\Theta(K - \varpi)\Theta(K)}{H(K - \varpi)H(K)} = \frac{e^{(i\pi\varpi)/(2K)}}{k \operatorname{sn}(K - \varpi)},$$

the condition

$$iK' \left\{ Z(\varpi) + \frac{\operatorname{cn} \varpi \operatorname{dn} \varpi}{(\operatorname{sn} \varpi - \operatorname{sn} \varpi')} \right\} + \frac{i\pi\varpi}{2K} + \log \left\{ \frac{\operatorname{dn} \varpi}{k \operatorname{cn} \varpi} \times \frac{\operatorname{cn} \varpi \operatorname{dn} K}{\operatorname{dn} \varpi \operatorname{cn}(K + iK')} \right\} = 0,$$

or

$$Z(\varpi) + \frac{\operatorname{cn} \varpi \operatorname{dn} \varpi}{(\operatorname{sn} \varpi - \operatorname{sn} \varpi')} + \frac{\pi(\varpi + K)}{2KK'} = 0. \tag{73}$$

This equation determines ϖ' in terms of ϖ.

When there is symmetry, the analysis of Case III resembles that of the general configuration in Case II, so far as Ω is concerned, except that the condition giving ϖ' is found by integrating from K to $K + iK'$ instead of from $-K$ to K. The form of w has one parameter fewer. The system

has two degrees of freedom, corresponding analytically to the parameters k^2 and ϖ.

A further simplification occurs if $\varpi = 0$. We then find from (73) $\operatorname{sn} \varpi' = 2K'/\pi$, as can be verified directly by observing that the function under the logarithm is $\operatorname{sn} u/(\operatorname{cn} u + \operatorname{dn} u)$. Since $\operatorname{sn} u$, $\operatorname{dn} u$ remain real and positive, while $am(\operatorname{cn} u) = -\frac{1}{2}\pi$, the amplitude of the expression will increase by $\frac{1}{2}\pi$ in the range $(K, K+iK')$.

Hence

$$\frac{dz}{dw} = e^{\Omega} = e^{\{\pi(u+K)\}/(4K')} \sqrt{\left\{\frac{(\operatorname{cn} u + \operatorname{dn} u)}{(-k' \operatorname{sn} u)}\right\}}, \qquad w = \frac{B \operatorname{sn} u}{(1+\operatorname{sn} u)}. \tag{74}$$

The constant multiplier is found by making $(dz)/(dw) = 1$ when $u = -K$, corresponding to the point at infinity. But even in this special case the quadratures cannot be effected in terms of known functions.

7. Case IV.—*Two-Wake Problem.* (i) *Symmetrical Case.*

It is not difficult to construct the boundaries of the Ω figure for Case IV; but if we try to solve the problem by a direct application of the Schwarz-Christoffel formula, the result is a degenerate case (which is noticed in the next section). The problem may, however, be attacked indirectly through the case of symmetry. By making a change of variable, the original figure can be represented on a half-plane, and it is found that the Schwarz-Christoffel formula must be modified by introducing in the numerator a quadratic with imaginary roots. The root in the upper half-plane corresponds to the branch point of the two-sheeted Riemann surface in the Ω-figure. The method and the following solution are due to Prof. Love.

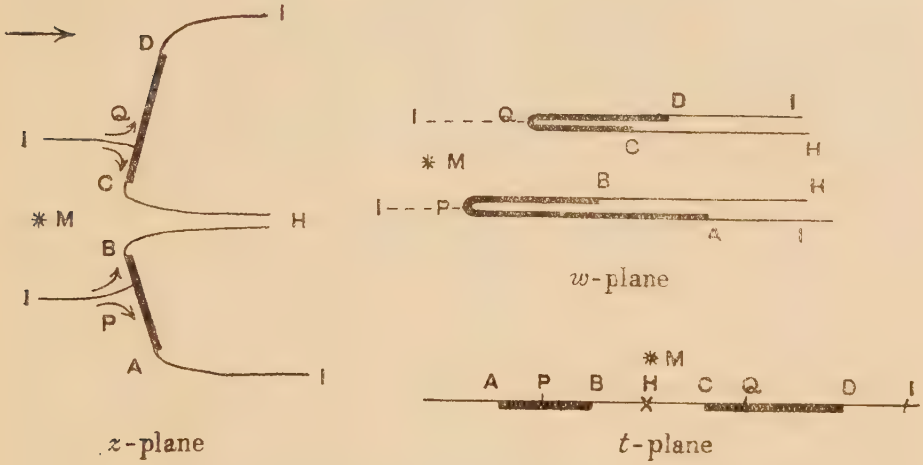
In the case of symmetry, the central stream-line may be regarded as a rigid boundary. We then have a known problem.* The ultimate velocity of the jet between the wall and the dead-water (see Fig. 7) is the same as that of the stream at infinity, because it is determined by the pressure in the dead-water, which is everywhere the same. Along the wall, the velocity sinks to a minimum at some point M and then increases again. The formulæ are

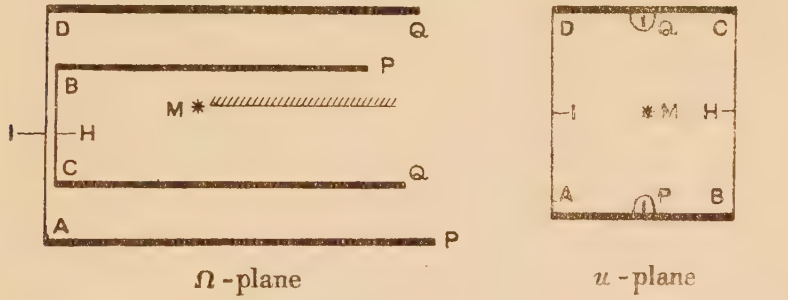
$$\frac{d\Omega}{ds} = \frac{A(s-m)}{(s-q)\sqrt{\{(1-s^2)(1-k^2s^2)\}}}, \qquad \frac{dw}{ds} = \frac{B(s-q)}{(s-1)(s+1)^2} \tag{75}$$

with the lettering of the figure. We have $m^2 < 1$, $k^2q^2 > 1$. The values of s at H (at infinity on the jet), C, D, I (at infinity outside the wake)

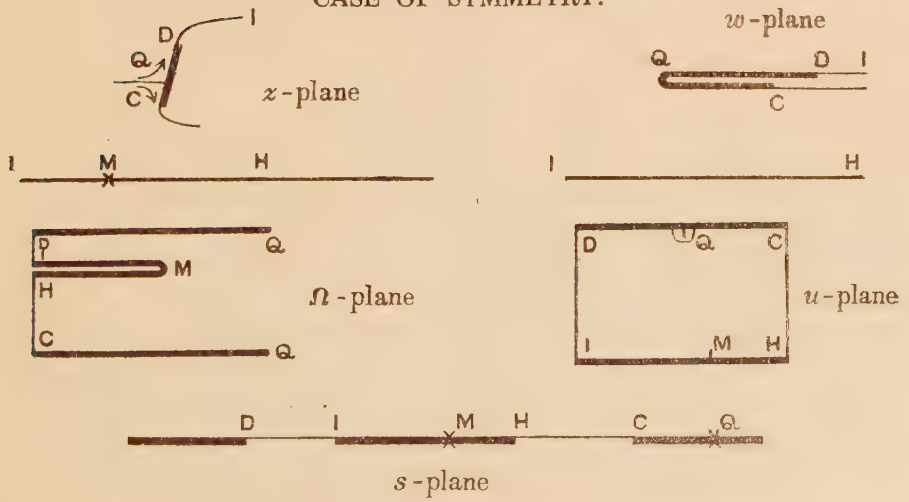
* Love, *Proc. Camb. Phil. Soc.*, Vol. 7 (1891), Problem 4.

FIG. 7. CASE IV.





CASE OF SYMMETRY.



are 1, $1/k$, $-1/k$, -1 respectively. Putting $s = \mathrm{sn}\,(u,\ k)$, $q = \mathrm{sn}\,\varpi$, $m = \mathrm{sn}\,\chi$, we have the region represented on a rectangle $u = K,\ K+iK',\ -K+iK',\ -K$. The point M lies on IH and Q on CD, so that χ and $(\varpi - iK')$ are real and numerically less than K. Since Ω increases by $i\pi$ at Q, the residue of $(d\Omega)/(du)$ is -1 at $u = \varpi$ (giving the value of A), while χ is found by making the velocities at H and I equal.

The analysis follows the lines of Case II (58) to (63) and gives

$$\frac{dz}{dw} = \frac{\Theta(u)\,H(K+\varpi)(\mathrm{dn}\,u\,\mathrm{cn}\,\varpi - \mathrm{cn}\,u\,\mathrm{dn}\,\varpi)}{\Theta(K)\,H(u+\varpi)\,k'\,\mathrm{cn}\,\varpi}, \tag{76}$$

$$w = B\left(\frac{\mathrm{sn}\,\varpi - 1}{4}\right)\log\left(\frac{\mathrm{sn}\,u+1}{\mathrm{sn}\,u-1}\right) - \frac{B(\mathrm{sn}\,\varpi+1)}{2(\mathrm{sn}\,u+1)} + \text{const.} \tag{77}$$

The figure has two degrees of freedom and the solution has the two parameters k^2 and ϖ. Now transform the s-plane homographically so that the line of symmetry IMH becomes the negative half of the real axis of ξ. Then, making $\xi = \eta^2$, we shall have the complete s-plane, cut along IMH, represented on a half-plane. The transformation is the inverse of that in (22). We have

$$s = \frac{1+\xi}{1-\xi}, \qquad \xi = \eta^2, \qquad n = \frac{m-1}{m+1} < 0,$$

$$r = \frac{q-1}{q+1} > 0, \qquad 0 < j = \frac{1-k}{1+k} < 1.$$

$$\frac{d\Omega}{d\eta} = \frac{A'(\eta^2 - n)}{(\eta^2 - r)\sqrt{\{(j-\eta^2)(1-j\eta^2)\}}} \tag{78}$$

$$w = \tfrac{1}{2}B(1-\mathrm{sn}\,\varpi)\log\eta + \tfrac{1}{4}B\eta^2(1+\mathrm{sn}\,\varpi) + \text{const.} \tag{79}$$

Now let $\eta = \sqrt{j}\,\mathrm{sn}\,(v,\ j), \qquad n = j\,\mathrm{sn}^2(\nu,\ j), \qquad r = j\,\mathrm{sn}^2(\rho,\ j),$ (80)

$$\frac{d\Omega}{dv} = \frac{A'(\mathrm{sn}^2 v - \mathrm{sn}^2 \nu)}{(\mathrm{sn}^2 v - \mathrm{sn}^2 \rho)}. \tag{81}$$

The constant A' can be found from the discontinuities of Ω. We have $0 < j\,\mathrm{sn}^2\rho < 1$, while ν is a pure imaginary. In the v-plane, the points $H,\ C,\ D,\ I$ correspond respectively to $v = 0,\ J,\ J+iJ',\ iJ'$. The imaginary axis is the line of symmetry, while the portions parallel to the real axis correspond to the free stream-lines. Completing the figure by reflexion in the imaginary axis, we obtain a representation of the complete configuration with the two obstacles. It will be more convenient to turn the axes through a right angle, so that (as in the preceding cases) the sides parallel to the real axis correspond to the rigid boundaries.

8. CASE IV.—*Two-Wake Problem.* (ii) *Unsymmetrical Case.*

The formula (78) shows that there is a point in the fluid where the velocity is stationary in magnitude and direction. We shall suppose this to be true in general. From a physical point of view, we see that along every stream-line the velocity is initially and also ultimately equal to unity and parallel to Ox. If it is not always unity (the maximum admissible value), there must be a point on the stream-line where it is a minimum; if it is not always parallel to Ox, there must be points of inflexion where it changes its curvature. Our singular point would arise as the intersection of the locus of the minima and the locus of the inflexions. This argument does not prove that there is one and only one such intersection, but that is the simplest and most natural hypothesis. Analytically, the point corresponds to the branch point where the two sheets of the Riemann surface in the Ω-plane give coincident points in the fluid. One such branch point at least is needed if the variable z is to be a continuous function of Ω and not two distinct branches of different functions.

The Ω-region is a double open rectangle, drawn on a two-sheeted Riemann surface, with a branch point at M, from which a branch-cut extends to the points PQ at infinity in the two sheets. The ϖ-region is a plane slit along two semi-infinite lines parallel to the axis of ϕ.

Transforming to a half-plane of t, we make A, B, C, D correspond respectively to $t = -1,\ 1,\ 1/k,\ -1/k$. This involves no loss of generality, since three constants can always be assigned arbitrarily. The values of t corresponding to the point at infinity on the jet and on the outer boundaries of the wake are h and j respectively. Other points are denoted by small letters corresponding to capitals in the figure. The branch point $m = m_1 + im_2$ lies in the upper half-plane of t, so that $m_2 \geqslant 0$.

We have

$$\frac{d\Omega}{dt} = A\frac{(t-m_1)^2+m_2^2}{(t-p)(t-q)\sqrt{\{(1-t^2)(1-k^2t^2)\}}}, \qquad \frac{dw}{dt} = B\frac{(t-p)(t-q)}{(t-h)(t-j)^3} \tag{82}$$

where $p^2 < 1, \quad k^2q^2 > 1, \quad k < kh < 1, \quad -1 < kj < -k.$

Let $t = \mathrm{sn}\,(u, k), \quad p = \mathrm{sn}\,\varpi, \quad q = \mathrm{sn}\,\varpi', \quad h = \mathrm{sn}\,\sigma, \quad j = \mathrm{sn}\,\rho.$

Then
$$\frac{d\Omega}{du} = \frac{F}{\mathrm{sn}\,u - \mathrm{sn}\,\varpi} + \frac{G}{\mathrm{sn}\,u - \mathrm{sn}\,\varpi'} + H \tag{83}$$

where F, G, H are real.

The values of u corresponding to M and its conjugate are given by equating this to zero. The values of F and G are found by making the

residues at $u = \varpi$ and $u = \varpi'$ equal to -1, because Ω has a discontinuity πi at these points. Hence

$$F = -\operatorname{cn}\varpi \operatorname{dn}\varpi, \qquad G = -\operatorname{cn}\varpi' \operatorname{dn}\varpi'. \tag{84}$$

The value of H is determined by a physical condition. Since each wake is a simply connected space at uniform pressure, the velocity on the two stream-lines bounding it are equal. And since the velocity at infinite distances in the jet must tend to be uniform, the velocity is the same on all the free surfaces. Making the velocities at A, B equal, the analysis will be similar to that of Case I, save that we have an arbitrary constant H at our disposal, instead of a function of ϖ, ϖ' of known form. We find

$$\Omega = -P'u - \log QR + \text{const.} \tag{85}$$

where

$$P' = Z(\varpi) + Z(\varpi') - H \tag{86}$$

and Q, R are as in (28), (29). Instead of (31), we have the condition that the real part of P' shall vanish. Since $(\varpi' - iK')$ is real, and H is real, we get

$$H = Z(\varpi) + Z(\varpi' - iK') + \frac{\operatorname{cn}(\varpi' - iK')\operatorname{dn}(\varpi' - iK')}{\operatorname{sn}(\varpi' - iK')}, \tag{87}$$

and P' thus reduces to $-(\pi i)/(2K)$. Taking exponentials,

$$\frac{dz}{dw} = \text{const.} \times \frac{H(u+\varpi)\,H(u+\varpi')\,e^{(i\pi u)/(2K)}}{\Theta^2(u)(\operatorname{cn} u \operatorname{dn}\varpi - \operatorname{dn} u \operatorname{cn}\varpi)(\operatorname{cn} u \operatorname{dn}\varpi' - \operatorname{dn} u \operatorname{cn}\varpi')} \tag{88}$$

$$= \text{const.} \times \frac{H(u+\varpi')(\operatorname{cn} u \operatorname{dn}\varpi + \operatorname{dn} u \operatorname{cn}\varpi)}{H(u-\varpi)(\operatorname{cn} u \operatorname{dn}\varpi' - \operatorname{dn} u \operatorname{cn}\varpi')}\, e^{(i\pi u)/(2K)}, \tag{89}$$

where a certain function of ϖ has been taken into the constant.

The five parameters k^2, ϖ, ϖ', ρ, σ are not independent. For we have to express the condition that the jet has asymptotes parallel to the current at infinity. If this were not the case, the jet would, from a certain point onwards, make a greater angle with Ox than the tangent of one of the outer stream-lines, which it would inevitably cross. There would be no objection to this in a purely theoretical problem, if the jet could be placed in a different sheet of a Riemann surface. But, physically, the solution is not to be reconciled with the present hypothesis, though it clearly shows where our hypothesis lacks generality. We therefore postulate that

$(dz)/(dw)$ shall take equal values at $u = \rho$ and $u = \sigma$, so that

$$\frac{H(\rho+\varpi')\,H(\sigma-\varpi)(\mathrm{cn}\,\rho\,\mathrm{dn}\,\varpi+\mathrm{dn}\,\rho\,\mathrm{cn}\,\varpi)(\mathrm{cn}\,\sigma\,\mathrm{dn}\,\varpi'-\mathrm{dn}\,\sigma\,\mathrm{cn}\,\varpi')}{H(\rho-\varpi)\,H(\sigma+\varpi')(\mathrm{cn}\,\rho\,\mathrm{dn}\,\varpi'-\mathrm{dn}\,\rho\,\mathrm{cn}\,\varpi')(\mathrm{cn}\,\sigma\,\mathrm{dn}\,\varpi+\mathrm{dn}\,\sigma\,\mathrm{cn}\,\varpi)}$$

$$= e^{\{i\pi(\sigma-\rho)\}/(2K)}. \tag{90}$$

With this restriction, the solution has four degrees of freedom. The relative inclination of the obstacles depends on the parameters ϖ, ϖ', and k^2 only. In the formula (85), put $u = K$ for B, and $u = -K+iK'$ for D, on the corresponding upper edges of the obstacles. Since the absolute velocities are equal to unity at each point, we find

$$\Omega_B-\Omega_D = \frac{\pi i}{2K}(2K-iK')$$

$$+\log\left\{\frac{H(K+\varpi')\,H(-K+iK'-\varpi)\,\mathrm{cn}\,\varpi\,\mathrm{dn}\,\varpi'}{H(K+\varpi)\,H(-K+iK'-\varpi')(-1)\,\mathrm{cn}\,\varpi'\,\mathrm{dn}\,\varpi}\right\}$$

$$= \pi i+\frac{\pi K'}{2K}+\log\left\{\frac{H(K+\varpi')\,\Theta(K+\varpi)\,e^{\{(\pi i)/(2K)\}(\varpi+\varpi')}\,\mathrm{cn}\,\varpi\,\mathrm{dn}\,\varpi'}{H(K-\varpi)\,\Theta(-K+\varpi')(-1)\,\mathrm{cn}\,\varpi'\,\mathrm{dn}\,\varpi}\right\}$$

$$= \pi i+\frac{\pi K'}{2K}+\frac{\pi i}{2K}(\varpi+\varpi')+\log\left\{-\frac{\mathrm{sn}\,(\varpi'+K)\,\mathrm{cn}\,\varpi\,\mathrm{dn}\,\varpi'}{\mathrm{sn}\,(\varpi+K)\,\mathrm{cn}\,\varpi'\,\mathrm{dn}\,\varpi}\right\}. \tag{91}$$

Hence $$\Omega_B-\Omega_D = \frac{\pi i}{2K}(\varpi+\varpi'-iK')+\pi i+\log(-1). \tag{92}$$

Hence $$\theta_B-\theta_D = \frac{\pi}{2K}(\varpi+\varpi'-iK') \tag{93}$$

correct to a multiple of 2π.

In conclusion, let us try to obtain a solution without modifying the usual form of the Schwarz-Christoffel formula. We must have

$$F+G = 0 \quad \text{and} \quad H = 0$$

or $$\mathrm{cn}\,\varpi\,\mathrm{dn}\,\varpi+\mathrm{cn}\,\varpi'\,\mathrm{dn}\,\varpi' = 0 \quad \text{and} \quad Z(\varpi)+Z(\varpi')+\frac{i\pi}{2K} = 0. \tag{94}$$

Rationalizing and rejecting the solution $\mathrm{sn}^2\varpi = \mathrm{sn}^2\varpi'$, because ϖ is real and numerically less than K, while ϖ' has the imaginary part iK', we are left with

$$k^2(\mathrm{sn}^2\varpi+\mathrm{sn}^2\varpi') = 1+k^2. \tag{95}$$

Solutions of (94) and (95) are given by

$$\varpi \equiv \pm K, \quad \varpi' \equiv iK' \pm K \pmod{4K, 2iK'}. \tag{96}$$

We will prove that there is no alternative solution. We may regard (95) as an equation defining ϖ' as a function of ϖ. Consider the expression—

$$H(\varpi, \varpi') \equiv Z(\varpi) + Z(\varpi' - iK') + \frac{\operatorname{cn}(\varpi' - iK') \operatorname{dn}(\varpi' - iK')}{\operatorname{sn}(\varpi' - iK')}. \tag{97}$$

This is an odd function of the pair of variables $(\varpi' - iK')$ and ϖ, which changes sign by becoming infinite for the values (0, 0). If the expression is to vanish, as required by (94), the variables must be congruent to values numerically less than K, which have opposite signs. If the function vanishes, say, for a positive value of ϖ less than K, then its derivative must vanish for some point within the range $0 < \varpi < K$, the function being continuous and differentiable. Now differentiating (95) and reducing by means of (94), we get

$$\operatorname{sn}\varpi \, d\varpi = \operatorname{sn}\varpi' \, d\varpi'.$$

Hence $$dH = \left(\operatorname{dn}^2\varpi - \frac{E}{K}\right) d\varpi + \left(\operatorname{dn}^2\varpi' - \frac{E}{K}\right) \frac{\operatorname{sn}\varpi}{\operatorname{sn}\varpi'} \, d\varpi.$$

Since this is to vanish and since $\operatorname{sn}\varpi + \operatorname{sn}\varpi' \neq 0$, we get

$$k^2 \operatorname{sn}\varpi \operatorname{sn}\varpi' = 1 - (E/K) > 0. \tag{98}$$

The smallest numbers congruent to ϖ and $(\varpi' - iK')$ must therefore have the same sign, which is not possible. Hence there is no alternative solution.

We can easily see that the solution is degenerate; for the branch points lie at the ends of the obstacles, which are found to lie parallel to the direction of the current at infinity. Since the right-hand member of (83) is identically zero, Ω is constant everywhere, and equal to zero (say). Hence $(dz)/(dw) = 1$, and $z = w + \text{constant}$.

The planes of z and w are indistinguishable, except that we may, if we choose, produce to infinity the cuts parallel to the obstacles in the w-plane. This renders the w-region simply connected.

In this case, the only parameter of any importance is the inclination of the line joining the bows of the obstacles to the stream at infinity. Theoretically, however, we may consider also the lengths of the obstacles, These three parameters correspond to functions of the constants k^2, ρ, and σ, which appear in the solution.

The condition (90), which these must satisfy, now reduces to an identity, for we have

$$\frac{H(\rho+iK'-K)\,H(\sigma-K)\,\mathrm{dn}\,\sigma\,\mathrm{cn}\,\rho}{H(\rho-K)\,H(\sigma+iK'-K)\,\mathrm{dn}\,\rho\,\mathrm{cn}\,\sigma}$$

$$= e^{\{i\pi(\sigma-\rho)\}/(2K)}\frac{\Theta(\rho+K)\,H(\sigma+K)\,\mathrm{dn}\,\sigma\,\mathrm{cn}\,\rho}{\Theta(\sigma+K)\,H(\rho+K)\,\mathrm{dn}\,\rho\,\mathrm{cn}\,\sigma}$$

$$= e^{\{i\pi(\sigma-\rho)\}/(2K)}. \tag{99}$$

We have, therefore, three independent parameters in the solution. In (82) we put $p = 1$, $kq = -1$, and (neglecting a trivial constant factor) we have

$$\frac{dz}{dt} = \frac{dw}{dt} = \pm\frac{(1-t)(1+kt)}{(h-t)(j-t)^3}. \tag{100}$$

The lengths of the obstacles are

$$l_1 = \int_{-1}^{1}\left|\frac{dz}{dt}\right|dt, \qquad l_2 = \int_{1/k}^{\infty}+\int_{-\infty}^{-1/k}\left|\frac{dz}{dt}\right|dt. \tag{101}$$

It is interesting to note the complete correspondence of the analytical formulæ with the geometry of the figure, in this trivial degenerate case.

THE AVERAGE VALUE OF A FUNCTIONAL*

By NORBERT WIENER.

[Received February 27th, 1922.—Read March 9th, 1922.]

1. The notion of a mean or average of a quantity is most familiar to us in those cases where the quantity averaged is regarded as a function of an independent variable, the range of which consists either of a discrete set of values or of a manifold of a finite number of dimensions. Such general theories of the average or the integral as have been formulated have, however, been founded on a much wider basis, and have concerned themselves with an independent variable subject to very few restrictions. Papers by Fréchet,† Daniell,‡ and the author§ may be mentioned in this connection.

Among the simplest ranges more complex than space of a finite number of dimensions are space of a denumerable infinity of dimension and function-space. Daniell‖ has discussed averages in space of the former type, while there is a very elegant theory of the average of a function of a line due to Gâteaux¶ and Lévy.** E. H. Moore†† has also done some work which should be mentioned in this connection. These theories, however, while they give very natural definitions of averages in function-space, by no means exhaust the field. Other definitions are possible, and may in some cases be useful. In three previous papers‡‡

* Presented to the American Mathematical Society, December 1920.

† M. Fréchet, *Bull. Soc. Math. de France*, Vol. 43 (1915), p. 248.

‡ P. J. Daniell, "A General Form of Integral", *Annals of Mathematics*, Vol. 19 (1918), p. 279.

§ N. Wiener, *Annals of Mathematics*, Vol. 21 (1920), p. 66.

‖ P. J. Daniell, *Annals of Mathematics*, Vol. 21 (1919), p. 281.

¶ R. Gâteaux, *Bull. Soc. Math. de France*, Vol. 47 (1919), p. 47.

** P. Lévy, *Leçons d'analyse fonctionnelle*, Paris, 1922.

†† Cf. T. H. Hildebrandt, *Bull. American Math. Soc.*, Vol. 24 (1918), p. 201.

‡‡ N. Wiener, "The Average of an Analytic Functional", *Proc. Nat. Acad. Sci., Washington*, Vol. 7, pp. 253–260; N. Wiener, "The Average of an Analytic Functional and the Brownian Movement", *ibid.*, pp. 294–298; N. Wiener, "Differential-Space", *Jour. Math. and Phys. Mass. Inst. Technology*, Vol. 2, pp. 131–174.

the author has developed the properties of a type of function-space essentially different from the familiar type, and has given a partial discussion of the theory of integration in this space. It is the purpose of the present paper to give a self-contained discussion of this type of integration which shall be in many respects more complete than those which he has yet developed. In order that the content of this paper be intelligible by itself, a certain amount of repetition from the other papers has been unavoidable, as is also a certain amount of preliminary discussion which treats of well-known points in classical probability theory.

2. Let us consider a particle free to wander along the X-axis. Let the probability that it wander a given distance in a given time be independent

(1) of the position from which it starts to wander,

(2) of the time when it starts to wander,

(3) of the direction in which it wanders.

It may then be shown* that the probability that after a time t it has wandered from the origin to a position lying between $x = x_1$ and $x = x_2$ is

$$(1) \qquad \frac{1}{\sqrt{\{\pi c t\}}} \int_{x_1}^{x_2} e^{-(x^2/ct)}\, dx,$$

where c is a certain constant which we may reduce to 1 by a proper choice of units. This choice we shall make. The exponential form of the integral needs no comment, while the mode in which t enters results from the fact that

$$(2) \qquad \frac{1}{\sqrt{\{\pi(t_1+t_2)\}}} \int_{x_1}^{x_2} e^{-\{x^2/(t_1+t_2)\}}\, dx$$

$$= \frac{1}{\sqrt{\{\pi t_1\}}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{\{\pi t_2\}}} \int_{x_1}^{x_2} e^{-\{(y-x)^2/t_2\}}\, dy \right] e^{-(x^2/t_1)}\, dx.$$

This identity is tantamount to the statement that the probability that after a period t_1+t_2 the particle has wandered a distance between x_1 and x_2 is the total compound probability that it be anywhere at time t_1 and that it then wander to a position between x_1 and x_2 in a subsequent period of length t_2. This fact lies at the bottom of all that follows.

* A. Einstein, *Annalen der Physik*, Vol. 17, p. 905.

3. We shall say that the particle of § 2 has its displacements normally distributed. Let us represent the history of such a particle by an equation $x = \Phi(t)$, Φ being a continuous function. If we suppose the particle to start from rest, we shall have $\Phi(0) = 0$. We shall consider in what follows particle-histories or time-paths in which the range of t is from 0 to 1.

There are certain assemblages of time-paths to which we can immediately assign a measure, a probability. These assemblages are obtained by restricting the position of the particle at certain specified times, finite in number, to certain specified finite or infinite intervals. An example is the set of all time-paths $x = \Phi(t)$, such that

$$x_{11} \leqslant \Phi(t_1) \leqslant x_{12};$$

$$x_{21} \leqslant \Phi(t_2) \leqslant x_{22};$$

$$\dots \qquad \dots \qquad \dots$$

$$x_{n1} \leqslant \Phi(t_n) \leqslant x_{n2}.$$

$$(0 \leqslant t_1 \leqslant t_2 \leqslant \dots \leqslant t_n \leqslant 1.)$$

This assemblage of time-paths may be considered as possessing the compound probability that after a time t, the particle shall occupy a position between x_{11} and x_{12}, that by time t_2 it shall have wandered from whatever position it shall have occupied at time t_1 to a position between x_{21} and x_{22}, and so on. By (1), this compound probability is

$$(3) \quad \frac{1}{\sqrt{\{\pi^n t_1(t_2-t_1)(t_3-t_2)\dots(t_n-t_{n-1})\}}} \times \int_{x_{11}}^{x_{12}} \int_{x_{21}}^{x_{22}} \dots \int_{x_{n1}}^{x_{n2}} e^{-[(\xi_1^2/t_1)+\{(\xi_2-\xi_1)^2/(t_2-t_1)\}+\dots+\{(\xi_n-\xi_{n-1})^2/(t_n-t_{n-1})\}]} d\xi_1\, d\xi_2 \dots d\xi_n.$$

We shall call the assemblage of time-paths which we have just specified an *interval* and shall term (3) its *weight*. We shall allow any x_{k1} to be $-\infty$ and any x_{k2} to be $+\infty$. It is clear that the weight of any interval consisting of the logical sum of a finite number of mutually exclusive intervals is the sum of the weights of its component intervals; and that if *every* x_{k1} is $-\infty$, and *every* x_{k2} is $+\infty$, the weight of the interval is 1.

4. Under certain circumstances, if an assemblage can be divided into sub-assemblages with assignable weights, it is possible to define the mean of a function ranging over the assemblage in question. These circumstances have been set forth by the author in an earlier paper already noted. The mean thus defined, which was a Daniell integral, presupposed

that the assemblage could be divided into successive collections I_n, each of a finite number of weighted intervals, satisfying the following conditions:

(a) Every interval of I_{n+1} forms part of one interval I_n and of one only;

(b) The weight of an interval of I_n is the sum of the weights of the component intervals in I_{n+1};

(c) If S_n consists of all the elements in a number of intervals of I_n and S_{n+1} is contained in S_n, then either there is an element common to every S_n, or the total weight of S_n approaches 0 with $1/n$.

For the purposes of the present paper, we shall define the intervals of I_n in the following somewhat arbitrary manner: we shall take for our intervals intervals in the sense of § 3, letting

$$t_h = h/2^n \quad (1 \leqslant h \leqslant 2^n),$$

$$x_{1h} = \tan(k_h \pi/2^n),$$

$$x_{2h} = \tan\left((k_h+1)\pi/2^n\right),$$

k_h being some positive or negative integer. It will be seen that there are 2^n such segments (x_{1h}, x_{2h}), running from $-\infty$ to $+\infty$, that every segment correlated with I_n is made up of a finite number of segments correlated with I_{n+1}, and that if H is any finite positive quantity, and ϵ is any positive quantity, then there is an m such that $n > m$, the segments correlated with I_n between $-H$ and H are all less in length than ϵ. The accompanying figure shows an interval of I_3. The interval is indicated by the vertical black lines.

The black horizontal segments form an interval of I_3 containing the curve drawn. The line $OPQRS$ satisfies any condition of equal continuity fulfilled by curves of the interval.

I_n itself is now defined as the assemblage of $(2^n)^{2^n}$ intervals which are obtained by associating a k_h with each h from 1 to 2^n inclusive. The weight of the intervals of I_n is defined as in § 3. Conditions (a) and (b) will clearly be satisfied.

Condition (c) requires a consideration somewhat more careful. In this connection, let us make use of the function

$$\psi(u) = 2^{1-k/4}/(1-2^{-\frac{1}{4}}),$$

where k is the largest integer such that $1/2^k > u$. Clearly

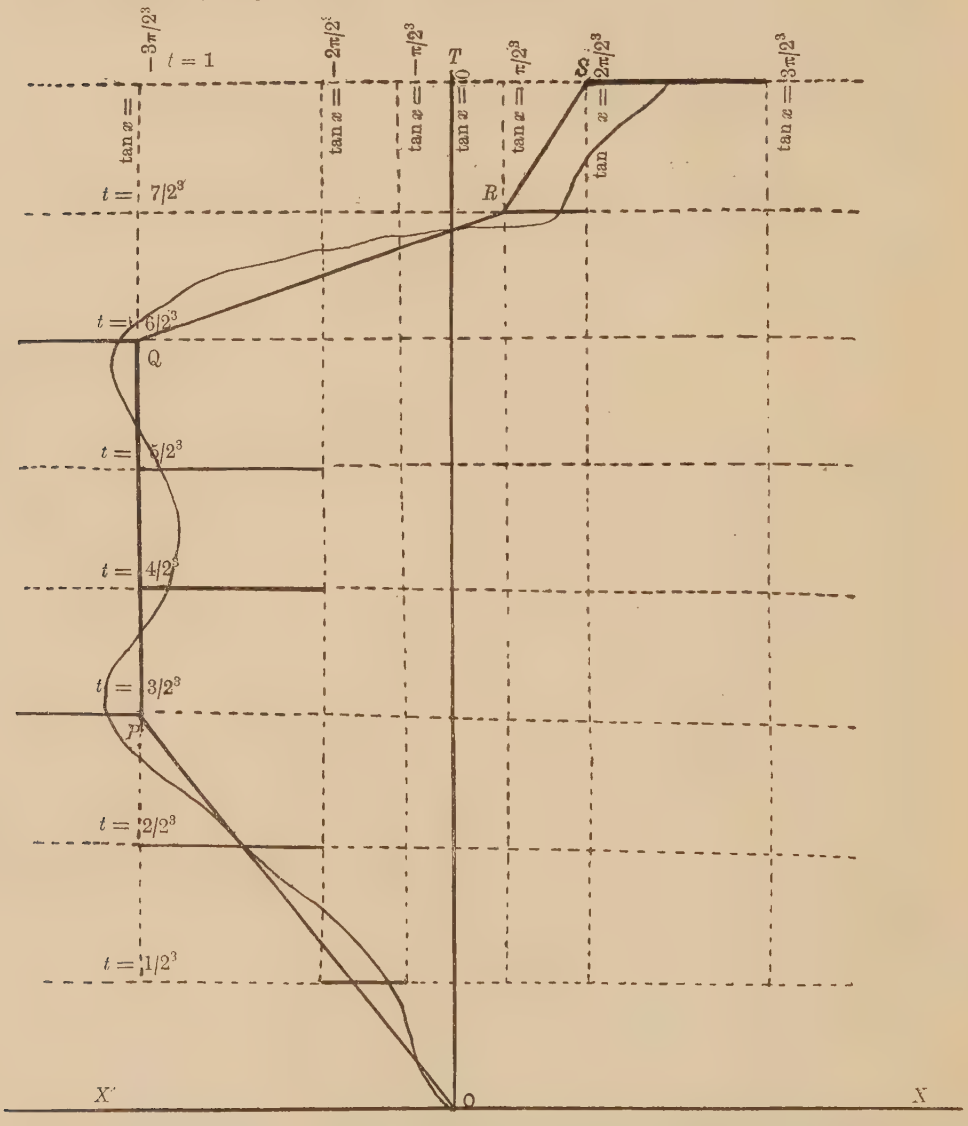
$$\lim_{u \to 0} \psi(u) = 0. \tag{4}$$

Now consider the functions $f(t)$ such that $f(0) = 0$ and that for all values

of t_1 and t_2 between 0 and 1, inclusive,

$$|f(t_1)-f(t_2)| \leqslant h\psi(|t_1-t_2|), \tag{5}$$

where h is a given positive integer. These functions are manifestly what Ascoli* calls equally continuous, and are bounded as a set. They therefore



form what Fréchet† calls a compact set, and there is no difficulty in

* Ascoli, "Le curve limiti di una varietà data di curve", *Lincei*, Vol. 18, pp. 521–586.

† M. Fréchet, "Sur quelques points du calcul fonctionnel", *Rend. Cir. Mat. Pal.*, Vol. 22, pp. 6, 37.

showing that every limit-function of the set satisfies the same condition of equal continuity as the members of the set, and hence belongs to the set,* which is closed. Now an interval is closed, and hence those functions in an interval satisfying our condition of equal continuity form a closed set. Hence, if every S_n contains intervals which contain functions satisfying the given condition of equal continuity, there is a function common to every S_n.† Then (*c*) is satisfied.

On the other hand, suppose that some S_k contains no function satisfying the given condition of equal continuity. That is to say, given any function $f(t)$ from S_k, suppose that there are two numbers t_1 and t_2 between 0 and 1, inclusive, such that

$$|f(t_1)-f(t_2)| > h\psi(|t_1-t_2|). \tag{6}$$

We shall show that by choosing h sufficiently large, the total weight possible for S_k may be made arbitrarily small. This will be sufficient to complete the proof of (*c*).

To show this, let us represent t_1 as the binary fraction

$$0 \cdot a_1 a_2 \dots a_n \dots$$

and t_2 as the binary fraction

$$0 \cdot b_1 b_2 \dots b_n \dots .$$

Let t_3 be a number whose binary expansion may be made to agree with that of t_1 up to and including a_j and with that of t_2 up to and including b_j. Choose t_3 in such a manner that j may be as large as possible.‡ In general t_3 will admit of a terminating expression. The interval from t_1 to t_3 will then be expressible in the form

$$0 \cdot 00 \dots 0 c_{j+1} c_{j+2} \dots ,$$

where there are j consecutive 0's and every c is 0 or 1. The interval from t_3 to t_2 may be expressed in a similar manner. In other words, the interval from t_1 to t_2 may be reduced to the sum of a denumerable set of intervals from terminating binaries to adjacent terminating binaries of the same number of figures, such that there are not more than two inter-

* *Ibid.*, p. 6.

† *Ibid.*, p. 7.

‡ This may necessitate the use of an expression for t_3 ending in 111... to agree with the smaller of the quantities t_1 and t_2, and of a terminating expression for t_3 to agree with the larger.

vals in the set of magnitude $1/2^{j+k}$, where k is any positive integer, and such that every interval is of one of these sizes.

Now clearly

$$|t_2-t_1| \geqslant 2^{-1-j}. \tag{7}$$

Hence, by the definition of ψ,

$$|f(t_1)-f(t_2)| > h \cdot 2^{1-(j-1)/4}/(1-2^{-1/4}), \tag{8}$$

which we may write in the form

$$|f(t_1)-f(t_2)| > 2h \sum_{j+1}^{\infty} 2^{-n/4}. \tag{9}$$

If we make use of the fact that f is continuous, and correlate the terms of (9) with the digits of the steps from t_3 to t_1 and t_2, we obtain the result that for some interval from $m/2^{j+k}$ to $(m+1)/2^{j+k}$, where m and k are integers and $0 \leqslant m < 2^{j+k}$, we shall have

$$\left| f\left(\frac{m}{2^{j+k}}\right) - f\left(\frac{m+1}{2^{j+k}}\right) \right| > h \cdot 2^{-(j+k)/4}. \tag{10}$$

Writing i for $j+k$, this becomes

$$\left| f(m2^{-i}) - f\left((m+1)2^{-i}\right) \right| > h \cdot 2^{-i/4}. \tag{11}$$

Every function satisfying (6) satisfies (11) for some m and i. We may go further and say that any interval consisting entirely of functions satisfying (6) has all its functions satisfy (11) for the same value of i. This follows immediately from the fact that, as shown in the diagram, every interval possesses what we may term a most continuous time-path, which fulfils any condition of equal continuity fulfilled by any time-path in the interval.

We shall now show that all functions fulfilling (11) for any one of a finite number of values of i may be enclosed in a finite set of intervals the total weight of which may be made to vanish as h becomes infinite. To begin with, let i and m be fixed, and let n be so great that the segments to which $f(m2^{-i})$ and $f\left((m+1)2^{-i}\right)$ are confined are all less in magnitude than ϵ, over a range running from $-E$ to $+E$. That this can be done is clear from the definition of I_n. If we divide all intervals containing functions satisfying (11) into those going beyond $\pm E$ for $t = m2^{-i}$ or $t = (m+1)2^{-i}$ and all others, we get for the total weight of these intervals

a quantity not exceeding

$$(12)\quad P = \frac{2}{\sqrt{(\pi m 2^{-i})}} \int_E^\infty e^{(-2ix^2)/m}\,dx + \frac{2}{\sqrt{\{\pi(m+1)2^{-i}\}}} \int_E^\infty e^{(-2ix^2)/(m+1)}\,dx$$
$$+ \frac{2}{\sqrt{(\pi 2^{-i})}} \int_{(h\,.\,2^i-2\epsilon)}^\infty e^{-2ix^2}\,dx$$
$$= \frac{2}{\sqrt{\pi}} \left\{ \int_{E\sqrt{(2^i/m)}}^\infty e^{-x^2}\,dx + \int_{E\sqrt{\{(2^i)/(m+1)\}}}^\infty e^{-x^2}\,dx + \int_{(h\,.\,2^{i/4}-2(i+2)/2\epsilon)}^\infty e^{-x^2}\,dx \right\}.$$

We may prove without difficulty that given any positive η, by choosing n sufficiently great, and hence ϵ sufficiently small and E sufficiently large,

$$(13)\quad P < \eta + \frac{2}{\sqrt{\pi}} \int_{h\,.\,2^{i/4}}^\infty e^{-x^2}\,dx < \eta + \frac{2}{\sqrt{\pi}} \int_{h\,.\,2^{i/4}}^\infty e^{-x}\,dx$$

if $h > 1$. Hence

$$(14)\quad P < \eta + \frac{2}{\sqrt{\pi}}\, e^{-h\,.\,2^{i/4}}.$$

It may readily be shown that η may be chosen once for all values of m. Hence if we let m range from 1 to $2^i - 1$, the total probability that (11) should be satisfied for a given i is less than

$$(15)\quad P_i = 2^i \eta + \frac{2^{i+1}}{\sqrt{\pi}}\, e^{-h\,.\,2^{i/4}}.$$

Let us have $\eta < 1/(h\,.\,2^{2i})$, and choose an appropriate n. Then

$$(16)\quad P_i < \frac{1}{h\,.\,2^i} + \frac{2^{i+1}}{\sqrt{\pi}}\, e^{-h\,.\,2^{i/4}}.$$

Since, for a sufficiently large x, $e^x > x^p$, if we take $p = 8$ and let i be sufficiently large,

$$(17)\quad P_i < \frac{1}{h\,.\,2^i} + \frac{2}{\sqrt{\pi}} \cdot \frac{1}{h^8\,.\,2^i}.$$

It is thus clear that ΣP_i converges to a limit which vanishes as h becomes infinite.

Now, it follows from the definition of P_i that the weight of S_k is less than ΣP_i if every function in S_k satisfies (6). This completes the proof of (c), and allows us to apply at once the theorems contained in the author's earlier paper, and hence those due to Daniell.

Before we proceed further, it may be remarked that none of the results already obtained are affected if we discard from consideration all functions satisfying equation (11) for all values of h. There is no interval which

consists exclusively of such functions. We shall, accordingly, consider them discarded in all that follows.

We shall designate as a *step-functional* a functional* F which is constant over every interval of some I_n. A step-functional is allowed to be multiple-valued for arguments common to several intervals. If i_k be used to represent the general interval from I_n, we shall say that the *average value* of F is the quantity

$$A\{F\} = \sum_k F\{f\}\, p(i_k), \tag{18}$$

where f is a function belonging to i_k and $p(i_k)$ is the weight of i_k. It follows from the results of my previous paper that A is what Daniell terms an I-integral.†

5. A brief résumé of Daniell's main definitions and results in so far as they apply to the operation A is now in order. The class of all step-functionals plays the rôle of what he calls T_0. The class which is the T_1 of his terminology consists of all those functionals F which are limits of non-decreasing sequences of step-functionals

$$F_1 \leqslant F_2 \leqslant \ldots \leqslant F_n \leqslant \ldots.$$

We then make the definition

$$A\{F\} = \lim A\{F_n\}. \tag{19}$$

For *any* functional G, $\dot{A}\{G\}$ is defined as the lower bound of $A\{F\}$ for all functionals F belonging to T_1 and such that $F \geqslant G$ for all arguments. $\underset{\cdot}{A}\{G\}$ is defined by

$$\underset{\cdot}{A}\{G\} = -\dot{A}\{-G\}. \tag{20}$$

If

$$\dot{A}\{G\} = \underset{\cdot}{A}\{G\} = \text{finite}, \tag{21}$$

their common value is by definition $A\{G\}$, and we call G summable. Clearly every function in T_0 is summable.

The most important theorem which Daniell proves reads as follows:

If $\{F_n\}$ is a sequence of summable functions with the limit F, and if a summable functional G exists such that for all n

$$|F_n| \leqslant G,$$

* Cf. Fréchet, *loc. cit.*, p. 4.

† *A General Form of Integral*, p. 280.

then F is summable, and

$$A\{F\} = \lim A\{F_n\}. \tag{22}$$

In addition to this, A is a linear positive operator.

6. Let us form as in § 4 sets of intervals σ_h of total weight not exceeding ΣP_i containing all the continuous functions satisfying (11) for specified values of h. Let $\tau_1 = \sigma_1$, let τ_2 consist of those parts of intervals of σ_2 that lie in intervals of τ_1, and in general let τ_n consist of those parts of intervals of σ_n that lie in intervals of τ_{n-1}. Let G be a functional that differs from a step-functional only in the fact that it is 0 for every function in τ_n. I say that it is summable. For let us assume that the original step-functional F is constant over the intervals of I_h. Arrange the intervals of τ_n, which form a denumerable set, in a progression, and let F_m be the step-functional which differs from F by being 0 over the first m of these. Clearly $F_m \leqslant |F|$ and $\lim F_m = G$, so that G is summable.

Next let H be a functional which differs from a bounded continuous* functional only in that it vanishes for all functions in τ_n. Then H is summable. For let F_m be that functional which is constant over each interval of I_m, excluding functions in τ_n (for which it vanishes), which assumes in each interval of I_m the upper bound of H for an f in that interval and not in τ_n. Since an interval is closed and since the set of all time-paths not in τ_n is extremal,† the region to which f is here restricted is extremal, and the upper bound of H is actually the largest value assumed by H in the region in question.‡ Since for later and later I_m's the portions of intervals not in τ_n come to have smaller and smaller maximum difference between any two functions they contain,

$$\lim F_m = H. \tag{23}$$

As $F_m \leqslant |H|$, and as H is bounded, it follows that H is summable.

Finally—and this is our cardinal theorem—*any bounded continuous functional defined over all time-paths is summable.* Let F be such a functional. Consider the sequence $\{F_n\}$, where F_n differs from F only by being 0 over τ_n. F_n is summable by the last theorem. $F = \lim F_n$, as there is no function of the region which we consider which is common to every τ_n. Furthermore, every F_n is not greater than the constant step-functional $\max |F|$. Thus our theorem follows at once.

* Fréchet, *loc. cit.*, p. 7.
† Fréchet, *loc. cit.*, p. 37.
‡ Fréchet, *loc. cit.*, p. 29.

7. We shall now search for a more explicit expression for $A\{F\}$. In the first place, by definition (3), the weight of an interval J of I_n is

$$(24)\quad \pi^{-2^{n-1}}\, 2^{n.\,2^{n-1}} \int_J \cdots \int \exp\left\{-2^n\left[x_1^2+\sum_{j=2}^{2^n}(x_j-x_{j-1})^2\right]\right\} dx_1 \cdots dx_{2^n},$$

and this weight is also by definition $A\{G\}$, where G is 1 over J and 0 elsewhere. It follows at once that if F is any step-functional assuming constant values over intervals of I_n,

$$(25)\quad A\{F\} = \pi^{-2^{n-1}}\, 2^{n.\,2^{n-1}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F\{f_{x_1,\ldots,x_{2^n}}\} \times \exp\left\{-2^n\left[x_1^2+\sum_{j=2}^{2^n}(x_j-x_{j-1})^2\right]\right\} dx_1 \cdots dx_{2^n},$$

where $f_{x_1,\ldots,x_{2^n}}(t)$ represents the broken line with corners successive at $(0, 0)$, $(1/2^n, x_1)$, $(2/2^n, x_2)$, $\ldots$, $(1, x_{2^n})$.

By a theorem of Daniell which we have already given, this formula may be extended to all bounded functionals $F\{f\}$ that are only dependent on the values of f for arguments of the form $k/2^n$, and that are limits of uniformly bounded sequences of step-functionals constant over the intervals of I_n. It is in particular true of all bounded continuous functionals for which $F\{f\} = F\{f_{x_1,\ldots,x_{2^n}}\}$.

Under this supposition we may elide, as it were, any x_k in (25) such that a change in x_k does not alter $F\{f_{x_1,\ldots,x_{2^n}}\}$. If we write

$$(26)\quad \{\Psi(t_1, \ldots, t_n)(x_1, \ldots, x_n)\}(t)$$

for that broken line with corners successively at $(0, 0)$, (t_1, x_1), $\ldots$, (t_n, x_n), then

$$(27)\quad A\{F\} = \pi^{-n/2} \prod_1^n (t_k - t_{k-1})^{-\frac{1}{2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F\Big\{\{\Psi(t_1, \ldots, t_n)(x_1, \ldots, x_n)\}(t)\Big\} \times \exp\left\{-\frac{x_1^2}{t_1} - \frac{(x_2-x_1)^2}{t_2-t_1} - \cdots - \frac{(x_n-x_{n-1})^2}{t_n-t_{n-1}}\right\} dx_1 \cdots dx_n,$$

if $t_1, \ldots, t_n = 1$ are terminating positive binaries, arranged in order of magnitude, and $F\{f\}$ is any continuous, bounded functional dependent only on $f(t_1), \ldots, f(t_n)$.

The restriction that the binaries $t_1, \ldots, t_n$ terminate is inessential. Let U_h be the set of all time-paths satisfying (5). Let

$$(28)\quad F'\{f\} = F\Big\{(t_1, \ldots, t_n)\Big(f(t_1), \ldots, f(t_{k-1}), f(t_k+\delta), f(t_{k+1}), \ldots, f(t_n)\Big)\Big\}.$$

In the first place there is a $\phi(\delta, h)$ such that $\lim_{\delta \to 0} \phi(\delta, h) = 0$, and that

(29)
$$\left| \Psi(t_1, \ldots, t_n)\left(f(t_1), \ldots, f(t_n)\right) - \Psi(t_1, \ldots, t_n)\left(f(t_1), \ldots, f(t_k+\delta), \ldots, f(t_n)\right) \right| \leqslant \phi(\delta, h)$$

for every f in U_h. In the second place, U_h is bounded and extremal,* so that F is uniformly continuous over U_h.† Hence there is a Φ such that $\lim_{\delta \to 0} \Phi(\delta, h) = 0$ and that

(30) $$|F'\{f\} - F\{f\}| \leqslant \Phi(\delta, h)$$

over U. Now, we may write

(31) $$|F'\{f\} - F\{f\}| = Q_1\{f\} + Q_2\{f\},$$

where $Q_1\{f\}$ equals $|F'\{f\} - F\{f\}|$ over U_h and 0 elsewhere, while Q_2 equals 0 over U_h and $|F' - F|$ elsewhere. Hence

(32)
$$\begin{aligned} |A\{F'\} - A\{F\}| &= |A\{F' - F\}| \\ &\leqslant A\{|F' - F|\} \\ &\leqslant \dot{A}\{Q_1\} - \dot{A}\{Q_2\} \\ &\leqslant \dot{A}\{Q_1\} + 2 \max |F| . A\{R\}, \end{aligned}$$

where $R = 1$ over σ_h and $R = 0$ elsewhere. By definition, σ_h is the sum of a denumerable set of intervals, and hence R is the limit of a bounded set of step-functionals R_n and is summable to the limit $A\{R_n\}$.‡ By the discussion following (17), it may be shown that this limit is no greater than ΣP_i, which vanishes as h becomes infinite. Choose h so that

(33) $$2 \max |F| . A\{R\} < \epsilon/2,$$

and then choose δ so that $\Phi(\delta, h) < \epsilon/2$. Then by (32)

(34) $$|A\{F'\} - A\{F\}| < \epsilon.$$

It follows that $|A\{F'\} - A\{F\}|$ approaches 0 with δ.

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* Fréchet, *loc. cit.*, p. 37.

† Fréchet, *loc. cit.*, p. 29.

‡ Daniell, *A General Form of Integral*, 7 (7).

Now let $0 \leqq t_1 \leqq \ldots \leqq t_n = 1$, and let every t but t_k be a terminating binary. Let $t_k + \delta$ terminate. Let $t_j' = t_j$ if $j \neq k$ and let $t_k' = t_k + \delta$. By (27) and (28),

(35)

$$A\{F'\} = \pi^{-(n/2)} \prod_1^n (t_j' - t_{j+1}')^{-\frac{1}{2}} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} F'\left\{\{\Psi(t_1', \ldots, t_n')(x_1, \ldots, x_n)\}(t)\right\}$$

$$\times \exp\left\{-\frac{x_1^2}{t_1'} - \ldots - \frac{(x_n - x_{n-1})^2}{t_n' - t_{n-1}'}\right\} dx_1 \ldots dx_n$$

$$= \pi^{-(n/2)} \prod_1^n (t_j' - t_{j+1}')^{-\frac{1}{2}} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} F\left\{\{\Psi(t_1, \ldots, t_n)(x_1, \ldots, x_n)\}(t)\right\}$$

$$\times \exp\left\{-\frac{x_1^2}{t_1'} - \ldots - \frac{(x_n - x_{n-1})^2}{t_n' - t_{n-1}'}\right\} dx_1 \ldots dx_n.$$

Hence

(36)
$$\left| A\{F'\} - \pi^{-(n/2)} \prod_1^n (t_j - t_{j+1})^{-\frac{1}{2}} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} F\left\{\{\Psi(t_1, \ldots, t_n)(x_1, \ldots, x_n)\}(t)\right\} \right.$$

$$\left. \times \exp\left\{-\frac{x_1^2}{t_1} - \ldots - \frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}}\right\} dx_1 \ldots dx_n \right|$$

$$\leqq \max |F| \left| \Delta \left\{ \pi^{-(n/2)} \Pi (t_j - t_{j+1})^{-\frac{1}{2}} \right.\right.$$

$$\left.\left. \times \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \exp\left\{-\frac{x_1^2}{t_1} - \ldots - \frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}}\right\} dx_1 \ldots dx_n \right\} \right|$$

where the Δ is taken for a change of δ in t_k. As the expression under the Δ is a continuous function of the t's (so long as no two are equal), the left-hand side of (36) vanishes with δ. Combining this fact with (34), we see that (27) is valid for all continuous bounded functionals of f that depend only on $f(t_1), \ldots, f(t_n)$. By a continuation of this process we can make (27) hold when none of the t_j's need be terminating binaries.

Now let F be *any* bounded continuous functional of time-paths. Let us write

(37) $$F_{t_1, \ldots, t_n}\{f\} = F\left\{\Psi(t_1, \ldots, t_n)\big((t_1), \ldots, f(t_n)\big)\right\}.$$

Let Φ equal 0 on σ_h and F elsewhere, and let $\Phi_{t_1, \ldots, t_n}$ equal to 0 on σ_h and $F_{t_1, \ldots, t_n}$ elsewhere. Each of these functionals is summable by a theorem of Daniell,* being the "logical product" of a summable functional such as

* *A General Form of Integral*, p. 280.

F and the limit of a denumerable set of step-functionals—*i.e.* the functional which is 0 on σ_h and $\max|F|$ elsewhere.* Furthermore

$$\lim_{\max(t_{i+1}-t_i)\to 0} \Phi_{t_1,\ldots,t_n} = \Phi. \tag{38}$$

Hence, as the $\Phi_{t_1,\ldots,t_n}$'s form a bounded set,†

$$\lim_{\max(t_{i+1}-t_i)\to 0} A\{\Phi_{t_1,\ldots,t_n}\} = A\{\Phi\}. \tag{39}$$

Moreover, $|A\{F-\Phi\}|$ and $|A\{F_{t_1,\ldots,t_n}-\Phi_{t_1,\ldots,t_n}\}|$ both exist, and are both less than or equal to $\max|F|.A\{R\}$ {Cf. 32)}. Hence both these quantities vanish as h becomes infinite.

From these facts it may be concluded that if F be a bounded continuous functional,

$$\begin{aligned} A\{F\} = \lim_{\max(t_{i+1}-t_i)\to 0} \pi^{-(n/2)} \prod_1^n (t_k-t_{k-1})^{-\frac{1}{2}} \\ \times \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F\{\{\Psi(t_1,\ldots,t_n)(x_1,\ldots,x_n)\}(t)\} \\ \times \exp\left\{-\frac{x_1^2}{t_1}-\cdots-\frac{(x_n-x_{n-1})^2}{t_n-t_{n-1}}\right\} dx_1 \ldots dx_n. \end{aligned} \tag{40}$$

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* *Ibid.*, 7 (5).
† *Ibid.*, 7 (7).

ON DIRICHLET MULTIPLICATION OF INFINITE SERIES.

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In a former paper I have discussed the convergence of the Cauchy product of semi-convergent series.* The general conditions deduced in that paper are applicable to series with complex terms and to the product of any number of such series.† In the present paper a further advantage of these general conditions is established; for it is proved that these conditions are sufficient to ensure the convergence not only of the Cauchy product but also of the more general Dirichlet product. This theorem thus resembles the three classical theorems on the multiplication of series (Cauchy's, Mertens, and Abel's), because these theorems also hold for all types of Dirichlet multiplication and not merely for the Cauchy type.‡

The results previously obtained have been simplified in form as well as generalised in their application. Although the new conditions of convergence differ only slightly in form from those of my former paper, they possess the advantages of being more symmetrical and also somewhat less stringent. It will be proved that the following conditions are sufficient to ensure the convergence of any type of Dirichlet product of the convergent series Σa_r, Σb_r, ..., Σh_r, Σk_r.

$$\left.\begin{array}{llll} \left|\sum_n^\infty a_r\right| . \sum_1^n |b_r| & \ldots \sum_1^n |k_r| & \to 0 \\ \left|\sum_n^\infty b_r\right| . \sum_1^n |c_r| & \ldots \sum_1^n |a_r| & \to 0 \\ \ldots \quad \ldots & \ldots \quad \ldots & \\ \left|\sum_n^\infty k_r\right| . \sum_1^n |a_r| & \ldots \sum_1^n |h_r| & \to 0 \end{array}\right\} \qquad (1)$$

as $n \to \infty$.

* *Proc. London Math. Soc.*, Ser. 2, Vol. 19, pp. 57–74.

† See conditions (11) of the paper referred to, Theorem C is not, in general, true for Dirichlet multiplication; for in the more general type of product conditions (15), though sufficient, are no longer necessary conditions of convergence.

‡ See *The General Theory of Dirichlet Series*, by Hardy and Riesz, p. 63.

In addition to the extension of previous results to the case of Dirichlet multiplication the present paper also contains a generalisation of a theorem due to Hardy and Littlewood.* These authors have proved that the Cauchy product of any number of convergent series converges if the nth term of each series is of the order n^{-1}. Their proof is not of a very direct character as it depends on Cesàro's method of summation. When I communicated a more elementary proof of this theorem to Prof. Hardy he suggested that a similar argument might establish a more general result. He suggested that the Dirichlet product of type λ of any number of series converges if the nth term of each series is of order $(\lambda_n - \lambda_{n-1})/\lambda_n$. This I found to be the case.

Thus it will be established that there are two kinds of condition which ensure the convergence of the Dirichlet product of semi-convergent series regardless of the number of such series involved. The first kind refers to the order of the remainder after n terms of each of the several series. If this order is in each case that of $n^{-1}(\log n)^{\alpha}$ the product series converges for all types of Dirichlet multiplication.† The second kind refers to the order of the nth term of each series and has been already stated.

We shall begin with certain definitions and an explanation of the notation adopted.

Dirichlet Multiplication.

We are led to the Cauchy rule for the multiplication of the infinite series Σa_r, Σb_r, ..., Σk_r by arranging the formal product of the power series $\Sigma a_r x^r$, ..., $\Sigma k_r x^r$ in powers of x and then putting $x = 1$. In like manner we are led to the Dirichlet rule of multiplication by arranging the formal product of the Dirichlet series

$$\Sigma a_r e^{-\alpha_r s}, \ \ldots, \ \Sigma k_r e^{-\kappa_r s}$$

in ascending order of the sums $\alpha+\beta+\ldots+\kappa$ and then putting $s = 0$.‡ The resulting product is called the Dirichlet product of type $(\alpha, \beta, \ldots, \kappa)$.§

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 11 (1912).

† It is easily proved that when the remainders are of this order that conditions (i) are satisfied. See below, page 477.

‡ See Landau's *Handbuch*, p. 751.

§ For the purpose of the present paper it is convenient to frame a definition in terms of *several* series. When $\alpha_r = \beta_r = \ldots = \kappa_r = \lambda_r$ the product may be described more briefly as of type (λ).

Obviously Cauchy multiplication is the special case of Dirichlet multiplication corresponding to

$$a_r = \beta_r = \ldots = \kappa_r = r.$$

Explanation of Notation.

The symbol (a) is used instead of the words "the suffix of a."

$S_m(a \ldots k)$ denotes the partial sum of the Dirichlet product of type $(a, \beta, \ldots, \kappa)$ of the convergent series $\Sigma a_r, \Sigma b_r, \ldots, \Sigma k_r$ corresponding to

$$a+\beta+\ldots+\kappa \leqslant m.$$

$S_m(a \ldots k)$ is thus the sum of those terms of the multiple series $\Sigma a_A b_B \ldots h_H k_K$ for which

$$a_A+\beta_B+\ldots+\kappa_K \leqslant m.$$

$G_n^m(a \ldots e\,;\ f \ldots k)$ denotes the sum of those terms of $S_m(a \ldots k)$ for which

$$(a) > n, \ \ldots, \ (e) > n, \ (f) \leqslant n, \ \ldots, \ (k) \leqslant n.$$

$G_n^m(a \ldots e\,;\ f_F \ldots k_K)$ denotes the sum of those terms of $S_m(a \ldots k)$ for which $(a) > n, \ldots, (e) > n, \ (f) = F, \ldots, (k) = K.$

$\Gamma_n^m(a \ldots e\,;\ f \ldots k)$ is the sum of that group of terms of $S_m(a \ldots k)$ which satisfy the conditions $(a) > n, \ldots, (e) > n.$

$\rho_n(a)$ denotes the upper bound of $|\sum_v^\infty a_r|, \quad v \geqslant n.$

$\rho_n(a\,;\ b \ldots e)$ denotes the upper bound of $\rho_v(a) \sum_1^v |b_r| \ldots \sum_1^r |e_r|, \quad v \geqslant n.$

In order to economise the notation we shall, after the manner of Borel, use the same letter R in a whole series of inequalities to denote a positive constant, not necessarily the same in all inequalities in which it occurs. For the same reason a single symbol will be used to denote both a group of terms and the sum of the group.

Before dealing with the problem of the convergence of the product of several series we shall examine, in some detail, conditions for the convergence of the product of two series.

Consider the Dirichlet product of type (a, β) of the convergent series $\Sigma a_r, \Sigma b_r$. $S_\mu(ab)$ denotes in our notation the sum of those terms of the double series $\Sigma a_r b_s$ for which $a+\beta \leqslant \mu$. Select a term $a_m b_n$ of $S_\mu(ab)$

such that neither $a_{m+1}b_n$ nor $a_m b_{n+1}$ belong to $S_\mu(ab)$ and write

$$S_\mu(ab) - \sum_1^m a_r \sum_1^n b_r = D_\mu. \tag{2}$$

Take a pair of positive integers u and v, such that $u \leqslant m$, $v \leqslant n$, and consider the group of terms of D_μ for which $(a) \leqslant u$. If $r \leqslant u$ the sum of those terms of D_μ which possess the factor a_r is

$$a_r(b_{n+1} + \ldots + b_e),$$

where $\alpha_r + \beta_e \leqslant \mu < \alpha_r + \beta_{e+1}$. Clearly the modulus of this sum is not greater than $2\rho_n(b)|a_r|$. It readily follows that the modulus of the sum of those terms of D_μ for which $(a) \leqslant u$ is not greater than

$$2\rho_n(b) \sum_1^u |a_r|.$$

Similarly the modulus of the sum of those terms of D_μ for which $(b) \leqslant v$ is not greater than

$$2\rho_m(a) \sum_1^v |b_r|.$$

Consider now those terms of D_μ for which $(a) > u$ and $(b) > n$. By reasoning similar to that used above it is easily proved that the modulus of their sum is not greater than

$$2\rho_u(a) \sum_{n+1}^N |b_r|,$$

where $$\alpha_u + \beta_N \leqslant \mu < \alpha_u + \beta_{N+1}.$$

In like manner the modulus of the sum of those terms of D_μ for which $(b) > v$ and $(a) > m$ is not greater than

$$2\rho_v(b) \sum_{m+1}^M |a_r|,$$

where $$\alpha_M + \beta_v \leqslant \mu < \alpha_{M+1} + \beta_v.$$

Combining the results we have

$$|D_\mu| \leqslant 2\left[\rho_n(b) \sum_1^u |a_r| + \rho_m(a) \sum_1^v |b_r| + \rho_r(b) \sum_{m+1}^M |a_r| + \rho_u(a) \sum_{n+1}^N |b_r|\right], \tag{3}$$

because the classifications of the terms of D_μ carried out above clearly dissect D_μ into four mutually exclusive classes.

Suppose now that m and n are selected so that*

$$a_m \asymp \beta_n \quad \text{as} \quad (m, n) \to \infty .$$

Since Σa_r, Σb_r converge,

$$\rho_n(b) \to 0, \quad \rho_m(a) \to 0 \quad \text{as} \quad (m, n) \to \infty .$$

Hence we can select u and v as functions of n and m respectively so that

$$\rho_n(b) \sum_1^u |a_r| \to 0, \quad \rho_m(a) \sum_1^v |b_r| \to 0,$$

when $(m, n) \to \infty$ and $(u, v) \to \infty$.

Since $\rho_v(b) \to 0, \quad \rho_u(a) \to 0 \quad \text{as} \quad (u, v) \to \infty$,

it follows from (3) that $D_\mu \to 0 \quad \text{as} \quad \mu \to \infty$

if $\sum_{m+1}^{M} |a_r| \quad \text{and} \quad \sum_{n+1}^{N} |b_r|$

remain bounded as $(m, n) \to \infty$. This is secured if the two following conditions are satisfied:

$$a_n = O\left(\frac{a_n - a_{n-1}}{a_n}\right), \qquad b_n = O\left(\frac{\beta_n - \beta_{n-1}}{\beta_n}\right). \tag{4}$$

For we have when these conditions are satisfied

$$\begin{aligned} \sum_{m+1}^{M} |a_r| &= O\{(a_{m+1} - a_m)/a_m + \ldots + (a_M - a_{M-1})/a_M\} \\ &= O\{(a_M - a_m)/a_m\} \\ &= O(a_M/a_m) \\ &= O(1), \end{aligned}$$

since $a_m \ll a_M \ll \mu \quad \text{and} \quad a_m \asymp \beta_n \asymp a_m + \beta_n \asymp \mu.$

Similarly $\sum_{n+1}^{N} |b_r| = O(1).$

Thus $S_\mu(ab) \to \sum_1^\infty a_r \cdot \sum_1^\infty b_r$ as $\mu \to \infty$ when the equations (4) are satisfied.

* The symbol $\asymp$ implies that $a_m/\beta_n < R$ and $\beta_n/a_m < R$ as $(m, n) \to \infty$. See Hardy's *Orders of Infinity*. By the notation $(m, n) \to \infty$ we mean that m and n tend to infinity steadily, each passing through all integer values.

Hence the Dirichlet product of type (α, β) converges when these conditions are fulfilled.*

By taking $u = m$ and $v = n$ it is easily seen that (3) becomes

$$|D_\mu| \leqslant 2\rho_n(b) \sum_1^m |a_r| + 2\rho_m(a) \sum_1^n |b_r|$$

and hence $$S_\mu(ab) \to \sum_1^\infty a_r . \sum_1^\infty b_r \quad \text{as} \quad \mu \to \infty,$$

if $$\left| \sum_m^\infty a_r \right| . \sum_1^n |b_r| \to 0, \qquad \left| \sum_n^\infty b_r \right| . \sum_1^m |a_r| \to 0 \tag{5}$$

as $(m, n) \to \infty$. Thus any type of Dirichlet product of Σa_r, Σb_r converges if the conditions (5) are satisfied as $(m, n) \to \infty$. In particular any type of Dirichlet product of Σa_r and Σb_r converges if

$$\left| \sum_1^n a_r \right| . \sum_1^n |b_r| \to 0, \qquad \left| \sum_1^n b_r \right| . \sum_1^n |a_r| \to 0, \quad \text{as} \quad n \to \infty. \tag{6}$$

We shall now proceed to consider the multiplication of several series. In order to establish the sufficiency of conditions (1) the following theorem will be required.

Theorem (A).

$$|G_n^m(a \dots e;\ f_F \dots k_K)| < R[\rho_n(a;\ b \dots e) + \dots + \rho_n(e;\ a \dots d)]|f_F| \dots |k_K|$$

where R is a positive constant depending only on the series $\Sigma a_r, \dots, \Sigma e_r$, and independent of m and n.

It is obviously an immediate consequence of this theorem that

$$|G_n^m(a \dots e;\ f \dots k)| < R[\rho_n(a;\ b \dots e \dots k) + \dots + \rho_n(e;\ a \dots d \dots k)]$$

for all values of m and n.

Denote the number of letters in the first set of letters in the symbol G_n^m, namely $a \dots e$, by p. We shall prove (A) by proving (i) that (A) holds when $p = 1$, and (ii) that the truth of (A) for $p < p'$ implies its truth for $p < p' + 1$.

* Hardy proved this theorem on the assumption that

$$\alpha_n - \alpha_{n-1} = O(\alpha_n), \quad \beta_n - \beta_{n-1} = O(\beta_n).$$

Proc. London Math. Soc., Ser. 2, Vol. 10 (1911), pp. 396–405. Rosenblatt, *Jahresbericht der deutschen Math. Vereinegung*, Vol. 23 (1914), pp. 80–84, shows that this restriction is unnecessary.

The fundamental characteristic of Dirichlet multiplication on which a great deal of the argument of the present discussion rests can be expressed thus: if $a_A \dots k_K$ belongs to the partial sum $S_m(a \dots k)$ of the Dirichlet product of the series Σa_r, ..., Σk_r, then $a_{A'} \dots k_{K'}$ also belongs to this partial sum if

$$A \geqslant A', \ \dots, \ K \geqslant K'.$$

This readily follows from the definition of Dirichlet product of type $(\alpha, \beta, \dots \kappa)$ when we bear in mind the monotone character of the functions of r, a_r, β_r, ..., κ_r. For the value of $\alpha+\beta+\dots+\kappa$ corresponding to the suffixes A', B', ..., K' is not greater than its value corresponding to the suffixes A, B, ..., K.

If then $a_A \dots e_E \dots f_F$ belongs to $G_n^m(a;\ b_B \dots k_K)$ $a_{A'} \dots e_E \dots k_K$ also belongs to this group when $A \geqslant A' > n$. Denote by N the greatest value of (a) in terms of the group $G_n^m(a;\ b_B \dots k_K)$. Then clearly

$$G_n^m(a;\ b_B \dots k_K) = \left(\sum_{n+1}^{N} a_r \right) b_B \dots k_K.$$

But
$$\left| \sum_{n+1}^{N} a_r \right| \leqslant 2\rho_n(a).$$

Hence
$$|G_n^m(a;\ b_B \dots k_K)| \leqslant 2\rho_n(a) |b_B| \dots |k_K|.$$

Thus (A) is established for $p = 1$.

To prove (ii).

We shall consider the value of $|G_n^m(a \dots et;\ f_F \dots k_K)|$ when $p = p'$ assuming that (A) holds when $p < p'$.

Denote by n' the greatest value of i for which $a_i \dots e_i t_i f_F \dots k_K$ is a member of the group $G_n^m(a \dots et;\ f_F \dots k_K)$ and write

$$G_n^m(a \dots et;\ f_F \dots k_K) = Q + \left(\sum_{n+1}^{n'} a_r \dots \sum_{n+1}^{n'} e_r \sum_{n+1}^{n'} t_r \right) f_F \dots k_K.^*$$

Clearly every term of

$$\left(\sum_{n+1}^{n'} a_r \dots \sum_{n+1}^{n'} e_r \cdot \sum_{n+1}^{n'} t_r \right) f_F \dots k_K$$

belongs to the group of terms $G_n^m(a \dots et;\ f_F \dots k_K)$, and therefore Q is the sum of those terms of this group for which at least one of the p' numbers (a), ..., (e), (t) is greater than n' and one at least is less than $n'+1$. These results follow readily from the definition of n' and from the fundamental characteristic of Dirichlet multiplication mentioned above.

* If $n' = n$, $G_n^m(a \dots et;\ f_F \dots k_K) = Q$.

Consider a sub-group of the group of terms Q. Form this sub-group so that for each member of it*

$$(a) > n',\ \ldots,\ (d) > n',\ \ldots,\ (e) \leqslant n',\ (t) \leqslant n'.$$

If this sub-group be denoted by $Q(a \ldots d)$ it is easily seen that

$$Q(a \ldots d) = G_{n'}^{m'}(a \ldots d;\ \ldots et) f_F \ldots k_K$$

where $$m' = m - (\phi_F + \ldots + \kappa_K).$$

Since the number of letters in the set $a \ldots d$ is less than p' it follows from our assumption that (A) holds when $p < p'$ that

$$|G_{n'}^{m'}(a \ldots d;\ \ldots et)| < R[\rho_{n'}(a;\ b \ldots et) + \ldots + \rho_{n'}(d;\ a \ldots et)],$$

R being a positive constant independent of m and n. Hence from

$$|Q(a \ldots d)| < R[\rho_n(a;\ b \ldots et) + \ldots + \rho_n(d;\ a \ldots et)]\,|f_F| \ldots |k_K|$$

for $n' \geqslant n$. Now Q can evidently be divided into a finite number of mutually exclusive groups of terms of which $Q(a \ldots d)$ may be regarded as typical. Hence we conclude that

$$|Q| < R[\rho_n(a;\ b \ldots et) + \ldots + \rho_n(t;\ a \ldots e)]\,|f_F| \ldots |k_K|.$$

Moreover it is readily seen that

$$\left| \left(\sum_{n+1}^{n'} a_r \ldots \sum_{n+1}^{n'} t_r \right) f_F \ldots k_K \right| < R[\rho_n(a;\ b \ldots t)]\ |f_F| \ldots |k_K|$$

where R is a constant independent of n. Therefore

$$|G_n^m(a \ldots t;\ f_F \ldots k_K)| < R[\rho_n(a;\ b \ldots t) + \ldots + \rho_n(t;\ a \ldots e)]\,|f_F| \ldots |k_K|$$

for all values of m and n. Thus (A) is established.

Convergence of the Dirichlet product.

n being the greatest value of i for which $a_i \ldots k_i$ belongs $S_m(a \ldots k)$, write

$$S_m(a \ldots k) - \sum_1^n a_r \ldots \sum_1^n k_r = D_m. \qquad (7)$$

Evidently the Dirichlet product converges if $D_m \to 0$ as $m \to \infty$. Now D_m is the sum of those terms of $S_m(a \ldots k)$ for which at least one of the

* In order to avoid introducing too many letters, $a \ldots d$ are taken to represent any number of letters and not merely four. Similarly, $\ldots$, (e), (t) may represent several letters.

numbers (a), ..., (k) is greater than n and at least one is less than $n+1$. Thus

$$D_m = \Sigma G_n^m, \tag{8}$$

where the summation extends over all the G_n^m's got by dividing the set of letters $a, \ldots, k$ into two groups, there being always at least one letter in the first group and at least one in the second group. This is so because a term of D_m must be a member of one at least of the G_n^m groups (for it must have at least one suffix greater than n) and it cannot belong to more than one of the groups. Moreover a term of G_n^m cannot be contained in

$$\sum_1^n a_r \ldots \sum_1^n k_r$$

and so must belong to D_m.

Since D_m can be divided into a finite number of mutually exclusive groups of terms of which the group $G_n^m(a \ldots e\,;\, f \ldots k)$ may be regarded as typical we infer from (A) that

$$|D_m| < R[\rho_n(a\,;\ b \ldots k) + \ldots + \rho_n(k\,;\ a \ldots h)]$$

for all values of n. Hence $D_m \to 0$ as $m \to \infty$ if the following conditions are satisfied

$$\left.\begin{aligned} &\rho_n(a) \sum_1^n |b_r| \ldots \sum_1^n |k_r| \to 0 \\ &\rho_n(b) \sum_1^n |c_r| \ldots \sum_1^n |a_r| \to 0 \\ &\quad \ldots \qquad\quad \ldots \qquad\quad \ldots \\ &\rho_n(k) \sum_1^n |a_r| \ldots \sum_1^n |h_r| \to 0 \end{aligned}\right\} \tag{9}$$

as $n \to \infty$.

Thus the conditions (1) are sufficient to ensure the convergence of the Dirichlet product because the conditions (1) and (9) are equivalent. To prove this equivalence it is only necessary to bear in mind the definitions of $\rho_n(a)$, ..., $\rho_n(k)$ and to observe that $\sum_1^n |a_r|$, ..., $\sum_1^n |k_r|$ are monotone non-decreasing functions of n.

In particular it follows that the q-th Dirichlet power of the convergent series Σa_r converges if

$$\left|\sum_n^\infty a_r\right| \left\{\sum_1^n |a_r|\right\}^{q-1} \to 0 \quad \text{as} \quad n \to \infty.$$

Applications.

As a first application of the convergence theorem which has been established consider the convergence of the q-th "ordinary" Dirichlet power of the series

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\ldots.$$

Since in this case

$$\sum_{n}^{\infty} a_r = O(n^{-1}) \quad \text{and} \quad \sum_{1}^{n} |a_r| = O(\log n)$$

it is easily seen that the condition (10) is satisfied for all integer values of q. Thus every Dirichlet power of this series converges. This theorem is a special case of the following more general theorem.

If the n-th remainder of each of a number of series is of the order $n^{-1}(\log n)^{\alpha}$ every Dirichlet product of these series converges.

To prove this it is only necessary to observe that if

$$\sum_{n+1}^{\infty} a_r = O\{n^{-1}(\log n)^{\alpha}\}$$

then

$$|a_n| = \left| \sum_{n}^{\infty} a_r - \sum_{n+1}^{\infty} a_r \right| = O\{n^{-1}(\log n)^{\alpha}\}$$

and

$$\sum_{1}^{n} |a_r| = O\{(\log n)^{\alpha+1}\}.$$

Hence series $\Sigma a_r, \ldots, \Sigma k_r$, whose remainders are of the order specified above, satisfy conditions (1), since

$$n^{-1}(\log n)^{k} \to 0 \quad \text{as} \quad n \to \infty$$

for all values of k.

Consider next the convergence of the q-th Dirichlet power of

$$\Sigma\,(-)^{n+1} n^{-s}.$$

In this case we have, obviously,

$$\left| \sum_{n}^{\infty} (-)^{r+1} r^{-s} \right| = O(n^{-\alpha}), \qquad \sum_{1}^{n} |n^{-s}| = O(n^{1-\alpha})$$

where α is the real part of s (assumed positive). Hence the q-th Dirichlet power converges if

$$\alpha > (q-1)/q.$$

We shall now prove the following theorem:

The Dirichlet product of type λ *converges if the* n*-th term of each series is of the order* $(\lambda_n-\lambda_{n-1})/\lambda_n$.

The proof will depend on the following theorem:

THEOREM B.—

$$\Gamma_n^m(a\dots e;\ f\dots k)\to 0 \quad \text{as} \quad n\to\infty$$

when $a_n, b_n, \dots, k_n$ *are each of the order* $(\lambda_n-\lambda_{n-1})/\lambda_n$ *and* m *is of the order* λ_n.

It will first be proved that the convergence theorem follows from (B). From our definitions it is easily seen that the G_n^m's and the Γ_n^m's are connected by a series of linear equations of which the following are typical.

$$G_n^m(a\dots h;\ k)=\Gamma_n^m(a\dots h;\ k)$$

$$G_n^m(a\dots g;\ hk)=\Gamma_n^m(a\dots g;\ hk)-G_n^m(a\dots gh;\ k)-G_n^m(a\dots gk;\ h)$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$G_n^m(a\dots e;\ f\dots k)=\Gamma_n^m(a\dots e;\ f\dots k)-\Sigma G_n^m$$

where the summation on the last equation extends over all the G_n^m's got by taking all possible combinations of the letters $f\dots k$ as the second set of letters in the symbol G_n^m, one of these letters at least being omitted in each combination.

The above system of equations, with similar ones, constitute a series of reduction formulæ which enable to express any G_n^m, for example $G_n^m(a\dots e;\ f\dots k)$ linearly in terms of the Γ_n^m's. For these equations enable us to reduce step by step the number of letters in the second set of letters of the G_n^m's in terms of which $G_n^m(a\dots e;\ f\dots k)$ is expressed. D_m of equations (7) and (8) can therefore be expressed linearly in terms of the Γ_n^m's, and we can write

$$S_m(a\dots k)-\sum_1^n a_r\dots\sum_1^n k_r=\Sigma\Gamma_n^m.$$

Assume now that the multiplication is of type λ, and moreover that we have

$$a_n=O\{(\lambda_n-\lambda_{n-1})/\lambda_n\};\ \dots,\ k_n=O\{(\lambda_n-\lambda_{n-1})/\lambda_n\}. \qquad (10)$$

It is readily seen that $m = O(\lambda_n)$ and hence if (B) holds

$$S_m(a \dots k) \to \sum_1^\infty a_r \dots \sum_1^\infty k_r$$

as $m \to \infty$. Thus the truth of (B) implies the convergence of the product of type λ.

Theorem (B) is proved by induction. Denote by p the number of letters of the first set occurring in the symbol $\Gamma_n^m(a \dots e\,;\, f \dots k)$, and let q denote the number of letters in the second set. To establish (B) we shall prove (i) that it holds when $p = 1$ and $q = 1$, and (ii) that the truth of (B) for $p < p'$, $q < q'$ implies its truth when $p = p'$ and $q = q'$, p' and q' being any pair of positive integers.

To prove (i).

Making use of the analysis given above in discussing the convergence of the product of a pair of series it is readily seen that

$$|\Gamma_n^m(b\,;\, a)| \leqslant 2\Big[\rho_u(a) \sum_{n+1}^N |b_r| + \rho_n(b) \sum_1^u |a_r|\Big],$$

where $$\lambda_u + \lambda_N \leqslant m < \lambda_u + \lambda_{N+1}, \quad u \leqslant n.$$

Since the conditions (10) are assumed we have

$$\sum_{n+1}^N |b_r| = O(\lambda_N/\lambda_n).$$

Hence, if $m = O(\lambda_n)$, $\lambda_N = O(\lambda_n)$

and $$\sum_{n+1}^N |b_r| = O(1).$$

Moreover $\rho_u(a) \to 0$ as $u \to \infty$, $\rho_n(b) \to 0$ as $n \to \infty$.

Hence $\Gamma_n^m(b\,;\, a) \to 0$ as $n \to \infty$,

when $m = O(\lambda_n)$.

To prove (ii).

We assume that (B) holds for $p < p'$, $q < q'$. Suppose also that the conditions (10) are satisfied and that $m = O(\lambda_n)$. Consider

$$\Gamma_n^m(ab \dots e\,;\, f \dots k),$$

where there are p' letters in the set $ab \dots e$ and r letters in the set $f \dots k$, r being less than q'.

The sum of these terms of $\Gamma_n^m(ab\dots e;\ f\dots k)$ which possess a_A as a factor is

$$a_A\,\Gamma_n^{m-\xi}(b\dots e;\ f\dots k)$$

where $\xi = \lambda_A$. From our assumption it follows that

$$\Gamma_n^{m-\xi}(b\dots e;\ f\dots k)\to 0 \quad \text{as} \quad n\to\infty$$

since $m-\xi = O(\lambda_n)$. Denoting by ϵ_n the upper bound of $|\Gamma_n^{m-\xi}|$ as A takes on all possible values we have

$$|\Gamma_n^m(ab\dots e;\ f\dots k)| \leqslant \epsilon_n \sum_{n+1}^{N'} |a_r|$$

where $$\lambda_{N'}+(p'-1)\lambda_n \leqslant m < \lambda_{N'+1}+(p'-1)\lambda_n.$$

Now $$\sum_{n+1}^{N'} |a_r| = O(\lambda_{N'}/\lambda_n)$$

and $$\lambda_{N'} = O(m) = O(\lambda_n).$$

Hence $$\sum_{n+1}^{N'} |a_r| = O(1).$$

Therefore $$\Gamma_n^m(ab\dots e;\ f\dots k)\to 0 \quad \text{as} \quad n\to\infty.$$

The assumptions made above therefore imply the truth of (B) for $p = p'$, $q < q'$, and hence for any value of p combined with any value of q less than q'. It also follows that these assumptions ensure the convergence of the product of type λ of q convergent series which satisfy the conditions (10) provided q is less than q'.

Now consider $\Gamma_n^m(a;\ b\dots k)$. Here $p = 1$ and we assume $q = q'$. Delete from $\Gamma_n^m(a;\ b\dots k)$ those terms of the multiple series $\Sigma a_A\dots k_K$ for which

$$\lambda_{(b)}+\dots+\lambda_{(k)} \leqslant \lambda_w < m-q'\lambda_n.$$

Denote this group of terms by W. Clearly, if $a_A b_B\dots k_K$ belongs to W, $a_{A'} b_B\dots k_K$ also belongs to W when $A \geqslant A' > n$. Hence the sum of those terms of W which involve the factor $b_B\dots k_K$ is

$$b_B\dots k_K(a_{n+1}+\dots+a_v)$$

where v is the greatest value of (a) which occurs in terms involving $b_B\dots k_K$. Now the modulus of this sum is not greater than

$$2|b_B|\dots|k_K|\rho_n(a).$$

Grouping the other terms of W similarly we conclude that

$$|W| \leqslant 2\rho_n(a) \sum_1^w |b_r| \dots \sum_1^w |k_r|.$$

Choose w as a function of n in such a way that $w \to \infty$ as $n \to \infty$ while

$$\rho_n(a) \sum_1^w |b_r| \dots \sum_1^w |k_r| \to 0 \quad \text{as } n \to \infty.$$

Consider the terms of $\Gamma_n^m - W$. Grouping together the terms involving a_t, where $t > n$ it is easily seen that their sum is

$$a_t [S_{m-\tau}(b \dots k) - S_\omega(b \dots k)]$$

where $\tau = \lambda_t$ and $\omega = \lambda_w$.

Clearly $\lambda_t \leqslant m - \lambda_w$, otherwise a_t would occur only in terms of W and therefore it could not occur as a factor in terms of $\Gamma_n^m - W$. Hence

$$m - \tau = m - \lambda_t \geqslant \lambda_w = \omega,$$

and we may write $m - \tau = \omega + \theta$, where θ is positive. Now since the series $\Sigma b_r, \dots, \Sigma k_r$ are in number $q' - 1$ their product of type λ converges and therefore

$$S_\omega(b \dots k) \to \sum_1^\infty b_r \dots \sum_1^\infty k_r$$

as $n \to \infty$. Hence

$$|S_{\omega+\theta}(b \dots k) - S_\omega(b \dots k)| \to 0$$

as $n \to \infty$. If then ϵ_ω denotes the upper bound of $|S_{\omega+\theta} - S_\omega|$ for $\theta \geqslant 0$

$$|a_t \{S_{m-\tau} - S_\omega\}| \leqslant |a_t| \epsilon_\omega.$$

Grouping the other terms of $\Gamma_n^m - W$ similarly we have

$$|\Gamma_n^m - W| \leqslant \epsilon_\omega \sum_{n+1}^{N''} |a_t|$$

where $$\lambda_{N''} = O(m) = O(\lambda_n).$$

Hence $$|\Gamma_n^m - W| \to 0 \quad \text{as} \quad n \to \infty.$$

But $$W \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore $$\Gamma_n^m \to 0 \quad \text{as} \quad n \to \infty.$$

Thus the truth of (B) for $p = 1$, $q < q'$ implies its truth for $p = 1$ and

$q = q'$ and therefore for $p = p'$ and $q = q'$.* Thus (ii) is proved and hence (B) is completely established.

Example.

As an example of the second theorem of convergence for the product of several series consider the q-th Dirichlet power of type λ of the series

$$\Sigma a_r \equiv \frac{1}{2 \log 2} \pm \frac{1}{3 \log 3} \pm \frac{1}{4 \log 4} \pm \cdots \pm \frac{1}{(n+1) \log (n+1)} \pm \cdots,$$

where the ambiguities of sign are determined in any manner which renders the series convergent.

Suppose $$\lambda_n = \log n.$$

Then clearly $$a_n = O\{(\lambda_n - \lambda_{n-1})/\lambda_n\},$$

and therefore every Dirichlet power of type λ converges.

* See above, pp. 479, 480.

NOTES ON THE USE OF THE SCHWARZ-CHRISTOFFEL TRANSFORMATION IN ELECTROSTATICS (AND HYDRODYNAMICS)

By H. W. RICHMOND.

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1.1. The Schwarz-Christoffel transformation* has the great merit that it provides a direct method which *must* lead to the solution of problems of certain types: we are not dependent upon a lucky guess of the suitable formula. It has the defect that the problems to which it can be applied (in which the boundaries of some of the conductors, or some of the lines of force and equipotentials must be straight lines)† are commonly not the most natural or the most important. A protruding angle on a conductor implies an infinite density of the distribution as given by a mathematical formula; in hydrodynamics it implies an infinite velocity and is inadmissible: such angles are hardly to be avoided if we are limited to rectilinear boundaries.

1.2. In certain particular problems pertaining to regions with curved boundaries, it is possible, by an adaptation of the Schwarz-Christoffel method, to know that the solution *can* be obtained, if ever it is thought worth while. Some of these appear to be of interest and are solved in the following paragraphs. If the result is complicated, at least we shall have learned that no other method can give an essentially different one: the formula may be bettered in details, but not intrinsically changed.

1.3. Suppose that a region with a curved boundary can be converted into a rectilinear polygon by a known conformal transformation; the Schwarz-Christoffel transformation, applied to the polygon, will map it

* The principles and many applications to two-dimensional problems in electrostatics are explained very fully in Sir J. J. Thomson's *Recent Researches in Electricity and Magnetism* (1893), Chapter 3; also Lamb, *Hydrodynamics*, p. 86; Darboux, *Théorie des Surfaces*, Vol. 1, pp. 176–180; Forsyth, *Theory of Functions* (2nd edition), pp. 632–643, and many papers in periodicals.

† The conductors are cylindrical, generated by lines parallel to Oz, but they and the whole field are defined by the section made by a perpendicular plane xOy: hence the terms of plane geometry are used.

upon a real axis, and the curvilinear area is thus mapped (by two stages) conformally upon a half-plane. Its electrostatic properties follow in the usual way. In using this obvious principle we here limit it to this: When $z(\equiv x+iy$, where x and y are Cartesian co-ordinates) is a simple function of $w(\equiv u+iv)$, the curves $u = \text{const.}$, $v = \text{const.}$ are familiar to us as equipotentials and lines of force in some electrostatic field. To any curvilinear region in the z-plane, bounded by equipotentials and lines of force, corresponds in the w-plane a rectilinear polygon, bounded by lines parallel either to the u-axis or the v-axis. This we map by Schwarz-Christoffel upon a half-plane. [The mapping is a purely geometrical process: it was *convenient* to describe u and v by reference to a familiar potential; but it was not *necessary*, and need have no connexion with the physical properties of the region discovered later.] The principle to be used may be stated in the form:

Areas bounded by equipotentials and lines of force of a known electrostatic field can be treated by the Schwarz-Christoffel method in the same systematic way as rectilinear polygons. The examples which follow sufficiently explain the process.

2.1. *Three examples from the field* $z = \cosh w$ (*confocal conics*), Fig. 2.1.—The curves $u = \text{const.}$ and $v = \text{const.}$ are confocal ellipses and hyperbolas, the foci S and S' being $(1, 0)$ and $(-1, 0)$. AOA', BOB',

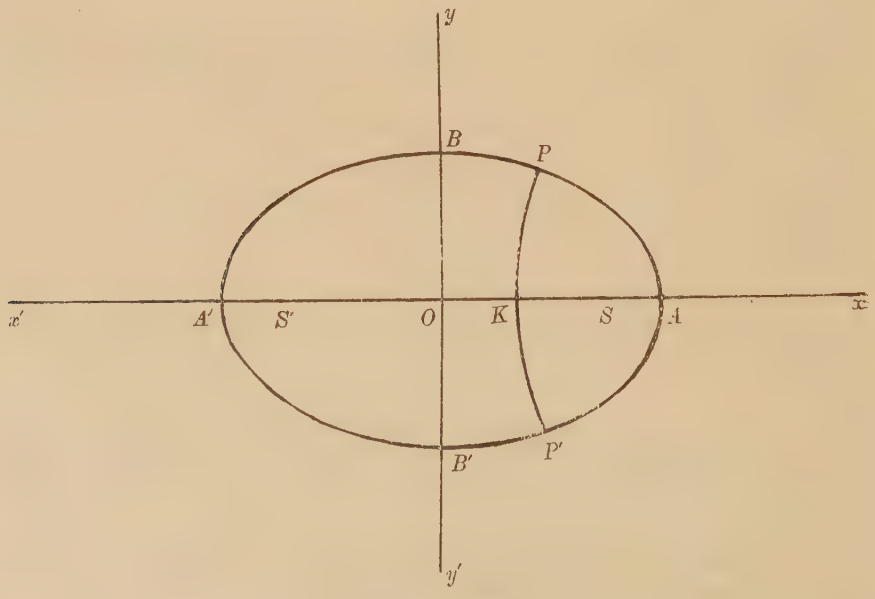

FIG. 2.1 (z-plane).

the major and minor axes of the ellipse $u = c$ are of lengths

$$AA' = 2a = 2\cosh c\,; \quad BB' = 2b = 2\sinh c.$$

Usually u is restricted to positive values, and v to a range from $-\pi$ to π. The curve $u = 0$ is the finite line SS'; $v = 0$ and $v = \pm\pi$ are respectively Sx and $S'x'$; $v = +\frac{1}{2}\pi$ and $v = -\frac{1}{2}\pi$ are the positive and negative sides of yOy'. By the principle of §1.3, it should be possible to find the field due to a charged cylinder whose cross section is any one of the three following shapes:

(i) the line $xABA'x'$ (§2.2);

(ii) the semi-ellipse $OBA'B'O$ (§2.3);

(iii) any symmetrical arc of an ellipse or hyperbola, *i.e.* an arc whose middle point is the end of a principal axis (§2.4);

for in all these the regions that have to be considered are bounded by curves along which either u or v is constant. The first is well known—an infinite plane crossed by a ridge in the form of a semi-ellipse.

2.2.* *Section of the cylinder is $xABA'x'$.*—On xA, $v = 0$ and u diminishes from infinity to c; on ABA', $u = c$ and v increases from 0 to π; on $A'x'$, $v = \pi$ and u increases from c to infinity. The (partly) curved line $xABA'x'$ in the z-plane is represented in the w-plane by the perimeter of the infinite rectangle formed by the lines $v = 0$, $u = c$, $v = \pi$ (Fig. 2.2). We map this on the real t-axis, supposing that t diminishes continuously and that it $= +1$ and -1 at the two corners, by the relation

$$E\frac{dw}{dt} = \frac{1}{(t-1)^{\frac{1}{2}}(t+1)^{\frac{1}{2}}} \tag{2.21}$$

or

$$t = \cosh(Ew+F). \tag{2.22}$$

But $t = 1$ when $w = c$, and $t = -1$ when $w = c+i\pi$; hence

$$t = \cosh(w-c) \tag{2.23}$$

Replacing t by the complex variable $W \equiv U+iV$, and eliminating w

* It is understood that, if t and α are real $(t-\alpha)^{\frac{1}{2}}$ is positive when $t > \alpha$, and is a positive quantity multiplied by i when $t < \alpha$.

from (2.23) and $z = \cosh w$, we find

$$W = z\cosh c - (z^2-1)^{\frac{1}{2}}\sinh c = az - b(z^2-1)^{\frac{1}{2}} \qquad (2.24)$$

or

$$z = W\cosh c + (W^2-1)^{\frac{1}{2}}\sinh c = aW + b(W^2-1)^{\frac{1}{2}}. \qquad (2.25)$$

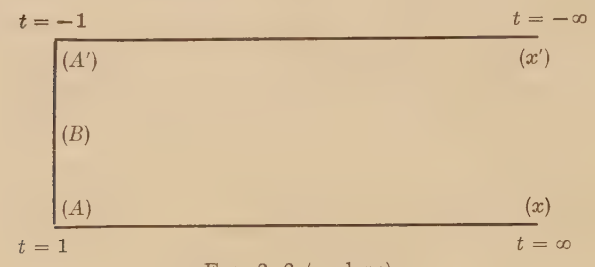

FIG. 2.2 (w-plane).

Here, if $V = 0$ and U diminishes from $+\infty$ to $-\infty$, the point z describes the contour $xABA'x'$. The potential at a point outside the conductor is V.

2.3. *The section of the cylinder is the semi-ellipse $OBA'B'O$.*—As a first step consider the line (curved in part) $xSOBA'x'$; it is represented in the w-plane by the line shown in Fig. 2.3. If the real variable t constantly diminishes and assumes the values α, β, γ, δ at the critical points

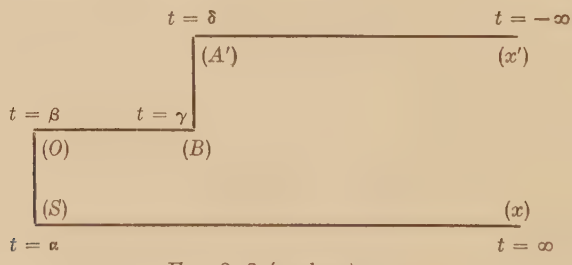

FIG. 2.3 (w-plane).

as indicated, the line $xSOBA'x'$ is mapped upon the real t-axis by the relations

$$z = \cosh w,$$

$$\frac{dw}{dt} = \frac{E(t-\gamma)^{\frac{1}{2}}}{(t-\alpha)^{\frac{1}{2}}(t-\beta)^{\frac{1}{2}}(t-\delta)^{\frac{1}{2}}}, \qquad (2.31)$$

$\alpha > \beta > \gamma > \delta$, and E is real and positive; see footnote, §2.2.

Secondly, consider $W = U + iV$, where U is the potential of the charged semi-elliptic cylinder. Let it be arranged that U shall vanish on the cylinder and be positive outside it, and that V shall increase by 2π

THE LONDON MATHEMATICAL SOCIETY.

BALANCE SHEET, 31st October, 1923.

Liabilities.	£	s.	d.	Assets	£	s.	d.
Lieut.-Col. Campbell's Fund	499	7	5	London, Midland & Scottish Railway £385 4 per Cent. Guaranteed Stock at cost ...	499	7	5
Life Compositions Fund	2806	16	6	India Stock £700. 7*s.* 1*d.* 3 per Cent. at cost	700	0	0
Invested Surplus Fund	700	0	0	Funding Loan £2500 4 per Cent. at cost ...	2000	0	0
Reserve Fund	63	0	1	Conversion Loan £1700 3½ per Cent. at cost	1291	6	6
Donations	317	17	0	Deposit at Bank	400	0	0
Legacy from late Sir A. B. Kempe, F.R.S. ...	200	0	0	Cash at Bank	43	18	10
Sundry Creditors	15	1	0				
De Morgan Medal Fund		1	2				
Revenue Account balance in hand	332	9	7				
	£4934	12	9		£4934	12	9

REVENUE ACCOUNT for the year 1st November, 1922, to 31st October, 1923.

	Year 1921-22.							Year 1921-22.					
	£	s.	d.	£	s.	d.		£	s.	d.	£	s.	d.
Balance from last year	187	15	8	*81*	*19*	*11*	Printing and Circulating *Proceedings*, &c.	751	8	6	*895*	*1*	*3*
Income tax returned	50	17	0	*51*	*7*	*2*	Purchases and Binding for Library ...	129	9	9	*4*	*10*	*9*
Dividends on—							Rent	30	0	0	*26*	*0*	*0*
Lord Rayleigh's Fund ...	34	16	10	*33*	*12*	*8*	Postages and Sundries	10	3	5	*2*	*15*	*8*
The Society's Investments	156	9	9	*135*	*9*	*6*	Balance in hand on 31st October, 1923...	332	9	7	*187*	*15*	*8*
Entrance Fees	46	4	0	*69*	*12*	*0*							
Annual Subscriptions	418	3	0	*384*	*6*	*4*							
Sale of *Proceedings*, &c.	352	5	0	*346*	*5*	*9*							
Advertisements revenue	7	0	0	*7*	*10*	*0*							
	£1253	11	3	*1110*	*3*	*4*		£1253	11	3	*1110*	*3*	*4*

REVENUE ACCOUNT of the DE MORGAN MEDAL FUND.

	£	s.	d.		£	s.	d.
Balance in hand 31st October, 1922 ...	6	0	9	Paid Messrs. Joseph Moore, Ltd., for De Morgan medal and bronze replica ...	9	0	9
Dividends received	2	5	5	Balance in hand 31st October, 1923... ...		1	2
Income tax returned		15	9				
	£9	1	11		£9	1	11

STATEMENT OF TRUST FUNDS. 31st October, 1923.

Lord Rayleigh's Fund consists of the following Investments:—

Great Indian Peninsula Railway £50. 8*s.* 8*d.* Annuity, Class B.
Do. do. do. do. £200 Guaranteed 3 per Cent. Stock.

This Fund is held on trust to apply the income as part of the general income of the Society.

The *De Morgan Medal Fund* consists of Great Western Railway £60 5 per Cent. Preference Stock.

8th November, 1923.

Signed on behalf of the Council:—

W. H. YOUNG, *President.*

A. E. WESTERN, *Treasurer.*

I report to the Members that I have obtained all the information and explanation I required as Auditor, and that I have examined the above Accounts with the books, and that, to the best of my information and of the explanations given to me, such Accounts are properly drawn up so as to exhibit a true and correct view of the state of the affairs of the Society as shown by the books of the Society.

ALLAN J. C. CUNNINGHAM,

Lt.-Col. late R.E., Auditor.

14th November, 1923.

in a circuit round the cylinder, vanishing at O and being π at A'. Then xSO is the line of force $V=0$, OBA' is the equipotential line $U=0$, $A'x'$ is the line of force $V=\pi$, and we see that to the contour $xSOBA'x'$ corresponds in the W-plane to an infinite rectangle formed by $V=0$, $U=0$, $V=\pi$. This is mapped upon the real t-axis, so that t has the same values β and δ at the angles as at the corresponding points in equation (2.31), by the substitution

$$t=\tfrac{1}{2}(\beta+\delta)+\tfrac{1}{2}(\beta-\delta)\cosh W. \qquad (2.32)$$

Such is the form of the solution: z is a function of W, defined by $z=\cosh w$, and equations (2.31) and (2.32). When t is eliminated

$$\frac{dw}{dW}=\frac{E(\cosh W-m)^{\frac{1}{2}}}{(\cosh W-l)^{\frac{1}{2}}} \qquad (2.33)$$

where $l=(2\alpha-\beta-\delta)/(\beta-\delta)$, $m=(2\gamma-\beta-\delta)/(\beta-\delta)$;
since $\alpha>\beta>\gamma>\delta$, $l>1$ while m lies between $+1$ and -1.
It is convenient to write $l=\cosh k$, $m=\cos\kappa$; thus

$$\frac{dw}{dW}=\frac{E(\cosh W-\cos\kappa)^{\frac{1}{2}}}{(\cosh W-\cosh k)^{\frac{1}{2}}} \qquad (2.34)$$

and $z=\cosh w$. It remains to determine E, l, m or E, k, κ as far as possible. Equation (2.31) and Fig. 2.3 show that w increases by $\frac{1}{2}i\pi$, c, $\frac{1}{2}i\pi$, while t diminishes from α to β, β to γ, γ to δ respectively. Hence we have three conditions

$$\tfrac{1}{2}\pi=E\int_0^k\frac{(\cosh U-\cos\kappa)^{\frac{1}{2}}}{(\cosh k-\cosh U)^{\frac{1}{2}}}\,dU,$$

$$c=E\int_0^\kappa\frac{(\cos V-\cos\kappa)^{\frac{1}{2}}}{(\cosh k-\cos V)^{\frac{1}{2}}}\,dV,$$

$$\tfrac{1}{2}\pi=E\int_\kappa^\pi\frac{(\cos\kappa-\cos V)^{\frac{1}{2}}}{(\cosh k-\cos V)^{\frac{1}{2}}}\,dV.$$

If we leave k and κ undetermined, and merely give to E such a value that the total increment of v in (2.31) as t diminishes from α to δ is π, formula (2.34) gives the field for a cylinder whose section is bounded by any one of the ellipses, and a branch of any one of the hyperbolas, such as $KPBA'B'P'K$. Apparently no material simplification ensues if the section is half an ellipse, the hyperbolic arc being replaced by a segment of yOy'.

2.4. *The section of the cylinder is a hyperbolic arc PKP'.*—Example (iii) of §2.1 includes three cases of which this is one: the others, where

the arc is elliptic, bisected by one or other of the principal axes, are very similar. To the path $xSKPKS'x'$ of a point in the z-plane, or rather to a path which borders on this without crossing it, corresponds in the w-plane a path such as is shown in Fig. 2.4; thus we map upon the real t-axis by the relation

$$\frac{dw}{dt} = \frac{E(t-\eta)}{(t-\alpha)^{\frac{1}{2}}(t-\beta)^{\frac{1}{2}}(t-\gamma)^{\frac{1}{2}}(t-\delta)^{\frac{1}{2}}}, \tag{2.41}$$

where $\alpha > \beta > \eta > \gamma > \delta$, and E is real and positive. Further, if U is the potential, zero on the conductor and positive outside, and if the lines

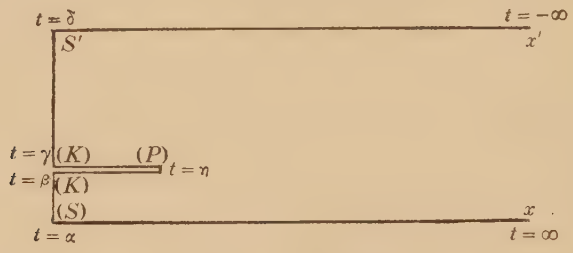

FIG. 2.4 (w-plane).

of force Kx and Kx' are $V = 0$ and $V = \pi$ respectively, we have, as in (2.32) and (2.33),

$$t = \tfrac{1}{2}(\beta+\gamma)+\tfrac{1}{2}(\beta-\gamma)\cosh W, \tag{2.42}$$

$$\frac{dw}{dW} = \frac{E(\cosh W-m)}{(\cosh W-l)^{\frac{1}{2}}(\cosh W-n)^{\frac{1}{2}}}, \tag{2.43}$$

$l > 1 > m > -1 > n$. Four conditions satisfied by F, l, m, n are derived from (2.41) and the dimensions of Fig. 2.4: it is unnecessary to quote them; the problem may be left in this form.

But the claim that these problems must be soluble by the Schwarz-Christoffel method has been made good. That the solutions involve elliptic integrals of all three kinds is a proof that no simple solution can be hoped for.

Incidentally, if in equations (2.31) and (2.41) we replace t by a complex variable, we have solved the problems of the theoretical flow of a stream of liquid past barriers shaped like the conductors, the stream being in the direction xOx'. But again elliptic integrals have to be introduced; and reduction of these to the Jacobian standard forms does not seem to throw new light on the solution.

2.5. As to special conics—parabolas, circles—excluded from §2.1 by the relation $z = \cosh w$, the same problems can be discussed by the same or by simpler methods, with the following results:

Parabolas.—$z = w^2$, $x = u^2 - v^2$, $y = 2uv$. The curves $u = \text{const.}$, $v = \text{const.}$ are parabolas with focus O and axis xOx'; a curve $u = c$ has its vertex A at the point $(c^2, 0)$ on the positive side of O: a curve $v = d$ has its vertex B at $(-d^2, 0)$ on the negative side.

(i) Analogue of (2.22)

$$z = w^2, \quad t = (w-c)^2, \text{ etc.} \tag{2.51}$$

(ii) Analogue of last paragraph of §2.3 (a cylinder whose section is the curve contained by $u = c$ and $v = d$),

$$\frac{dw}{dt} = \frac{E(t-\beta)^{\frac{1}{2}}}{(t-\alpha)^{\frac{1}{2}}(t-\gamma)^{\frac{1}{2}}} \quad (\alpha > \beta > \gamma). \tag{2.52}$$

$$\frac{dw}{dW} = (\cosh W - m)^{\frac{1}{2}} \quad (1 > m > -1). \tag{2.53}$$

(iii) Analogue of §2.4 (a cylinder whose section is an arc of the parabola $v = d$ bisected by Ox'),

$$\frac{dw}{dt} = \frac{E(t-\eta)}{(t-\alpha)^{\frac{1}{2}}(t-\beta)^{\frac{1}{2}}(t-\gamma)^{\frac{1}{2}}} \quad (\alpha > \beta > \eta > \gamma), \tag{2.54}$$

$$\frac{dw}{dW} = \frac{E(\cosh W - m)}{(\cosh W - l)^{\frac{1}{2}}} \quad (l > 1 > m > -1). \tag{2.55}$$

Circular Arc.—If we have a cylinder whose section is an arc of the circle of unit radius with centre at O, the arc subtending an angle 2α at the centre and having the point (1, 0) for its middle point, we can not only form equations analogous to (2.41) and (2.43) but integrate them. Thus we have for the field when such a cylinder is charged:

$$z = \frac{1 + e^W \sin \tfrac{1}{2}\alpha}{1 - e^{-W} \sin \tfrac{1}{2}\alpha}, \tag{2.56}$$

a result which can be identified with one previously obtained by Prof. Love.*

In place of (2.41) we have

$$z = \frac{1 + t \sin \tfrac{1}{2}\alpha + (t^2-1)^{\frac{1}{2}} \sin \tfrac{1}{2}\alpha}{1 + t \sin \tfrac{1}{2}\alpha - (t^2-1)^{\frac{1}{2}} \sin \tfrac{1}{2}\alpha}, \tag{2.57}$$

which, if a complex variable is substituted for t, gives the theoretical flow

* See §14, p. 358 of this volume.

of liquid past a barrier formed of a circular arc. [The results (2.56) and (2.57) were not originally obtained by me by the method of §2.4.]

Two other applications of §1.3 will be given.

3.1. THEOREM.—*A cylindrical conductor has for normal section a curve with two perpendicular axes of symmetry. If the field when the conductor is freely charged is known, it is possible to deduce the field when the same conductor bearing any charge is introduced with any orientation into a uniform field of force.*

The results of §2.2, which were derived from the known field due to a charged elliptic cylinder, bear other interpretations. The lines $V =$ constant of equations (2.24) and (2.25) are equipotentials not only in the problem of §2.2, but also when the cylinder is placed in a uniform field parallel to Oy; and they are stream-lines if it is placed in a uniform flow of liquid parallel to Ox.

Again, it is equally simple to deal with an infinite plane $y'Oy$ with a ridge BAB' running across it: we find

$$t = i \sinh (w-c), \tag{3.11}$$

$$W = ia(z^2-1)^{\frac{1}{2}}-ibz, \tag{3.12}$$

$$iz = a(W^2-1)^{\frac{1}{2}}+bW. \tag{3.13}$$

Here $V = 0$ gives the contour $y'B'ABy$: the curves $V =$ const. are equipotentials for this problem, and when the elliptic cylinder is in a uniform field parallel to Ox, or stream-lines in a flow parallel to Oy.

Further, fields may be superposed. If we superpose the fields of (2.24) and (3.12) we have the field when an elliptic cylinder is placed in a uniform electric field (or in a uniform flow of liquid) whose direction is normal to the axis of the cylinder, but is otherwise unrestricted. We express W (or U or V) in terms of z (or the coordinates x, y) and add the results multiplied by appropriate constants. In the electrostatic problem we can also include a charge upon the cylinder, by superposing the field of (2.1).

3.2. For an elliptic cylinder the results of these operations are well known: had it been otherwise they could have been derived as has been shown from the relation $z = \cosh w$, which gives the field when the cylinder is freely charged. The reasoning in no way depended upon the form of this relation, except that the coordinate axes xOx', yOy', being axes of symmetry, were lines of force. It could be applied to any closed

curve possessing the same symmetry. For if we know the field due to a charge on a cylinder whose section is such a curve, we are led to a relation

$$z(\equiv x+iy) = F(u+iv) = F(w) \qquad (3.21)$$

in which it can be arranged that the potential u shall be equal to c on the cylinder and greater than c outside it, and that v shall increase by 2π in a circuit round the curve, and shall vanish on Ox: v will be $\pi/2$, π, $3\pi/2$ on Oy, Ox', Oy' respectively. Equations (2.23) and (3.11) are established, and the argument proceeds to the end when the fields are superposed. This can be done explicitly if formal solutions for W in terms of z are possible; otherwise the superposition is implicit, or is carried out numerically.

Hence the theorem is established. The converse theorem is also true, that if the field is known when such a cylinder is placed in a uniform field of force, we can deduce the field when it is placed with a different orientation, or when it is freely charged.

3.3. The method of §2.2 can be applied to figures bounded by the ellipse and any of the lines Ax, By, $A'x'$, $B'y'$. For example, Ax may be continued to infinity giving a conductor like a semi-infinite plate or lamina, with the elliptic cylinder covering its edge. We have then $z = \cosh w$ and

$$W = \cosh \tfrac{1}{2}(w-c). \qquad (3.31)$$

Or the ellipse and the whole of the quadrant xOy' may form a conductor, shaped like a right-angled wedge with the ellipse covering the right angle like a cornice: then $z = \cosh w$ and

$$W = \cosh \tfrac{2}{3}(w-c). \qquad (3.32)$$

Similar results apply to the symmetrical cylinders of §3.2.

3.4. The same two-fold symmetry, it may be noted, is possessed by a pair of conductors, symmetrically placed with regard to yOy' and each bisected symmetrically by xOx'. It is sometimes possible to determine the field when the two cylinders bear equal charges, or equal and opposite charges. To limit the length of this paper, I confine myself to single conductors, and, as a final example, consider the field due to a charged (cylindrical) conductor whose normal section is bounded by two circular arcs intersecting at any angle: to avoid a difficulty, I derive this from a more general problem.

4.1. *The section of the cylinder is bounded by two circular arcs.*— Any two circles belong to a coaxal system, and this with the orthogonal

system can be treated by the complex variable. In Fig. 4.11 two circles intersect in SS': the line of centres is Ox, and SS' the radical axis is Oy. If OS is of length h, the values of z at S and S' are $+ih$ and $-ih$. Let

$$w = \log \frac{z+ih}{z-ih}, \tag{4.11}$$

then if $SP = r$, $S'P = r_1$, and, if θ, θ_1 are the inclinations of SP, $S'P$ to Ox

$$u = \log (r_1/r); \quad v = \theta_1 - \theta = S'PS. \tag{4.12}$$

The section of the conductor is $ASBS'A$, and on each of the circles which contain it v is constant. On Ax and Bx', u is zero; at S, u is infinite. To avoid difficulties at S, let a small circle EF orthogonal to the

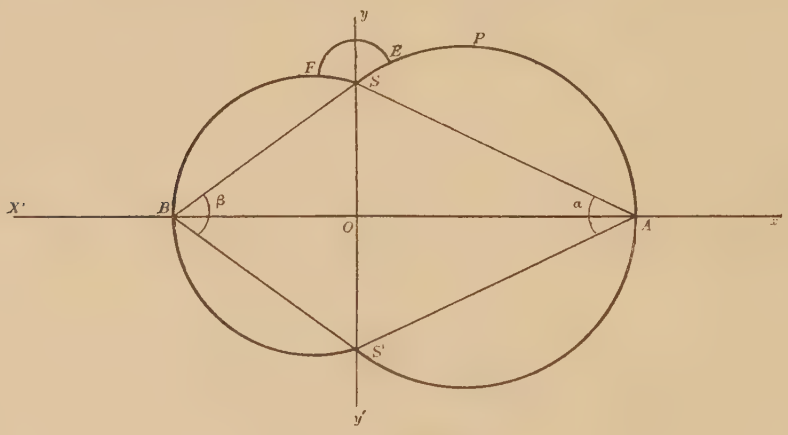

FIG. 4.11 (z-plane).

other two surround S, and consider the conductor increased by this. On the small circle u is constant, $= c$, say. On SAS' v is positive, and on SBS' it is negative: let α and $-\beta$ be the two values, the radii of the circles are cosec α and cosec β.

On xA, $u = 0$, v increases from 0 to α.

On AE, $v = \alpha$, u increases from 0 to c.

On EF, $u = c$, v diminishes from α to $-\beta$.

On FB, $v = -\beta$, u diminishes from c to 0.

On Bx', $u = 0$ v increases from $-\beta$ to 0.

Hence as z describes the contour $xAEFBx'$, w describes the perimeter of the rectangle shown in Fig. 4.12, which can be mapped by the Schwarz-

Christoffel process upon a real t-axis. The values of t (which steadily diminishes) at A, E, F, B may be assumed to be 1, k, $-k$, -1, where

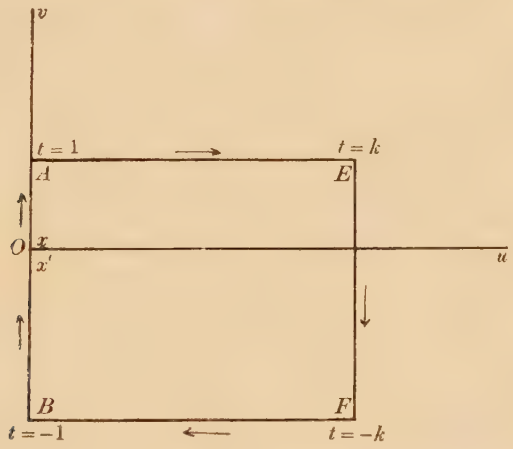

FIG. 4.12 (w-plane).

$1 > k > 0$, without restricting generality, and the formula of transformation must be

$$Ai\frac{dw}{dt} = \frac{1}{(t-1)^{\frac{1}{2}}(t-k)^{\frac{1}{2}}(t+k)^{\frac{1}{2}}(t+1)^{\frac{1}{2}}} = \frac{1}{(t^2-1)^{\frac{1}{2}}(t^2-k^2)^{\frac{1}{2}}}$$

where A is real and positive. Integrating,

$$1/t = \mathrm{sn}\,(B-Aiw) \pmod{k}.$$

If A and B are chosen so that $B-Aiw = -K$ at B where $w = -i\beta$ and $= K$ at A, where $w = ia$, we find

$$\frac{1}{t} = \mathrm{sn}\,\frac{K(\beta-a-2iw)}{\beta+a} \pmod{k}. \qquad (4.13)$$

Finally by considering E, where $w = c+ai$ and $t = k$, we have an equation to determine the modulus k; viz.

$$K'/K = 2c/(\beta+a). \qquad (4.14)$$

The contour $xAEFBx'$ is now mapped on a real axis. To obtain the field of a charged cylinder of which the curve is a section, we have to put $t = \cosh W$; this makes the potential U vanish on the cylinder, while $V = 0$ at A and $V = \pi$ at B.

4.2. *The field due to a charged cylinder whose section is the curve $ASBS'A$, formed of two circular axes.*—Suppose the small circle EF to

shrink to nothing: c must increase indefinitely, and, by (4.14), so must K'/K; k must tend to 0. K becomes $\frac{1}{2}\pi$, and sn becomes sin. Thus

$$\frac{1}{t} = \sin\frac{\pi(\beta-\alpha-2iw)}{2(\beta+\alpha)} = \cos\frac{\pi(\alpha+iw)}{\alpha+\beta}, \tag{4.21}$$

$$t = \cosh W, \tag{4.22}$$

whence

$$\tanh\tfrac{1}{2}W = \tan\frac{\pi(\alpha+iw)}{2(\alpha+\beta)}. \tag{4.23}$$

A better form is obtained if we write $w = 2i\lambda$. Equation (4.11) then gives

$$z = h\cot\lambda \tag{4.24}$$

and

$$\tanh\tfrac{1}{2}(U+iV) = \tan\frac{\pi(\alpha-2\lambda)}{2(\alpha+\beta)}. \tag{4.25}$$

Here U is the potential of the charged cylinder; it vanishes on the cylinder and is positive outside. V is zero on Ax, and is equal to π on Bx'. It will be observed that B in Fig. 4.11 may have any position to the left of A. If B is close to A, we have once more the circular arc of §2.56. In all cases there is a simple rule for the way in which the charge is divided between the two parts of the conductor, viz.

$$\frac{\text{charge on } SAS'}{\text{charge on } SBS'} = \frac{\text{angle } SBA}{\text{angle } SAB'}$$

which is in agreement with a result obtained by Prof. Love on p. 359 of this volume.

THE GENERAL SYMBOLIC NOTATION FOR THE PRINCIPLE OF DUALITY, AND ITS APPLICATION TO DETERMINANTS

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1. The following is the general symbolic theory answering to geometrical reciprocation for space of $(n-1)$ dimensions. The original notation is fully explained in my paper on "The Irreducible Concomitants of two Quadratics in n Variables," *Trans. Camb. Phil. Soc.*, Vol. 21 (1909), pp. 201–210. The present object is to establish a symmetrical symbolism adapted not only to survive projective but reciprocal transformations, in place of the original notation which refers entirely to point coordinates.

I have dealt with special cases of quaternary quadrics in *Proc. London Math. Soc.*, Ser. 2, Vol. 20 (1921), p. 473; also Vol. 21 (1922), p. 381.

2. Let $$(a_1 a_2 \dots a_n) \quad \text{and} \quad (a_1 a_2 \dots a_{n-r} u_1 \dots u_r), \tag{1}$$

$r = 1, 2, 3, \dots, (n-1)$, be the symbolic brackets required by the fundamental theorem of invariants completely to express a concomitant of one or more ground forms

$$f = a_x^p$$

where $$a_x = \sum_{i=1}^{n} a^{(i)} x^{(i)} \quad \text{and} \quad x^{(i)} = (u_1 u_2 \dots u_{n-1})^{(i)}.$$

Each of the n symbols in any of the symbolic brackets (1) is said to be of *currency* one. The $(n-1)$ symbols u express $(n-1)$ sets of cogredient variables (u) reciprocal to point coordinates (x) in space S_{n-1}; thus a set (u) may be said to express S_{n-2} coordinates. The group of n convolved symbols in $(a_1 \dots a_n)$ is said to be of currency n; the groups

$$a_1 a_2 \dots a_{n-r} \quad \text{and} \quad u_1 u_2 \dots u_r$$

are of currencies $(n-r)$ and r respectively. From this point of view the brackets (1) may be abbreviated thus

$$(a_1 a_2 \dots a_n) = a_n,$$

$$(a_1 \dots a_{n-r} u_1 \dots u_r) = (a_{n-r} \theta_r), \tag{2}$$

where the suffix to the Greek letter denotes its currency.

3. *Formal duality.*—Two symbols a_{n-r}, θ_r are said to be *formally dual* if the sum of their currencies is n. Such a relation between a and θ is therefore symmetrical.

As in what follows the symbols will only be used linearly, there is no need to treat separately the symbols a and u. It is therefore proposed to study the symbolic bracket

$$(\theta_{k_1} \theta_{k_2} \dots \theta_{k_p}), \quad \sum_{i=1}^{p} k_i = n, \tag{3}$$

and its dual, where θ_{k_i} is a convolution of k_i symbols a or u of currency one. In the dual symbolic bracket each symbol θ is replaced by its formal dual.

Characteristics.—Let bracket type (3) be said to have characteristic $(k_1, k_2, \dots, k_p)$; then type (1) has characteristic $(1, 1, 1, \dots, 1)$. The principle of duality requires the types of characteristics

$$(n-k_1, n-k_2, \dots, n-k_p) \quad \text{and} \quad (n-1, n-1, \dots, n-1)$$

in consonance with the above definition of formal duality. Now only for binary forms when $n = 2$ do these two dual characteristics belong to type (1). Accordingly we must extend the field of symbolic bracket factors so as to include new and more complex bracket types derived by duality from all possible characteristics of type (1) or (3). The general symbol for these is

$$(\theta_{n-k_1} \theta_{n-k_2} \dots \theta_{n-k_p}), \quad \Sigma k_i = n. \tag{4}$$

This symbol (4) is so far undefined except that it has currency $\Sigma(n-k_i) = (p-1)n$. Hence by the fundamental theorem it is expressible in terms each of $(p-1)$ factors (1).

Two further definitions are now useful. Let this number of factors $(p-1)$ be called the *rank* of the symbol (4), and let the number of symbols θ explicitly contained in any bracket (3) or (4) be the *range* of the bracket.

It is clear that the range of a bracket is unaltered by a dual transformation, whereas the rank is necessarily altered, if $n > 2$.

4. *The fundamental identities.*—Let the operator Ω_a, or more explicitly $\Omega_{a'a''a'''\dots a^{(p)}}$ denote the determinantal permutations of p sets of symbols a where

$$\left.\begin{aligned} &a'_1 a'_2 \dots a'_i = a' \\ &a''_1 a''_2 \dots a''_j = a'' \\ &\text{etc.} \end{aligned}\right\} \quad \begin{aligned} &i, j, \dots, l \text{ being the currencies of } a', a'', \dots, a^{(p)} \\ &\text{respectively,} \end{aligned}$$

then the fundamental identities* can be expressed in three groups, namely

$$\Omega_a(a'\theta'_{n-i})(a''\theta''_{n-j})\dots(a^{(p)}\theta^{(p)}_{n-l})$$

$$= \Omega_\phi(a'a''\dots a^{(p)}\phi_h)(\phi_j\theta''_{n-j})\dots(\phi_l\theta^{(p)}_{n-l}) \quad \text{(I)}$$

or $$= \Omega_\phi(a'a''\dots a^{(p)})(\phi_j\theta''_{n-j})\dots(\phi_l\theta^{(p)}_{n-l}) \quad \text{(II)}$$

or $$= 0, \quad \text{(III)}$$

where $\phi_j\dots\phi_l\phi_h = \theta'_{n-i}$ *and* $i+j+\dots+l < n$ *in* (I),

$\phi_j\dots\phi_l = \theta'_{n-i}$ *and* $i+j+\dots+l = n$ *in* (II),

$\phi_j\dots\phi_l = \theta'_{n-i}$ *and* $i+j+\dots+l > n$ *in* (III).

The number of terms on the left-hand side of each of these three relations is $(i+j+\dots+l)!/i!\,j!\dots l!$. On the right there are

$(n-i)!/(n-i-j\dots-l)!\,j!\dots l!$ in (I) and $(n-i)!/j!\dots l!$ in (II).

There are p different identities (I) arising from the p different ways in which θ may be chosen from the set $\theta', \theta'', \dots, \theta^{(p)}$.

5. If we write $h = n-i-j-\dots-l$ and $a_{n-h} = a'a''\dots a^{(p)}$, we may express the left-hand member of (I) by the single new bracket

$$(a_{n-h}\theta'_{n-i}\theta''_{n-j}\dots\theta^{(p)}_{n-l}) \quad (5)$$

which will complete the definition of bracket (4); for type (5), which is of range $(p+1)$, is a formal dual of $(\beta_h\beta_i\beta_j\dots\beta_l)$ where $h+i+j+\dots+l=n$; that is to say, it is a formal dual of a first-rank bracket. Thus we have

$$(a_{n-h}\theta'_{n-i}\theta''_{n-j}\dots\theta^{(p)}_{n-l}) = \Omega_a(a'\theta'_{n-i})(a''\theta''_{n-j})\dots(a^{(p)}\theta^{(p)}_{n-l}), \quad \text{(IV)}$$

where $a_{n-h} = a'a''\dots a^{(p)}$.

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* *Trans. Camb. Phil. Soc.*, Vol. 21, p. 209, § 21. Identities I, II, III here are equivalent to identity C.

But since $$(a_{n-h}\,\phi_h) = (-)^{h(n-h)}(\phi_h\,a_{n-h})$$

and $$\phi_h\,\phi_j\,\phi_k \ldots \phi_l = (-)^{(n-i-h)h}\,\phi_j\,\phi_k \ldots \phi_l\,\phi_h = (-)^{(n-i-h)h}\,\theta'_{n-i},$$

we may transform the right-hand member of (I) as follows:—

$$\Omega_\phi(a'\,a'' \ldots a^{(p)}\phi_h)(\phi_j\,\theta''_{n-j}) \ldots (\phi_l\,\theta^{(p)}_{n-l})$$
$$= (-)^{h(n-h)}\,\Omega_\phi(\phi_h\,a_{n-h})(\phi_j\,\theta''_{n-j}) \ldots (\phi_l\,\theta^{(p)}_{n-l})$$
$$= (-)^{hi}(\theta'_{n-i}\,a_{n-h}\,\theta''_{n-j} \ldots \theta^{(p)}_{n-l}).$$

In other words, θ' and a in (5) may be interchanged by attaching the sign $(-)^{hi}$ to the bracket. Furthermore the relation

$$(a_{n-h}\,\theta'_{n-i}\,\theta''_{n-j} \ldots \theta^{(p)}_{n-l}) = (-)^{ij}(a_{n-h}\,\theta''_{n-j}\,\theta'_{n-i} \ldots \theta^{(p)}_{n-l})$$

is true of the interchange of any pair among the θ's, as is seen by expanding as in (IV) these last two brackets.

Combining these results we see that bracket (5) is but for sign symmetrical in the $(p+1)$ symbols $a\theta' \ldots \theta^{(p)}$. For greater symmetry replace a, θ by $p+1$ symbols γ and let β denote the formal dual of γ. Then we have the theorem:

If $h+i+j+\ldots+l = n$, *the bracket types*

$$(\beta_h\beta_i \ldots \beta_l) \quad \textit{and} \quad (\gamma_{n-h}\,\gamma_{n-i} \cdots \gamma_{n-l})$$

are formal duals and satisfy the same determinantal relations, namely

$$(\beta_h\,\beta_i \ldots \beta_l) = (-)^{hi}(\beta_i\,\beta_h \ldots \beta_l) = (-)^{ij}(\beta_h\,\beta_j\,\beta_i \ldots \beta_l) = \ldots$$

and

$$(\gamma_{n-h}\,\gamma_{n-i} \cdots \gamma_{n-l}) = (-)^{hi}(\gamma_{n-i}\,\gamma_{n-h} \cdots \gamma_{n-l})$$
$$= (-)^{ij}(\gamma_{n-h}\,\gamma_{n-j}\,\gamma_{n-i} \ldots \gamma_{n-l}) = \ldots. \qquad (V)$$

6. *The dual of a first-rank bracket is irreducible.*—For let this first-rank bracket be $(a_{c_1}a_{c_2} \ldots a_{c_k})$ where $\Sigma c_i = n$, and its dual be $(\beta_{d_1}\beta_{d_2} \ldots \beta_{d_k})$ where $c_i + d_i = n$, $i = 1, 2, \ldots, k$.

Then we define the β bracket to be reducible if it can be expressed as a sum of terms such as

$$\lambda(\beta_{d_1}\beta_{d_2} \ldots \beta_{d_r})(\beta_{d_{r+1}}\beta_{d_k})$$

where none of the original β symbols is broken up, and where λ is numerical. Thus the β bracket is reducible only if

$$d_1+d_2+\ldots+d_r \equiv 0 \pmod{n}, \quad r < k,$$

that is if

$$c_1+c_2+\ldots+c_r \equiv 0 \pmod{n},$$

since $c_i+d_i \equiv 0 \pmod{n}$; and this is contrary to the hypothesis that $\Sigma c_i = n$, $i = 1$ to k. Hence both the first-rank bracket and its dual are irreducible.

7. *Rank of the dual bracket.*—Since $\Sigma c_i = n$, therefore

$$\Sigma d_i = kn - \Sigma c_i = (k-1)n;$$

thus the dual bracket has rank $(k-1)$ and range k, where k can be all integral values from 1 to n. When $k = n$, all the c_i's are unity and the dual pair of factors is

$$(a_1 a_2 \dots a_n) \quad \text{and} \quad (\alpha_{n-1} \alpha'_{n-1} \alpha''_{n-1} \dots).$$

When $k = 1$, the duals are (a_n) and (a_0) the latter being defined as unity.

When $k = 2$, the bracket $(a_r a_{n-r})$ has a dual of exactly the same type. Such brackets are said to be self-reciprocal or self-dual; and as this is the only bracket type for *binary* forms it follows that there is no theory of duality for them. If n is even and $n = 2r$, $r > 1$, we have the important self-dual type

$$(a_r \beta_r),$$

which suggests that the coordinates $(u_1 u_2 \dots u_r)$, or *central coordinates* exactly midway between point coordinates and S_{2r-2} coordinates, play an exceptional part in the geometry of space of $2r-1$ dimensions, as line coordinates do for ordinary space.

8. *Self-dual identities.*—The ternary identity

$$(ab \,.\, \rho\sigma) = a_\rho b_\sigma - a_\sigma b_\rho$$

is an example of a relation which formally survives the dual transformation. This is typical of a general class of factors

$$(a_1 a_2 a_3 \dots a_k \,.\, \psi_1 \psi_2 \dots \psi_k) \quad \text{or} \quad (a_k \psi_1 \dots \psi_k), \quad \text{where } k < n,$$

defined by (IV). Thus

$$(a_k \psi_1 \dots \psi_k) = \Omega_a (a_1 \psi_1)(a_2 \psi_2) \dots (a_k \psi_k) \qquad (6)$$

the currencies of a, ψ being 1, $n-1$ respectively.

Reciprocation will not alter this type of relation, consequently this identity is *self-dual*. If $k = n$ the left-hand member becomes

$$(a_1 a_2 \dots a_n)(\psi_1 \psi_2 \dots \psi_n).$$

9. We must distinguish between alternative duals of a given bracket factor. We revert to the general type (5); as it stands it has $(p+1)$ symbols α, θ, so that the formal dual is

$$(\beta_h \phi'_i \phi''_j \dots \phi_l^{(p)}).$$

If we write $\alpha_{n-h} = \alpha'_i \alpha''_j \dots$, there are now $2p$ explicit symbols α, θ, so that the formal dual is

$$(\beta'_{n-i} \beta''_{n-j} \dots \phi'_i \phi''_j \dots), \tag{7}$$

which is essentially of type (5) as is seen by expansion. Thus the bracket factor (7) is a *self-dual* type whereas (5) as written is not.

Once more, if in (5) $\alpha_{n-h} = \gamma_r \gamma'_q \gamma''_r \dots$; $p+q+r+\dots = n-h$ the dual is of type

$$(\delta_{n-p} \delta'_{n-q} \delta''_{n-r} \dots \phi'_i \phi''_j \dots). \tag{8}$$

Answering to all partitions of α and θ in (5), which express these $(p+1)$ symbols in lower currency, there are dual types: but the most important dual arises when α is expressed in symbols of currency one.

10. *Dual identities.*—Each of the fundamental identities contain symbols which undergo permutation, and possibly others which are constant. Let the permuted symbols be expressed in their original form of currency one; then let each symbol of the identity be formally reciprocated. The result will be a new equation between the symbols, which may be proved to be an identity. So the principle of duality applies to fundamental identities exactly as it applies to the elementary symbols.

The proof involves two formal lemmas, proofs of which are found elsewhere.* They may be quoted as follows:—

LEMMA I.—*If* $R(\theta_1 \theta_2 \dots \theta_{i-r}, \theta_{i-r+1} \dots \theta_i)$ *denote the operation* Ω_θ *applied to the* i *symbols* θ, *disregarding transpositions within either of the two groups* $\theta_1 \dots \theta_{i-r}$ *or* $\theta_{i-r+1} \dots \theta_i$, *then the effect of this operation followed*

* *Trans. Camb. Phil. Soc., ibid.*, pp. 204, 205.

by the operation $R(\theta_1\theta_2\dots\theta_i, \psi)$

is $(i+1)$ *times the operation*

$$R(\theta_1\theta_2\dots\theta_{i-r}, \theta_{i-r+1}\dots\theta_i\psi),$$

both θ *and* ψ *undergoing permutation.*

LEMMA II.—*If* $R(\phi_1\phi_2\dots\phi_r, \phi_{r+1}\dots\phi_j) = \Omega_\phi \quad (r<j)$

and $R(\phi_1\phi_2\dots\phi_r\phi_{r+2}\phi_{r+3}\dots\phi_j, \phi_{r+1}) = \Omega'_\phi,$

then $\Omega'_\phi\Omega_\phi = (r+1)\Omega''_\phi$

where $\Omega''_\phi = R(\phi_1\phi_2\dots\phi_r\phi_{r+1}, \phi_{r+2}\dots\phi_j).$

11. In what follows all the symbols θ, ϕ, ψ are of currency $(n-1)$ while we take $\sigma_r = s_1s_2\dots s_r$.

Starting with the product of two determinants we have the relation

$$(\psi_1\psi_2\dots\psi_n)(s_1s_2\dots s_n) = \Omega^n_s(\psi_1s_1)(\psi_2s_2)\dots(\psi_ns_n) \qquad \text{(VI)}$$

where the index n denotes the number of symbols s permuted. It is clear that here $\Omega^n_s = \Omega^n_\psi$.

Similarly, if $r<n$,

$$(\psi_1\psi_2\dots\psi_r \cdot s_1s_2\dots s_r) = \Omega^r_s(\psi_1s_1)\dots(\psi_rs_r) = \Omega^r_\psi(\psi_1s_1)\dots(\psi_rs_r). \qquad \text{(VII)}$$

Now let $s_1s_2\dots s_{n-1} = \psi_{n+1}$. If we substitute ψ_{n+1} in (VI) we obtain the relation

$$(\psi_1\psi_2\dots\psi_n)(\psi_{n+1}s_n) = \Omega'_\psi(\psi_1\psi_2\dots\psi_{n-1}\psi_{n+1})(\psi_ns_n)$$

where Ω'_ψ does not affect ψ_{n+1}. For Ω'_ψ produces n terms such as the one written down with (ψ_ns_n) for final bracket factor; and each of these terms expanded leads to $(n-1)!$ terms of the series on the right-hand side in (VI). With a slight rearrangement the result can take the alternative forms

$$\Omega^{n+1}_\psi(\psi_1\psi_2\dots\psi_n)(\psi_{n+1}s) = 0, \qquad \text{(VIII)}$$

or $$\Omega^j_\phi(\theta_1\theta_2\dots\theta_i\phi_2\phi_3\dots\phi_j)(\phi_1s) = \Omega^i_\theta(\theta_1\theta_2\dots\theta_{i-1}\phi_1\phi_2\dots\phi_j)(\theta_is), \qquad \text{(IX)}$$

where $\theta_1\theta_2\dots\theta_i$ replaces $\psi_1\psi_2\dots\psi_i$

and $\phi_1\phi_2\dots\phi_j$,, $\psi_{i+1}\psi_{i+2}\dots\psi_{n+1}$;

also $i+j = n+1.$

Replace θ_r by a_r and ϕ_r by b_r and s by σ_{n-1} throughout (VIII) and (IX), and we revert to the original identities, special cases of identities (I) and (II). Thus we have reciprocated in two simple cases; it now follows by

induction that the general case for *two* factors is true, namely if $r \leqslant i$ and $r \leqslant j$,

$$\Omega_\phi^j(\theta_1\theta_2\ldots\theta_i\,\phi_{r+1}\phi_{r+2}\ldots\phi_j)(\phi_1\ldots\phi_r\,.\,s_1s_2\ldots s_r)$$
$$= \Omega_\theta^i(\theta_1\ldots\theta_{i-r}\,\phi_1\ldots\phi_j)(\theta_{i-r+1}\ldots\theta_i\,.\,s_1s_2\ldots s_r), \qquad (X)$$

where $i+j = n+r$.

If $r = 1$, formula (X) becomes (IX) and is therefore true; so it is sufficient to show that, assuming (X) to be true, we may deduce the corresponding result when $r+1$ replaces r. Multiply (X) throughout by $(\theta_{i+1}s_{r+1})$ and operate with Ω_s^{r+1}, then, by (VII), formula (X) becomes

$$\Omega_\phi^j(\theta_1\theta_2\ldots\theta_i\,\phi_{r+1}\ldots\phi_j)(\phi_1\ldots\phi_r\,\theta_{i+1}\,.\,s_1s_r\ldots s_{r+1})$$
$$= \Omega_\theta^i(\theta_1\ldots\theta_{i-r}\,\phi_1\ldots\phi_j)(\theta_{i-r+1}\ldots\theta_i\,\theta_{i+1}\,.\,s_1s_2\ldots s_{r+1}),$$

where Ω_θ^i excludes θ_{i+1}.

Next operate on both sides with $R(\theta_1\theta_2\ldots\theta_i,\ \theta_{i+1})$. The result on the right-hand side by Lemma I is $(r+1)\Omega_\theta^{i+1}$; while on the left we obtain, by identity (IX),

$$\Omega'_\phi\Omega_\phi^j(\theta_1\ldots\theta_i\theta_{i+1}\phi_{r+2}\ldots\phi_j)(\phi_1\ldots\phi_r\phi_{r+1}\,.\,s_1s_2\ldots s_{r+1}),$$

where Ω_ϕ^j is the original $R(j-r,\ r)$ operation and Ω'_ϕ is the same as in Lemma II. Hence by this lemma $\Omega'_\phi\,\Omega_\phi^j = (r+1)\,\Omega_\phi^j$, where the latter operates on

$$(\theta_1\theta_2\ldots\theta_{i+1}\phi_{r+2}\ldots\phi_j)(\phi_1\ldots\phi_{r+1}\,.\,s_1\ldots s_{r+1}).$$

Combining these results, we have

$$\Omega_\phi^j(\theta_1\ldots\theta_{i+1}\phi_{r+2}\ldots\phi_j)(\phi_1\ldots\phi_{r+1}\,.\,s_1\ldots s_{r+1})$$
$$= \Omega_\theta^{i+1}(\theta_1\ldots\theta_{i-r}\phi_1\ldots\phi_j)(\theta_{i-r+1}\ldots\theta_{i+1}\,.\,s_1\ldots s_{r+1}),$$

where $(r+1)$ has replaced r of formula (X). Thus (X) is proved by induction, and holds if

$$r \leqslant i, \quad r \leqslant j, \quad i+j = n+r.$$

If $i = n$, we have

$$\Omega_\theta^n(\theta_1\ldots\theta_{n-j}\phi_1\ldots\phi_j)(\theta_{n-j+1}\ldots\theta_n\,.\,\sigma_j) = (\theta_1\theta_2\ldots\theta_n)(\phi_1\ldots\phi_j\,.\,\sigma_j); \qquad (XI)$$

and if $i > n$

$$\Omega_\theta^i(\theta_1\ldots\theta_n\,\phi_1\ldots\phi_j)(\theta_{n-j+1}\ldots\theta_n\,.\,\sigma_j) = 0. \qquad (XII)$$

A continued application of the results (X), (XI), (XII) to pairs of brackets, if θ is distributed in more than two brackets, gives results precisely dual to the original identities (I), (II), (III). The principle of duality is therefore established, and we have the theorem:

Every symbolic linear identity may be reciprocated, each symbol being replaced by its formal dual.

Since every relation or syzygy between concomitants is ultimately established by linear symbolic identities and the process of exchanging equivalent symbols, it follows that every rational symbolic relation of any degree may be reciprocated.

Application to the theory of determinants.

12. Now let the symbols take special values. We start with two determinants of order n

$$A = (a_1 a_2 \dots a_n), \qquad B = (b_1 b_2 \dots b_n),$$

and a dual determinant

$$\Theta = (\theta_1 \theta_2 \dots \theta_n).$$

Then
$$(b_1 b_2 \dots b_n)(\theta_1 \theta_2 \dots \theta_n) = \begin{vmatrix} (b_1\theta_1) \dots (b_1\theta_n) \\ (b_2\theta_1) \\ \vdots \\ (b_n\theta_1) \dots (b_n\theta_n) \end{vmatrix}. \tag{1}$$

Let α_r denote the co-factor of a_r in A, β_r that of b_r in B.

(i) Put $\theta_r = \alpha_r$, $b_r = a_r$ for all values of r, hence by (1)

$$(a_1 a_2 \dots a_n) = A^{n-1}; \tag{2}$$

a result well known as *Cauchy's Theorem.*

(ii) Put $\theta_r = \alpha_r$, $b_r \neq a_r$, then

$$BA^{n-1} = \begin{vmatrix} (b_1\alpha_1) \dots (b_1\alpha_n) \\ \dots \quad \dots \quad \dots \\ (b_n\alpha_1) \dots (b_n\alpha_n) \end{vmatrix}, \tag{3}$$

or *Bazin's Theorem* (1854).

(iii) Let A/a_r denote the symbol $(a_1 a_2 \dots a_{r-1} a_{r+1} \dots a_n)$ with a_r omitted from $(a_1 \dots a_r)$. Then

$$(a_1 a_2 \dots a_p \beta_1 \dots \beta_{n-p}) = \left(\frac{A}{a_1}\,\frac{A}{a_2} \dots \frac{A}{a_p}\,\frac{B}{b_{p+1}} \dots \frac{B}{b_{n-p}}\right)$$

$$= \Omega_b \frac{Ab_1'}{a_1} \cdot \frac{Ab_2'}{a_2} \dots \frac{Ab_p'}{a_p}\,\frac{Bb_{p+1}'}{b_{p+1}} \dots \frac{Bb_{n-p-1}'}{b_{n-p-1}},$$

where $$b_1' b_2' \dots b_{n-p-1}' = \frac{B}{b_{n-p}}.$$

Also $Bb_r'/b_s = 0$ unless $b_r' = b_s$, hence

$$(a_1 a_2 \dots a_p \beta_1 \dots \beta_{n-p}) = B^{n-p-1} \left\{ \Omega_b \frac{Ab_1}{a_1} \cdot \frac{Ab_2}{a_2} \dots \frac{Ab_p}{a_p} \right\}, \tag{4}$$

the Ω factor being a determinant of p rows and columns. This is Theorem B of a paper by Professor E. T. Whittaker,* while the next result (5) will be Theorem C.

(iv) Let $x = (u_1 u_2 \dots u_{n-1})$ and let $(ax)/u_r$ denote the effect of substituting a for u_r in x without disarranging the order of symbols: so that a occupies the r-th place in $(ax)/u_r$. Also let $\left\{\begin{matrix} a \\ u_r \end{matrix}\right\} x$ denote the same thing. Then for n factors $a, b, \dots, k$

$$(-)^{n-1} a_x b_x c_x \dots k_x = (u_1 u_2 \dots u_{n-1} \cdot a u_2 u_3 \dots u_{n-1} \cdot u_1 b u_3 \dots u_{n-1} \cdot \dots)$$

$$= \left(x \cdot \frac{ax}{u_1} \cdot \frac{bx}{u_2} \cdot \frac{cx}{u_3} \cdot \dots \cdot \frac{kx}{u_{n-1}}\right) \tag{5}$$

where groups of symbols between stops in the brackets are to be treated as a or θ in identity (IV), but each group is here of currency $n-1$. This follows at once from (IV) by writing $x = v_1 v_2 \dots v_{n-1} = u_1 u_2 \dots u_{n-1}$ and expanding the last bracket as

$$\Omega_r \left(v_1 \frac{ax}{u_1}\right)\left(v_2 \frac{bx}{u_2}\right) \dots \left(v_{n-1} \frac{kx}{u_{n-1}}\right);$$

for all terms vanish here except when $v_r = u_r$, $r = 1, 2, \dots$.

(v) More generally the series for

$$\left(a_1 a_2 \dots a_k \frac{b_{k+1} a_r}{a_{k+1}} \dots \frac{b_n a_r}{a_n}\right)$$

* *Proc. Edin. Math. Soc.*, Vol. 36, pp. 107–115.

where $r \neq k+1, \ldots w \neq n$, reduces to a single term

$$(-)^{n-k} A^{k-1} (b_{k+1} a_r) \ldots (b_n a_w)$$

or*

$$(-)^{n-k} A^{k-1} \begin{Bmatrix} b_{k+1} \\ a_r \end{Bmatrix} A \ldots \begin{Bmatrix} b_n \\ a_w \end{Bmatrix} A. \qquad (6)$$

This follows from (IV) by writing $a_1 = a_2 a_3 \ldots a_n$ and expanding; for

$$(a_i a_j) = 0 \text{ if } i \neq j \quad \text{and} \quad \left(a_i \frac{b_{k+t} a_r}{a_{k+t}}\right) = 0 \text{ if } i \neq k+t.$$

(vi) More generally take $n(n-1)/2$ arbitrary symbols c in the form $c_1, \gamma_2, \gamma_3, \ldots, \gamma_{n-1}$, where γ_r denotes r of the symbols c, and let

$$a_1 = a_{n-1}$$

$$a_2 = a_{n-2} a_{n-1}$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$a_{n-1} = a_1 a_2 \ldots a_{n-1};$$

then the same expansion gives the curious result

$$\left(a_{n-1} \cdot (a_{n-2} c_1) \cdot (a_{n-3} \gamma_2) \cdot \ldots (a_1 \gamma_{n-2}) \cdot \gamma_{n-1}\right)$$

$$= (a_{n-1} c_1)(a_{n-2} \gamma_2) \ldots (a_2 \gamma_{n-2})(a_1 \gamma_{n-1}), \qquad (7)$$

again a single term of $(n-1)$ factors for the whole series. Thus in quaternary forms

$$(abc \,.\, bcq \,.\, crs \,.\, tuv) = (abcq)(bcrs)(ctuv).$$

(vii) Once again, if the suffix denote the currency,

$$(a_i \beta_j \gamma_k \,.\, \beta_j \gamma_k \xi_h \cdot \gamma_k \eta_{h+i} \cdot \xi_{n-k}) = (a_i \beta_j \gamma_k \xi_h)(\beta_j \gamma_k \eta_{h+i})(\gamma_k \xi_{n-k}) \qquad (8)$$

where $h+i+j+k = n$, and the number of factors may clearly be generalized.

In these formulæ (1) to (8) a single determinant is expressed as a product of simpler determinants. The left member of (8) is not so obviously a determinant as the other bracket factors. That it is so, and that all the dual bracket types are determinants, may thus be shown:

* Cf. W. L. Ferrar, *London Math. Soc. Records* (Nov. 1922), p. vi.

(viii) The factor type (V), *e.g.* $(a_i\beta_j\gamma_k\,.\,\theta_{n-1}\phi_{n-j}\psi_{n-k})$ is a determinant of order rn, where r is the rank of the factor type. Thus we have

$$(-)^{\epsilon}(a_i\beta_j\gamma_k\,.\,\theta_{n-i}\phi_{n-j}\psi_{n-k})$$

$$=$$

$$\begin{vmatrix}
a_{11}\,a_{12}\ldots a_{1i}\,b_{11}\ \ldots\ b_{1j}\,c_{11}\ \ldots\ c_{1k}\,t_{11}\ \ldots\ t_{.\,n-i} & 0\ \ldots 0 & 0\ \ldots 0 \\
\vdots\quad\ \vdots\ \vdots\quad\ \vdots\ \vdots\quad\ \vdots\ \vdots\quad\ \vdots & \vdots\quad\ \vdots & \vdots\quad\ \vdots \\
a_{n1}\quad\ldots a_{ni}\,b_{n1}\ldots b_{nj}\,c_{n1}\ldots c_{nk}\,t_{n1}\ldots t_{n.\,n-i} & 0\ \ldots 0 & 0\ \ldots 0 \\
a_{11}\quad\ldots a_{1i}\,b_{11}\ \ldots\ b_{1j}\,c_{11}\ \ldots\ c_{1k}\,0\ \ldots 0 & u_{11}\ \ldots\ u_{1.\,n-j}\,0\ \ldots 0 & \\
\vdots\quad\ \vdots\ \vdots\quad\ \vdots\ \vdots\quad\ \vdots\ \vdots\quad\ \vdots & \vdots\quad\ \vdots\quad\ \vdots\quad\ \vdots & \vdots\quad\ \vdots \\
a_{n1}\quad\ldots a_{ni}\,b_{n1}\ldots b_{nj}\,c_{n1}\ldots c_{nk}\,0\ \ldots 0 & u_{n1}\ldots u_{n.\,n-j}\,0\ \ldots 0 & \\
a_{11}\quad\ldots a_{1i}\,b_{11}\ \ldots\ b_{ij}\,c_{11}\ \ldots\ c_{1k}\,0\ \ldots 0 & 0\ \ldots 0 & v_{11}\ \ldots v_{1.\,n-k} \\
\vdots\quad\ \vdots\ \vdots\quad\ \vdots\ \vdots\quad\ \vdots\ \vdots\quad\ \vdots & \vdots\quad\ \vdots & \vdots\quad\ \vdots \\
a_{n1}\quad\ldots a_{ni}\,b_{n1}\ldots b_{nj}\,c_{n1}\ldots c_{nk}\,0\ \ldots 0 & 0\ \ldots 0 & v_{n1}\ldots v_{n.\,n-k}
\end{vmatrix} \tag{9}$$

where

$$(t_1\ldots t_{n-i}) = \theta_{n-i}, \quad (u_1\ldots u_{n-j}) = \phi_{n-j}, \quad (v_1\ldots v_{n-k}) = \psi_{n-k},$$

$$(a_1\ldots a_i) = a_i, \qquad (b_1\ldots b_j) = \beta_j, \qquad (c_1\ldots c_k) = \gamma_k,$$

and $\epsilon = nj-ij-jk-ki$ (a complication which may be removed at the expense of symmetry by writing the columns of the determinant in the order a, t, b, u, c, v).

The duality, as between $\alpha\beta\gamma$ and $\theta\phi\psi$, is exhibited by suitable subtraction of rows in this determinant, bringing blocks of zeros into the a, b, c columns and leaving three-fold repetitions in the t, u, v columns.

The above study which develops from the Aronhold symbolic notation is evidently closely allied to Grassmann's "Ausdehnungslehre."* Sylvester and Sperling both considered determinants of essentially the same type as (9) above (see, for instance, Muir, *History of Determinants*, Vol. 2, pp. 58–62 and p. 108).

[*Added* February, 1924.—Since the above was written, a new treatise by Dr. Roland Weitzenböck—*Invariantentheorie* (Groningen, 1923)—has been published. The book deals with forms in n variables, and refers to

* *Collected Papers*, Vol. 1 (2); also Cf. Whitehead, *Universal Algebra*, Book 4, especially p. 191 and Ch. 3.

original works by Pascal, *Mem. Acc. d. Lincei*, Vol. 5 (1888), and by the author himself [*Wiener Berichte* (1913–1923)]. Pascal proves generally the theorem, due in binary forms to Gordan, that all symbolic reductions ultimately reduce to certain linear identities. This I utilize above in the last four lines of § 11. Also both authors recognize the principle of duality for characteristics

$$(1, 1, \ldots 1) \quad \text{and} \quad (n-1, n-1, \ldots n-1),$$

otherwise the extension of this principle to all first rank bracket forms appears to be new.

What I have called a symbol of *currency* $n-1$ is described as $(n-1)$-*fältig* in German.—H. W. T.]

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Vol. 21, page xliii, line 6 up. Prof. G. A. Miller has pointed out that C. Jordan's earliest published paper was on group theory, and it was published in 1861 (*Journal de l'Ecole polytechnique*, t. 22), and that another paper on the same subject appeared in the *Comptes Rendus* of 1864 (t. 58). His doctorate thesis was also published separately, and bears the date 1860.

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